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A STRUCTURAL PROPERTY OF PLANAR GRAPHS AND THE SIMULTANEOUS COLOURING OF THEIR EDGES AND FACES

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Let G be a planar graph with the minimal and maximal degrees $\delta(G)$ and $\Delta(G)$. respectively. Denote by $n_{s,t}$ the number of vertices in G having degree s and being incident with exactly t triangles, and let $n^*_{s,t} = \sum_{i \geqslant t} n_{s,i}$. It was proved in [5] that any 3-connected 5-regular planar graph has a vertex which is incident with more than three triangles. From [9], it follows under the same assumptions that $2n_{5,5} + n_{5,4} \geqslant 24$. Our first result is

Theorem 1. *Let G be a planar graph with $\delta(G) \geqslant 3$. Then*

$$6n_{3,3} + 5n_{3,2} + 4n_{3,1} + 3n_{3,0} + 4n_{4,4} + 3n_{4,3} + 2n_{4,2} + n_{4,1} + 2n_{5,5} + n_{5,4} \geqslant 24.$$

Proof. Without loss of generality, G may be assumed to be connected. It follows from the Euler formula that

$$\sum_{v \in V(G)} (s(v) - 6) + \sum_{k \geqslant 4} (2k - 6) F_k = -12, \tag{1}$$

where $s(v)$ is the degree of a vertex v, and F_k — the number of k-faces in G.

Indeed, we have:

$$V - E + F = 2; \tag{2}$$

$$2E = \sum_{v \in V(G)} s(v); \tag{3}$$

$$2E = \sum_{k \geqslant 3} kF_k. \tag{4}$$

To obtain (1), multiply the inequality (2) by 6, the inequality (4) by two and add them with (3).

Denote by $f_i(v)$ the number of i-faces incident with v, then (1) may be rewritten as

$$\sum_{v \in V(G)} (s(v) - 6 + \sum_{k \geqslant 4} (2k - 6) f_k(v)/k) = -12,$$

or

$$\sum_{v \in V(G)} g(v) = -12. \tag{5}$$

Denote by $V_{s,t}$ the set of those s-vertices which are incident with exactly t triangles. Clearly, if $v \in V_{s,t}$, then $g(v) \geqslant s - 6 + (s - t)/2 = h(s, t)$, so from (5) it follows that

$$-12 = \sum_{\{(s,t)\}} \sum_{v \in V_{s,t}} g(v) \geqslant \sum_{\{(s,t)\}} n_{s,t} h(s, t).$$

Hence,

$$-\sum_{\{(s,t):\, h(s,t) < 0\}} n_{s,t} h(s, t) \geqslant 12 + \sum_{\{(s,t):\, h(s,t) \geqslant 0\}} n_{s,t} h(s, t) \geqslant 12,$$

or

$$3n_{3,3} + \frac{5}{2} n_{3,2} + 2n_{3,1} + \frac{3}{2} n_{3,0} + 2n_{4,4} + \frac{3}{2} n_{4,3} + n_{4,2} + \frac{1}{2} n_{4,1} + n_{5,5} + \frac{1}{2} n_{5,4} \geqslant 12.$$

This completes the proof.

Corollary. *If G is a planar graph with $\delta(G) \geqslant 3$, then $3n^*_{3,0} + 2n^*_{4,1} + n^*_{5,4} \geqslant 12$.*

We now apply the Corollary to the problem of a simultaneous colouring the edges and the faces of planar graphs (any two edges and/or faces should be coloured with different colours provided they are adjacent or incident). Let $\chi_{ef}(G)$ denote the minimal number of colours needed to colour G in this way. In [4, 3] this problem was considered for 3- and 4-regular 3-connected planar graphs. For arbitrary planar graphs, by analogy with Kronk and Mitchem's conjecture $\chi_{vef}(G) \leqslant \Delta(G) + 4$ on the entire colouring [6], Mel'nikov conjectured [7, Problem 9, p. 543] that $\chi_{ef}(G) \leqslant \Delta(G) + 3$. It was proved in [1] that $\chi_{ef}(G) \leqslant 6$ if $\Delta(G) = 3$. The second result of the present note is

Theorem 2. *Let G be a planar graph without separating 3-cycles, then $\chi_{ef}(G) \leqslant \Delta(G) + 4$.*
Proof. In proving the Lemma below as well as the Theorem 2 itself, we use the concept of assigned colouring, introduced in [8] and [2].

Lemma. *Let a set $A(e)$ of admissible colours is assigned to every edge e of a planar graph G such that $|A(e)| \geqslant \Delta(G) + 2 + t(e)$, where $t(e)$ stands for the number of 3-faces incident with e. Then for every edge a colour admissible to it can be chosen so that the colours of any two adjacent edges would be distinct.*

Proof. Assuming the contrary, denote by G_0 a counterexample to the Lemma with the least number of edges. Let a set-system $A_0 = \{A_0(e): e \in E(G)\}$ contain no admissible edge colourings of G_0.

It is easily seen that $\delta(G_0) \geqslant 3$. If there were a vertex of degree 1 or 2 in G_0, then it would be incident with an edge e, adjacent to at most $1 + \Delta(G_0) - 1 = \Delta(G_0)$ edges. But then, by the minimality of G_0, the graph $G_0 - e$ might be coloured in accordance with A_0. Afterwards e might be coloured with that element of $A_0(e)$ which does not occur at the edges, adjacent to e.

By the Corollary, there exists in G_0 such a vertex v_0 such a vertex v_0 that at least one of the possibilities takes place:

(a) $s(v_0) = 3$;

(b) $s(v_0) = 4$ and v_0 is incident with a 3-face;

(c) $s(v_0) = 5$ *and* v_0 is incident with at least four 3-faces.

Denote by e_0 an edge of G_0 which, respectively,

(a′) is incident with v_0;

(b′) is incident with v_0 and at least one 3-face;

(c′) is incident with v_0 and two 3-faces.

Colour the edges of $G_0 - e_0$ in accordance with A_0, i.e. choosing the colour for every edge e from the set $A_0(e)$ so that the resulting colouring of the whole graph is admissible. It is easily verified that, in each of the situations (a)—(c), the edge e_0 is adjacent to at most $\Delta(G_0) + 1 + t(e_0)$ edges, so e_0 may be coloured with an element of $A_0(e_0)$ not occupied at the edges adjacent to e_0.

We have obtained an admissible edge colouring of G_0, chosen from A_0, which is a contradiction.

The Lemma is proved.

Now we are prepared to prove Theorem 2. First, colour all the nontriangular faces of a graph G with the colours 1, 2, 3, 4, 5. Next, for any edge e take as $A(e)$ the set of those colours $1, 2, \ldots, \Delta(G) + 3, \Delta(G) + 4$ which do not occur at the faces incident to e. By the Lemma, all the edges may be coloured in accordance with the assignement A. Finally, colour all the triangular faces of G: each of them is in contact with 6 edges and faces, whereas the total number of colours available is $\Delta(G) + 4 \geqslant 7$.

This completes the proof.

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ОДНО СТРУКТУРНОЕ СВОЙСТВО ПЛОСКИХ ГРАФОВ И СОВМЕСТНАЯ РАСКРАСКА ИХ РЕБЕР И ГРАНЕЙ

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Резюме

В любом плоском графе с минимальной степенью не меньше 3 найдется либо 3-вершина, либо 4-вершина, инцидентная теугольнику, либо 5-вершина, инцидентная четырем треугольникам. Ребра и грани любого плоского графа G без разделяющих 3-циклов с максимальной степенью $\Delta(G)$ можно совместно раскрасить в $\Delta(G) + 4$ цветов.