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# EXISTENCE RESULTS FOR IMPLICIT DIFFERENTIAL EQUATIONS

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ABSTRACT. The existence of solutions for certain systems of ordinary differential equations is studied when they are not solvable in the highest-order derivatives. Proofs of results are based on a theory of pseudomonotone operators and a generalized Leray-Schauder degree.

## 1. Introduction

In this paper, we shall study the initial value problem for two implicit systems of the forms

$$F(x,x',t)=0\,,\qquad x(0)=x_0\,, \tag{1.1}$$

$$G(x,x',y,t)=0\,,\qquad x(0)=x_0\,, \tag{1.2}$$

where $F\colon \mathbb{R}^{n+n+1}\to\mathbb{R}^n$, $G\colon \mathbb{R}^{n+n+m+1}\to\mathbb{R}^{n+m}$ are Carathéodory continuous [1; p. 76]. Problem (1.2) arises for example in modelling nonlinear electrical networks [13] by using Kirchhoff's laws. Problem (1.1) is a general implicit ordinary differential equation. The purpose of this paper is to derive Peano-like existence theorems for (1.1–2).

At the end of this note, the method used for proving existence results for (1.1–2) is applied also to the boundary value problems

$$F(x,x'',t)=0\,,\qquad x(-a)=x(a)=0\,, \tag{1.3}$$

$$G(x,x'',y,t)=0\,,\qquad x(-a)=x(a)=0\,, \tag{1.4}$$

where $F\colon \mathbb{R}^{n+n+1}\to\mathbb{R}^n$, $G\colon \mathbb{R}^{n+n+m+1}\to\mathbb{R}^{n+m}$ are Carathéodory continuous and $a>0$.

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The method of this paper, based on a theory of pseudomonotone operators and a generalized Leray-Schauder degree [2], is very similar to those used in [6], [7]. Implicit ordinary differential equations are also studied in [9], [10], [11], [14] by using a generalized degree for $A$-proper mappings. A theory of differential inclusions is applied in [3], [5], [8], [12] to treat equations like (1.1) and (1.3).

As examples, consider the problems

$$\begin{aligned} &y'' = g(y'',t) + h(t,y)\,, && 0 \le t \le 1\,, \\ &y(0) = y_0\,, \quad y'(0) = z_0\,, && y_0, z_0 \in \mathbb{R}^n\,, \end{aligned} \tag{1.5}$$

$$\begin{aligned} &y'' = g(y'',t) + h(t,y)\,, && 0 \le t \le 1\,, \\ &y(0) = y(1) = 0\,, \end{aligned} \tag{1.6}$$

where $g\colon \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ and $h\colon [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ are Carathéodory continuous, and moreover, $h$ is bounded on $[0,1] \times \mathbb{R}^n$.

Problems like (1.5–6) are studied by many authors. In [11], [14], when $n = 1$ and $z - g(z,t)$ is strictly monotone in $z \in \mathbb{R}$ uniformly with respect to (u.w.r.t. for short) $t \in [0,1]$. In [5], when $g(z,t)$ is nonexpansive in $z \in \mathbb{R}^n$ (see [4; p. 69]) u.w.r.t. $t \in [0,1]$. In [9], [10], when $n = 1$ and $g(z,t)$ is contractive in $z \in \mathbb{R}$ u.w.r.t. $t \in [0,1]$. In [8], when $g(z,t) = \tilde{g}\big(z, a(t)\big)$, where $\tilde{g}\colon \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ is continuous satisfying additional conditions (for instance, $\dim\{r \in \mathbb{R}^n \mid r = \tilde{g}(r,\tilde{a}) + b\} = 0$, $\forall \tilde{a} \in \mathbb{R}^k$, $\forall b \in \mathbb{R}^n$) and $a\colon \mathbb{R} \to \mathbb{R}^k$ is Lebesgue measurable. Finally, in [12], when $n = 1$, $g$, $h$ are independent of $t$, and for every $w \in \mathbb{R}$ the function $z \mapsto z - g(z) - h(w)$ changes the sign on $\mathbb{R}$, and $\operatorname{int}\{r \in \mathbb{R} \mid r = g(r) + h(w)\} = \emptyset$.

The results of this paper imply that (1.5) and (1.6) have weak solutions provided the mapping $z - g(z,t)$ is monotone in $z \in \mathbb{R}^n$ u.w.r.t. $t \in [0,1]$ and satisfying certain growth conditions in $z \in \mathbb{R}^n$ u.w.r.t. $t \in [0,1]$ (see (H1–2) below). Furthermore, if $g$, $h$ are, in addition, independent of $t$ satisfying $\operatorname{int}\{r \in \mathbb{R}^n \mid r = g(r)\} \ne \emptyset$ and $h(0) = 0$, then the result of [12; Theorem C] cannot be applied to those (1.5) with $n = 1$ neither [8; Main Theorem 1.1] is applicable to such (1.6). When $n = 1$, very simple equations with the above properties are explicitly given by

$$2|y'' - 1| - |y'' + 1| + 3y'' - 1 = \sin y$$

with either the initial or the boundary value conditions of (1.5) and (1.6).

Summarizing, our results do not follow from the above papers. On the other hand, those papers deal with much more general problems than we treat in this note. Finally, since the set of solutions of an equation with a continuous monotone operator is closed and convex, perhaps a method of [5] would give an alternative way for solving the problems of this paper.

## 2. Main results

Let $\langle \cdot , \cdot \rangle_p$ be a scalar product on $\mathbb{R}^p$ with the corresponding norm $|\cdot|_p$. Concerning (1.1), we assume the existence of a constant $M > 0$ such that

$$\begin{aligned}&\big\langle F(x,z_1,t) - F(x,z_2,t), z_1 - z_2 \big\rangle_n \geq 0\\ &\text{for any } z_1, z_2 \in \mathbb{R}^n \text{ and } |x|_n \leq M\,,\ \ |t| \leq M\,.\end{aligned} \tag{H1}$$

$$\begin{aligned}&0 < \varliminf_{|z|_n\to\infty} |F(x,z,t)|_n/|z|_n \leq \varlimsup_{|z|_n\to\infty} |F(x,z,t)|_n/|z|_n < \infty\\ &\text{uniformly with respect to } |x|_n \leq M\,,\ \ |t| \leq M\,.\end{aligned} \tag{H2}$$

The following definitions will be needed in what follows (see [2; p. 946]). Let $H$ be a Hilbert space with an inner product $(\cdot\,,\,\cdot)$, and let $\Omega$ be a bounded open convex subset of $H$. A mapping $T\colon \bar\Omega \to H$ is:

– *pseudomonotone* ($T \in PM$) if for any sequence $\{u_i\}_{i=1}^\infty \subset \bar\Omega$ with $u_i \rightharpoonup u \in \bar\Omega$ (weak convergence) and $\varlimsup_{i\to\infty} \big(T(u_i), u_i - u\big) \leq 0$, it follows that $T(u_i) \rightharpoonup T(u)$ and $\big(T(u_i), u_i\big) \to \big(T(u), u\big)$;

– *of class* $S_+$ ($T \in S_+$) if for any sequence $\{u_i\}_{i=1}^\infty \subset \bar\Omega$ with $u_i \rightharpoonup u \in \bar\Omega$ and $\varlimsup_{i\to\infty} \big(T(u_i), u_i - u\big) \leq 0$, it follows that $u_i \to u$;

– *bounded* if it takes any bounded set of $\bar\Omega$ into a bounded set.

It is not hard to see that $T \in PM \implies T + \varepsilon\mathbb{I} \in S_+ \ \ \forall \varepsilon > 0$, where $\mathbb{I}\colon H \to H$ is the identity map.

We are interested in weak solutions of (1.1–4) in the sense that their highest-order derivatives are integrable, and, in addition, they satisfy (1.1–4) almost everywhere with respect to $t$.

**THEOREM 2.1.** *Under the assumptions* (H1-2), *for any* $x_0 \in \mathbb{R}^n$ *satisfying* $|x_0|_n < M$, *there is an* $a > 0$ *such that problem* (1.1) *has a weak solution on* $(-a, a)$.

P r o o f. The assumption (H2) implies the existence of positive constants $\alpha$, $\beta$, $\gamma$ such that

$$\begin{aligned}&\beta\big(|z|_n^2 - \gamma\big) \leq |F(x,z,t)|_n^2 \leq \alpha\big(|z|_n^2 + 1\big)\\ &\text{for any } z \text{ and } \ |x|_n \leq M\,,\ \ |t| \leq M\,.\end{aligned} \tag{2.1}$$

By taking

$$H = L_2\big([-a,a], \mathbb{R}^n\big)\,,$$

$$T_\lambda(z)(t) = F\bigg(\lambda\Big(x_0 + \int_0^t z(s)\ \mathrm{d}s\Big), z(t), \lambda t\bigg)\,, \quad \lambda \in [0,1]\,,$$

problem (1.1) is equivalent to the equation $T_1(z) = 0$ in $H$. Let $(\cdot\,,\,\cdot)_{L_2} = \int\limits_{-a}^{a} \langle z_1(t), z_2(t)\rangle_n \, dt$ be the inner product on $H$ with the norm $|z|_{L_2} = \sqrt{(z,z)_{L_2}}$. We take

$$\Omega = \left\{ z \in H \mid \ |z|_{L_2} < 1 \right\}.$$

From $z \in \bar{\Omega}$, we obtain

$$\left| \int\limits_0^t z(s)\, \mathrm{d}s \right|_n \leq \sqrt{\left| \int\limits_0^t |z(s)|_n^2 \, \mathrm{d}s \right|} \sqrt{\left| \int\limits_0^t 1 \, \mathrm{d}s \right|} \leq \sqrt{|t|} \cdot |z|_{L_2} \leq \sqrt{a}\,.$$

Hence, for any $z \in \bar{\Omega}$, we have

$$\left| x_0 + \int\limits_0^t z(s)\, \mathrm{d}s \right|_n \leq |x_0|_n + \sqrt{a}\,.$$

If $|x_0|_n < M$, then we take a fixed $a$ such that

$$0 < a < \min\left\{ \left(M - |x_0|_n\right)^2, \frac{1}{2\gamma}, M \right\}.$$

Now we prove that $T_\lambda \in PM$. So let $\{z_i\}_{i=1}^\infty \subset \bar{\Omega}$ with $z_i \rightharpoonup z$ and $\overline{\lim\limits_{i\to\infty}} \left(T_\lambda(z_i), z_i - z\right)_{L_2} \leq 0$. Then $z \in \bar{\Omega}$, and the sequence $\left\{ \int\limits_0^t z_i(s)\, \mathrm{d}s \right\}_{i=1}^\infty$ converges to $\int\limits_0^t z(s)\, \mathrm{d}s$ in $H$. Since $\{z_i\}_{i=1}^\infty$ is bounded, and $T_\lambda$ is bounded by (2.1), we can assume $T_\lambda(z_i) \rightharpoonup \tilde{z} \in H$. Let $u \in H$ be arbitrary, then (H1) implies

$$\int\limits_{-a}^{a} \Big\langle F\Big(\lambda\Big(x_0 + \int\limits_0^t z_i(s)\, \mathrm{d}s\Big), z_i(t), \lambda t\Big) - F\Big(\lambda\Big(x_0 + \int\limits_0^t z_i(s)\, \mathrm{d}s\Big), u(t), \lambda t\Big)\,,\ z_i(t) - u(t) \Big\rangle_n \, \mathrm{d}t \geq 0\,.$$

Since $\int_0^t z_i(s)\,ds \to \int_0^t z(s)\,ds$ in $H$ and $z_i \rightharpoonup z$, we have

$$\begin{aligned}(T_\lambda(z_i), z_i - z)_{L_2} + (T_\lambda(z_i), z - u)_{L_2} &= (T_\lambda(z_i), z_i - u)_{L_2}\\ \geq \Big(F\Big(\lambda\Big(x_0 + \int_0^t z_i(s)\,ds\Big), u, \lambda\cdot\Big), z_i - u\Big)_{L_2}&\\ \to \Big(F\Big(\lambda\Big(x_0 + \int_0^t z(s)\,ds\Big), u, \lambda\cdot\Big), z - u\Big)_{L_2}&.\end{aligned} \tag{2.2}$$

By using $\overline{\lim}_{i\to\infty} (T_\lambda(z_i), z_i - z)_{L_2} \leq 0$ and $T_\lambda(z_i) \rightharpoonup \tilde{z}$, we have

$$(\tilde{z}, z - u)_{L_2} \geq \Big(F\Big(\lambda\Big(x_0 + \int_0^t z(s)\,ds\Big), u, \lambda\cdot\Big), z - u\Big)_{L_2}.$$

Setting $\omega v = z - u$ for $\omega > 0$, we obtain

$$(\tilde{z}, v)_{L_2} \geq \Big(F\Big(\lambda\Big(x_0 + \int_0^t z(s)\,ds\Big), z - \omega v, \lambda\cdot\Big), v\Big)_{L_2},$$

and letting $\omega \to 0^+$, we arrive at

$$(\tilde{z}, v)_{L_2} \geq (T_\lambda(z), v)_{L_2} \qquad \forall v \in H.$$

So $\tilde{z} = T_\lambda(z)$, i.e., $T_\lambda(z_i) \rightharpoonup T_\lambda(z)$. Furthermore, by taking $z = u$ in (2.2), we obtain $\underline{\lim}_{i\to\infty} (T_\lambda(z_i), z_i - z)_{L_2} \geq 0$. Hence $\overline{\lim}_{i\to\infty} (T_\lambda(z_i), z_i - z)_{L_2} \leq 0$ gives $\lim_{i\to\infty} (T_\lambda(z_i), z_i - z)_{L_2} = 0$, and consequently, $\lim_{i\to\infty} (T_\lambda(z_i), z_i)_{L_2} = (T_\lambda(z), z)_{L_2}$. The pseudomonotony of $T_\lambda$ is proved.

Now, (2.1) implies for $z \in \bar{\Omega}$ that

$$\begin{aligned}|T_\lambda(z)|^2_{L_2} &= \int_{-a}^{a} \Big|F\Big(\lambda\Big(x_0 + \int_0^t z(s)\,ds\Big), z(t), \lambda t\Big)\Big|^2_n\,dt\\ &\geq \beta \int_{-a}^{a} \big(|z(t)|^2_n - \gamma\big)\,dt = \beta\big(|z|^2_{L_2} - 2a\gamma\big).\end{aligned}$$

Hence, if $|z|_{L_2} = 1$ and $2a\gamma < 1$, then $T_\lambda(z) \neq 0$. Since $T_\lambda \in PM$, then $T_{\varepsilon\lambda} = T_\lambda + \varepsilon\mathbb{I} \in S_+$ for any $\varepsilon > 0$. It is clear that $T_{\varepsilon\lambda}(z) \neq 0$ for any $z \in \partial\Omega$ and a sufficiently small $\varepsilon > 0$. Consequently,

$$\deg(T_{\varepsilon 1}, \Omega, 0) = \deg(T_{\varepsilon 0}, \Omega, 0),$$

where deg is the generalized Leray-Schauder degree in the sense of [2]. We note that $T_{\varepsilon 0}(z) = F(0,z,0) + \varepsilon z$, and by (H1), we have

$$\Big((F(0,z_1,0)+\varepsilon z_1)-(F(0,z_2,0)+\varepsilon z_2),\, z_1-z_2\Big)_{L^2} \geq \varepsilon |z_1-z_2|_{L_2}^2 \qquad \forall\, z_1, z_2 \in H\,.$$

So $T_{\varepsilon 0}(z)$ is strongly monotone, and consequently, it is a homeomorphism (see [4; p. 100]). Finally, by using

$$\begin{aligned}|T_{\varepsilon 0}(z)|_{L_2}^2 &= |T_0(z)+\varepsilon z|_{L_2}^2 \geq \big(|T_0(z)|_{L_2} - \varepsilon |z|_{L_2}\big)^2 \geq \tfrac{1}{2}|T_0(z)|_{L_2}^2 - \varepsilon^2 |z|_{L_2}^2 \\ &\geq \tfrac{1}{2}\beta\big(|z|_{L_2}^2 - 2a\gamma\big) - \varepsilon^2|z|_{L_2}^2 = \frac{\beta}{2}\Big(\Big(1-\frac{2\varepsilon^2}{\beta}\Big)|z|_{L_2}^2 - 2a\gamma\Big)\,,\end{aligned}$$

we see that for any sufficiently small $\varepsilon > 0$, the equation $T_{\varepsilon 0}(z) = 0$ has a unique solution that is in $\Omega$. Hence we obtain that $\deg(T_{\varepsilon 0}, \Omega, 0) \neq 0$, and so $T_{\varepsilon 1}(z) = 0$ has a solution $z_\varepsilon \in \Omega$ for any sufficiently small $\varepsilon > 0$. By using $T_1 \in PM$ and letting $\varepsilon \to 0^+$, we obtain the solvability of $T_1(z) = 0$ in $\Omega$. The proof is finished. □

Concerning (1.2), we assume the existence of a constant $M > 0$ such that

$$\begin{aligned}&\big\langle G(x,z_1,y_1,t) - G(x,z_2,y_2,t),\, (z_1,y_1)-(z_2,y_2)\big\rangle_{n+m} \geq 0 \\ &\text{for any } (z_1,y_1),(z_2,y_2) \in \mathbb{R}^{n+m} \text{ and } |x|_n \leq M\,,\quad |t| \leq M\,.\end{aligned} \tag{A1}$$

$$\begin{aligned}0 < &\varliminf_{|(z,y)|_{n+m}\to\infty} |G(x,z,y,t)|_{n+m}/|(z,y)|_{n+m}\,, \\ &\varlimsup_{|(z,y)|_{n+m}\to\infty} |G(x,z,y,t)|_{n+m}/|(z,y)|_{n+m} < \infty \\ &\text{uniformly with respect to } |x|_n \leq M\,,\quad |t| \leq M\,.\end{aligned} \tag{A2}$$

**THEOREM 2.2.** *Under the assumptions* (A1–2), *for any* $x_0 \in \mathbb{R}^n$ *satisfying* $|x_0|_n < M$, *there is an* $a > 0$ *such that problem* (1.2) *has a weak solution on* $(-a,a)$.

Proof. By taking

$$H = L_2([-a,a], \mathbb{R}^{n+m})$$

$$T_\lambda(z,y)(t) = G\Big(\lambda\Big(x_0 + \int_0^t z(s)\,\mathrm{d}s\Big), z(t), y(t), \lambda t\Big)\,, \qquad \lambda \in [0,1]\,,$$

problem (1.2) is equivalent to the equation $T_1(z,y) = 0$ in $H$. Now we can repeat the proof of Theorem 2.1. The proof is finished. □

**Remark 2.3.** If $M$ can be arbitrarily large under the conditions (H1–2), respectively (A1–2), then problem (1.1), respectively (1.2), has a weak solution on any finite interval with any initial value $x_0$.

Finally, we note that the method used in the above proofs can be directly applied to the boundary value problems (1.3) and (1.4). Indeed, let $\psi_a$ be the Green's function of the problem $x \to x''$, $x(-a) = x(a) = 0$. Then, by taking

$$
\begin{aligned}
H &= L_2([-a,a],\mathbb{R}^n)\,,\\
T_\lambda(z)(t) &= F\Big(\lambda \int_{-a}^{a} \psi_a(t,s)z(s)\,\mathrm{d}s,\, z(t),\, \lambda t\Big)\,, \qquad \lambda \in [0,1]\,,
\end{aligned}
$$

for (1.3), respectively

$$
\begin{aligned}
H &= L_2([-a,a],\mathbb{R}^{n+m})\\
T_\lambda(z,y)(t) &= G\Big(\lambda \int_{-a}^{a} \psi_a(t,s)z(s)\,\mathrm{d}s,\, z(t), y(t),\, \lambda t\Big)\,, \qquad \lambda \in [0,1]\,,
\end{aligned}
$$

for (1.4), problem (1.3), respectively (1.4), is equivalent to the equation $T_1(z)=0$, respectively $T_1(z,y) = 0$, in $H$. Moreover, it is not hard to see that

$$
\sup_{t\in[-a,a]} \Big| \int_{-a}^{a} \psi_a(t,s)h(s)\,\mathrm{d}s \Big|_n \le \sqrt{8a^3}\,|h|_{L_2} \qquad \forall\, h \in L_2([-a,a],\mathbb{R}^n)\,.
$$

Now, similarly as above, for Theorems 2.1–2 we obtain:

**THEOREM 2.4.** *The assumptions* (H1–2), *respectively* (A1–2), *imply the existence of a weak solution for* (1.3), *respectively* (1.4), *for any sufficiently small* $a > 0$.

**Remark 2.5.** If $M$ can be arbitrarily large under the conditions (H1–2), respectively (A1–2), then problem (1.3), respectively (1.4), has a weak solution for any $a > 0$.

**Remark 2.6.** Similarly, we can prove the solvability of the problems

$$
\begin{aligned}
F(x,x'',t) &= 0\,, \qquad x(0) = x_0\,,\ \ x'(0) = z_0\,,\\
G(x,x'',y,t) &= 0\,, \qquad x(0) = x_0\,,\ \ x'(0) = z_0\,,
\end{aligned}
$$

where $F$, respectively $G$, satisfies (H1–2), respectively (A1–2).

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