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ON A PROPERTY OF RIESZ SPACES

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Riesz spaces E with the property that every positive linear operator from E to an Archimedean Riesz space is sequentially order-continuous have been investigated by many authors. A complete characterization of such spaces has been given by D. H. Fremlin. His theorem says that they are exactly the Riesz spaces in which every relatively uniformly closed ideal is a σ-ideal (i.e. admits suprema and infima of monotonic sequences). My purpose in this paper is to investigate Riesz spaces E with a stronger property, namely that every positive linear operator from E to an Archimedean Riesz space maps order convergent sequences to relatively uniformly convergent sequences. It turns out that the above mentioned condition continues to be sufficient as well as necessary. Then I shall show that certain known types of Riesz spaces (e.g. Riesz spaces with the diagonal property or with the property that disjoint order-bounded sequences are stable) have the described property. In the second section of this paper I relate this result to others concerning the order and topological structures of Riesz spaces.

My terminology will follow [1] or [2].

I

Definition 1.1. *Let E be a Riesz space. A sequence x_n of the elements of E order-converges to an element x in E if there is a decreasing sequence $u_n \in E$ with infimum 0 such that $|x_n - x| \leq u_n$ for each n. A sequence x_n is relatively uniformly convergent to an x in E if there is an $e \in E^+$ and a real sequence converging to 0 such that $|x_n - x| \leq a_n\, e$ for every n.*

Definition 1.2. *Let E and F be Riesz spaces. A linear operator T: $E \rightarrow F$ is said to be positive if $Tx \leq Ty$ in F whenever $x \leq y$ in E. T is sequentially order-continuous if* inf *$Tx_n = Tx$ in F whenever x_n is a decreasing sequence with infimum x in E. T is strongly sequentially order-continuous if $x_n \downarrow x$ implies the relatively uniform convergence of Tx_n to Tx in F.*

Definition 1.3. *A Riesz space E has the σ-property if every countable set in E is included in a principal ideal of E.*

Theorem 1.1. *Let E be a Riesz space Then these are equivalent:*

(i) *every order-convex relatively uniformly closed set is closed for the operations of taking infima and suprema of monotonic sequences;*

(ii) *every relatively uniformly closed ideal in E is a σ-ideal;*

(iii) *every positive linear operator from E to an Archimedean Riesz space with σ-property is strongly sequentially order-continuous.*

Proof. (i)$\to$(ii) is obvious. (ii)$\to$(i) is proved in [2] th. 1 3A. I shall prove that (i) implies (iii)

Let T be a positive linear operator from E to F, where F is an Archimedean Riesz space with σ-property, x_n be a sequence decreasing to 0 in E and consider $A=\{x;\ \exists n\, Tx \geq T(x_n)\}$. Then A is an order-convex set including all x_n. The order-convexity of A follows easily from the fact that $x \in A$ and $y \in E^+$ imply the existence of a natural number n such that $T(x+y) \geq Tx \geq Tx_n$. Hence $x+y \in A$. Consider now the set $B=\{x \in E;\ \exists$ a sequence $y_n \in A;\ Ty_n \xrightarrow{ru} Tx\}$ This is an order-convex set including A. For $x \in B$ and $y \in E^+$ imply that there is a sequence $y_n+y \in A$ such that $T(y_n+y) = Ty_n + Ty \xrightarrow{ru} Tx + Ty = T(x+y)$. I shall show that B is relatively uniformly closed, i.e. $x_n \in B$, $x_n \xrightarrow{ru} x$ implies $x \in B$. For each n there is a sequence $x_n^k \in A$ such that $T(x_n^k) \xrightarrow{ru} T(x_n)$ with k tending to infinity, i.e. there is a real sequence a_n^k converging to O with respect to k and an element $u_n \in F$ such that $|T(x_n^k)-T(x_n)| \leq a_n^k u_n$ for each k. Since F has the σ-property there is an $u \in F$ such that $u_n \leq K(n)u$ for each n, where $K(n)$ is a function from N to N, N the set of natural numbers. Denoting $a_n^k K(n)$ by b_n^k we obtain that there is a real sequence $c_n \downarrow 0$ such that for each n there is a $k(n)$ with $c_n \geq b_n^{k(n)}$. This is because the real numbers have the diagonal property, which is precisely the fact stated above (see also the definition 1.4). From this we obtain the existence of a sequence $x_n^{k(n)}$ of the elements of A such that $|T(x_n^{k(n)})-T(x_n)| \leq c_n u$ for each n. The rest of the proof follows from the fact that

$$|T(x_n^{k(n)})-T(x)| \leq |T(x_n^{k(n)})-T(x_n)| + |T(x_n)-T(x)| \leq c_n u + d_n v,$$

where the existence of a real sequence $d_n \downarrow 0$ and an element $v \in F^+$ follows from the above assumption.

By the condition (i) $0 \in B$, i.e. there exists a sequence $y_n \in A$ such that $T(y_n) \xrightarrow{ru} T(0)=0$. Since $y_n \in A$ for each n we obtain for a subsequence, say $x_{k(n)}$ of x_n that $T(y_n) \geq T(x_{k(n)}) \xrightarrow{ru} 0$, so $T(x_n) \xrightarrow{ru} 0$, as required.

(iii)$\to$(i). Let T be a positive linear operator from E to an Archimedean Riesz space F. Let x_n be a decreasing sequence in E with zero infimum. Consider the ideal A generated by x_1. We have inf $x_n = 0$ in A. The restriction of T to A will be denoted by T_1. This is a positive linear operator to an Archimedean Riesz space F_1 = the ideal generated by $T(x_1)$ in F. This space has the σ-property. By the condition (iii) T_1 is strongly sequentially order-continous, i.e. $x_n \downarrow 0$ implies $T(x_n) = T_1(x_n) \xrightarrow{r.u} 0$ in F_1, so $T(x_n) \xrightarrow{r.u} 0$ in F. Thus we have proved that E has the sequential order-continuity property (for this definition see [2]). The result follows now from [2], th. 1.3A.

Lemma 1.1. *Each Banach lattice has the σ-property.*

Proof. See [3].

Corollary 1.1. *Let E be a Riesz space with the property that every relatively uniformly closed ideal in E is a σ-ideal. Then every positive linear operator from E to a Banach lattice is strongly sequentially order-continuous.*

Definition 1.4. *A Riesz space E has the diagonal property if whenever x_{nk} is a double sequence in E such that $x_{nk} \downarrow 0$ for each n, there is a sequence $x_n \downarrow 0$ in E such that for each n there is a k with $x_n \geq x_{nk}$.*

Definition 1.5. *Sequential order-convergence is relatively uniform in E if $x_n \downarrow 0$ implies $x_n \to 0$ relatively uniformly.*

$$(x_n \xrightarrow{r.u} 0).$$

Lemma 1.2. ([1], th. 70.2.) *An Archimedean Riesz space E has the diagonal property if and only if it has the σ-property and sequential order-convergence is relatively uniform in E.*

Lemma 1.3. ([2], prop. 3.4.) *If E is an Archimedean Riesz space, then E has the diagonal property if and only if it has the σ-property and the sequential order--continuity property.*

Proposition 1.1. *If E is an Archimedean Riesz space then E has the diagonal property if and only if it has the σ-property and every positive linear operator from E to an Archimedean Riesz space with σ-property is strongly sequentially order-continuous.*

Proof. If E has the diagonal property then, by lemma 1.2. it has the σ-property and sequential order-convergence is relatively uniform. It follows easily that each positive linear operator on E has the property stated in the proposition.

Conversely, if E has the σ-property then by the hypothesis the identical mapping I: $E \to E$ maps order-convergent sequences to relatively uniformly sequences, i.e. sequential order-convergence is relatively uniform in E.

Definition 1.6. *Disjoint order-bounded sequences are stable in E if* $x_n \to 0$ *relatively uniformly whenever* x_n *is a disjoint order-bounded sequence in E.*

Proposition 1.2. *If E is a Riesz space in which disjoint order-bounded sequences are stable, then every positive linear operator from E to an Archimedean Riesz space with σ-property is strongly sequentially order-continuous.*

Proof Follows from [2], prop. 3.3.

II

Definition 2.1. *A topology on a Riesz space E is compatible if* E^+ *is closed with respect to this topology.*

A topology on E is locally solid if the solid neighbourhoods of 0 form a local base.

Proposition 2.1. *Let E be a Riesz space. Then these are equivalent:*

(i) *every order-convex relatively uniformly closed set in E is closed for the operations of taking infima and suprema of monotonic sequences;*

(ii) *whenever F is an Archimedean Riesz space with a locally solid, locally convex topology and T: E→F is a positive linear operator, then* $\inf x_n = 0$ *implies* $T(x_n) \to 0$ *in the topology of F.*

Proof. (i)→(ii). Let f be a positive linear functional on F. In view of [2], th. 1.3A, we obtain that $\inf x_n = 0$ implies $f\,T(x_n) \to 0$ in F, since $f\,T$ is a positive linear functional on E. Since owing to the local solidness of F each continuous linear functional on F is the difference between two positive linear functionals, we have that $T(x_n) \to 0$ in the weak topology of F, so $T(x_n) \to 0$ in the topology of F, as required.

(ii)→(i). Let T be a positive linear operator from E to an Archimedean Riesz space F Then there exists a locally solid, locally convex topology on F (see [4]). By the condition (ii) we have that $T(x_n) \to 0$ in this topology whenever $\inf x_n = 0$ in E. From this we obtain $\inf T(x_n) = 0$ in F, since the topology is compatible. The result follows from [2], th. 1.3A.

Proposition 2.2. *Let E be a Riesz space. Then*

(i) *every order-convex relatively uniformly closed set in E is closed for the operations of taking infima and suprema of monotonic sequences implies*

(ii) *whenever F is an Archimedean Riesz space and T: E→F is a positive linear operator, then* $x_n \downarrow 0$ *in a compatible topology of E implies* $\inf T(x_n) = 0$.

Proof. Since E^+ is closed with respect to the topology of E, we have that $\inf x_n = 0$ (see [5], prop. 3.1.14). The result then follows from [2], th. 1.3A, since the condition (i) implies that E has the sequential order-continuity property.

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ОБ ОДНОМ СВОЙСТВЕ ПРОСТРАНСТВ РИССА

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Резюме

В работе обобщаются некоторые результаты Д. Фремлина. Даются необходимые и достаточные условия, при которых всякий положительный линейний оператор обладает свойством усиленной o-непрерывности.