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THE REFLEXIBLE HYPERMAPS OF CHARACTERISTIC -2

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(*Communicated by Martin Škoviera*)

ABSTRACT. In the first part, we classify the reflexible (finite) hypermaps with one and two hyperfaces. From this classification, we derive that there is only one (up to an isomorphism) non-orientable reflexible hypermap with one hyperface of valency n, which is a projective hypermap ($\chi = 1$), and there are only two non-orientable reflexible hypermaps with two hyperfaces, both projective. In the second part, we classify the reflexible hypermaps on surfaces of characteristic -2. On orientable surfaces, [Breda d'Azevedo, A. J.—Jones, G. A.: *Rotary hypermaps of genus* 2, European J. Combin. (Submitted)] already gives a classification of rotary hypermaps of genus 2. It follows that there are 43 hypermaps, all reflexible, ten of which are maps. We show that on non-orientable surfaces, there are fifteen of them, twelve of which are maps.

1. Introduction

1.1. Topological hypermaps.

Topologically, a *hypermap* $\mathcal{H}$ is an imbedding (without crossings) of a connected trivalent graph $\mathcal{G}$ in a connected (not necessarily compact) surface $\mathcal{S}$, possibly non-orientable or with boundary ($\partial\mathcal{S} \neq \emptyset$), such that:

1. Each face (connected component of $\mathcal{S} \setminus \mathcal{G}$) is homeomorphic to an open disc or a half-disc.
2. The faces are labelled 0, 1 and 2 (or shaded black, grey and white) so that each vertex of $\mathcal{G}$ is incident with three faces carrying different labels (or shades).
3. If $\partial\mathcal{S} \neq \emptyset$, $\mathcal{G}$ intersects each connected component of $\partial\mathcal{S}$.

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4. No vertex of $\mathcal{G}$ lies on the boundary $\partial\mathcal{S}$.
5. Each edge e of $\mathcal{G}$ joins two distinct vertices unless $e \cap \partial\mathcal{S} \neq \emptyset$ in which case e has only one vertex.
6. For each face f, $\overline{f} \cap \mathcal{G}$ is connected.

Condition 5 states that the graph $\mathcal{G}$ has no loops and that free edges (edges with only one vertex) only occur when they intersect the boundary. For compact surfaces Condition 6 may be obtained from Condition 1, so Condition 6 does nothing new to finite hypermaps. However, for non-compact surfaces Condition 1 does not imply Condition 6. For instance, Figure 1.a is an imbedding not permitted by Condition 1, yet Figures 1.b and 1.c are of imbeddings in the non-compact tube and in the plane, respectively, both satisfying Conditions 1, 2, 3, 4 and 5. Although these two imbeddings are combinatorially similar, the imbedding 1.b does not satisfy Condition 6.

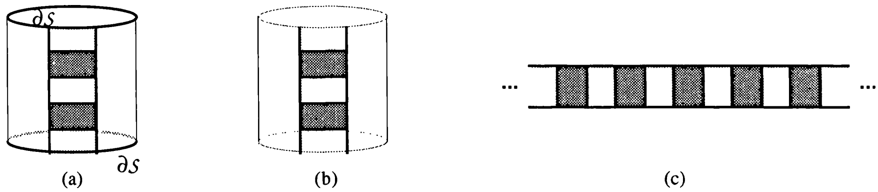

FIGURE 1.

The vertices of $\mathcal{G}$ are *blades*, while the faces labelled 0, 1 and 2 are, respectively, *hypervertices*, *hyperedges* and *hyperfaces* of $\mathcal{H}$. Each edge of $\mathcal{G}$ is incident with two faces carrying different labels, so we label each edge with the missing label. The edge-labelled graph formed in this way induces three permutations r_0, r_1 and r_2 on the set of blades Ω: r_i $(i = 0, 1, 2)$ permutes the two blades incident with each i-edge (edge labelled i) not meeting the boundary and fixes the unique blade incident with each i-edge meeting the boundary (Figure 2). These permutations r_i satisfy $r_i^2 = 1$ and since the graph $\mathcal{G}$ is connected, they generate a transitive group G of permutations of Ω, which we call the *Ω-group* of $\mathcal{H}$. If Ω is finite, we say that $\mathcal{H}$ is a finite hypermap.

FIGURE 2.

1.2. Klein hypermaps.

Let $\mathcal{S}$ be the hyperbolic plane, and Δ be the reflection group generated by the three reflections R_0, R_1 and R_2 on the sides of a hyperbolic triangle T with zero internal angles. Δ is isomorphic to the free group $C_2 * C_2 * C_2$ with presentation

$$\langle R_0, R_1, R_2 \mid R_0^2 = R_1^2 = R_2^2 = 1\rangle ,$$

so we have a transitive permutation representation $\pi\colon \Delta \to G \leq S_\Omega$ defined by $R_i \to r_i$. Δ acts properly discontinuous on $\mathcal{S}$, and $\mathcal{S}/\Delta$ is just the triangle T, a fundamental polygon for Δ. Take the barycentric subdivision of T, and label 0, 1 and 2 the resulting three regions (Figure 3).

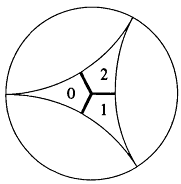

FIGURE 3. The hyperbolic triangle T in the Poincaré model of the hyperbolic plane.

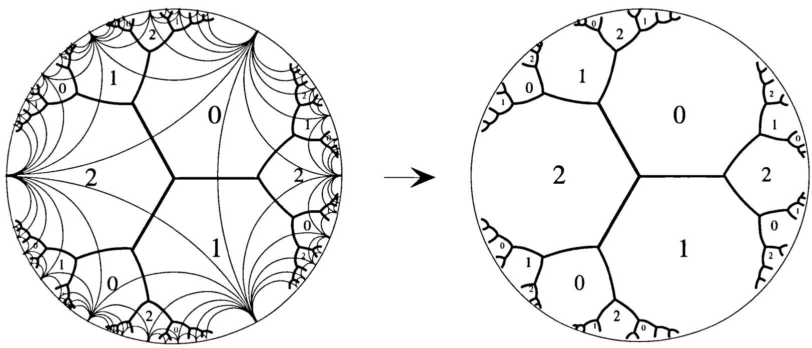

FIGURE 4. The hypermap δ.

The action of Δ on T produces a graph $\mathcal{G}_\Delta$ imbedded in $\mathcal{S}$, which can be seen as the Cayley graph for Δ in respect to the generators R_i, together with a face-labelling satisfying 2. This imbedding is a hypermap that we denote by δ (Figure 4); its i-faces are the connected union of the regions labelled i, and its Ω-group is isomorphic to Δ.

For any subgroup H of Δ, the orbit-space $\mathcal{S}/H$ is a Klein surface[1)], possibly with boundary. Imbedded in this surface is the graph $\mathcal{G}_H = \mathcal{G}_\Delta/H$, regarded as the Schreier coset graph for H in Δ with respect to the generators R_i. The face-labelling of δ is transferred to this imbedding inducing a hypermap δ/H. Write $\Omega H \equiv \Delta/H$ for the set of vertices of $\mathcal{G}_H$. The Ω-group of δ/H is isomorphic to the quotient group Δ/H^* generated by H^*R_0, H^*R_1 and H^*R_2, where H^* is the core of H in Δ.

Let P_H be a fundamental polygon for H in Δ arising from a Schreier transversal for H in Δ. The surface $\mathcal{S}/H$ has boundary if and only if some reflection fixes an edge of P_H ([HS]), or equivalently, some conjugate of R_i belongs to H. $\mathcal{S}/H$ is orientable if and only if the edges of P_H are only paired (identified) by orientation preserving elements or by reflections (boundary components) ([HS]). So, if $\mathcal{S}/H$ has no boundary, then $\mathcal{S}/H$ is orientable if and only if H is a subgroup of the even subgroup $\Delta^+ = \langle R_1R_2, R_2R_0\rangle$ of index 2 in Δ.

Let us call the hypermap δ/H arising in this way a *Klein* hypermap.

1.3. Algebraic hypermaps.

To a topological hypermap $\mathcal{H}$ we associate an algebraic sequence $(G, \Omega, r_0, r_1, r_2)$ composed of the Ω-group G, the set of blades Ω, and the generators r_0, r_1, r_2 defined before. Each such sequence gives rise to a Klein hypermap: fix an element α in Ω, let G_α be the point-stabilizer of α in G, and $H = \pi^{-1}\{G_\alpha\}$, the point-stabilizer of α in Δ. Then δ/H is a Klein hypermap with underlying graph $\mathcal{G}_H$ isomorphic to the underlying graph $\mathcal{G}$ of $\mathcal{H}$, and Ω-group Δ/H^* isomorphic to G. This subgroup H is called the *hypermap-subgroup* of $\mathcal{H}$.

Any sequence $(G, \Omega, r_0, r_1, r_2)$ where Ω is a set, G is a subgroup of the symmetric group S_Ω acting transitively on Ω, and r_0, r_1, r_2 are involutions generating G is called an *algebraic hypermap.*

To each Klein hypermap we have a surface triangulation induced by the triangle T. Thus, we may construct a Klein hypermap from an algebraic hypermap by taking 2-simplices T_ω for each blade $\omega \in \Omega$, each with a labelled barycentric subdivision, as we did with T, and joining them by their i-sides (sides opposite to the regions labelled i) according to the cycles of r_i (that is, T_ω is joined to $T_{\omega'}$ if $\omega r_i = \omega'$), so that their regions match up (Figure 5).

[1)] A Klein surface is a surface with a dianalytic structure [BEGG]; such structures enable one to define angles on surfaces including non-orientable surfaces and surfaces with boundary.

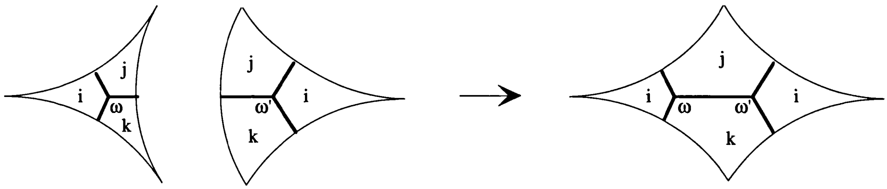

FIGURE 5.

The trivalent graph carried by the resulting surface (possibly with boundary) $\cup T_\omega$ is isomorphic to $\mathcal{G}$. If $\mathcal{H}$ is not finite, the extra Condition 6 ensures that the edge-labelled graph carries the same information to $\mathcal{H}$ as a Schreier coset graph of a point-stabilizer $H \leq \Delta$, so this construction may be carried forward; in this case, T_ω may have zero internal angles, or parallel sides.

1.4. Coverings.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two hypermaps, and let $(G_1, \Omega_1, x_0, x_1, x_2)$ and $(G_2, \Omega_2, y_0, y_1, y_2)$ be their associate algebraic hypermaps. A *covering* (or morphism) $\phi\colon \mathcal{H}_1 \to \mathcal{H}_2$ is a function ϕ from Ω_1 onto Ω_2, together with an epimorphism $\pi\colon G_1 \to G_2\colon x_i \mapsto y_i$, such that the following diagram is commutative,

$$\begin{array}{ccc} \Omega_1 \times G_1 & \xrightarrow{\cdot} & \Omega_1 \\ \big\downarrow{\scriptstyle \phi\times\pi} & & \big\downarrow{\scriptstyle \phi} \\ \Omega_2 \times G_2 & \xrightarrow{\cdot} & \Omega_2 \end{array}$$

that is, $\omega g\phi = \omega\phi g\pi$ for any $\omega \in \Omega_1$ and $g \in G_1$. Coverings correspond to inclusions $H_1 \leq H_2$ between their hypermap-subgroups.

A *symmetry* (or automorphism) of $\mathcal{H}$ is a bijective covering of $\mathcal{H}$ onto itself. The group of symmetries of $\mathcal{H}$, $\operatorname{Aut}\mathcal{H}$, acts on the set of blades Ω; if this action is transitive, we say that $\mathcal{H}$ is *reflexible* (or regular). $\operatorname{Aut}\mathcal{H}$ is isomorphic to $N_\Delta(H)/H$, where H is the hypermap-subgroup of $\mathcal{H}$, so $\operatorname{Aut}\mathcal{H}$ acts transitively on Δ/H if and only if $N_\Delta(H) = \Delta$. If $\mathcal{H}$ is reflexible, then H is normal in Δ, $\operatorname{Aut}\mathcal{H} \cong G$ and $\Omega \equiv G$, so the algebraic hypermap associated with $\mathcal{H}$ can be written as (G, G, r_0, r_1, r_2), where $G = \operatorname{Aut}\mathcal{H}$, and G acts on G by right multiplication.

1.5. The Walsh bijection.

Walsh [TW] introduced a bijection W from hypermaps $\mathcal{H}$ to bipartite maps $\mathcal{M}$ in the same surface $\mathcal{S}$. Any hypermap $\mathcal{H}$ corresponds to a bipartite map $\mathcal{M} = W(\mathcal{H})$ by contracting the 0- and 1-faces to points, conventionally coloured black and white. Conversely, any bipartite map $\mathcal{M}$, with its vertices coloured black and white, corresponds to a hypermap $\mathcal{H} = W^{-1}(\mathcal{M})$ by expanding the black vertices to 0-faces and the white vertices to 1-faces (Figures 6 and 7).

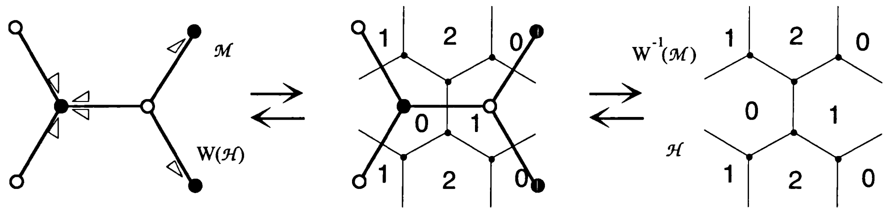

FIGURE 6. The Walsh bipartite map representation.

In the bipartite map representation of a hypermap $\mathcal{H}$, the blades are conventionally represented as 'one-blade arrows' pointing to black vertices (Figures 6 and 7). Figure 7 shows the Walsh bijection between bipartite maps and hypermaps with boundary.

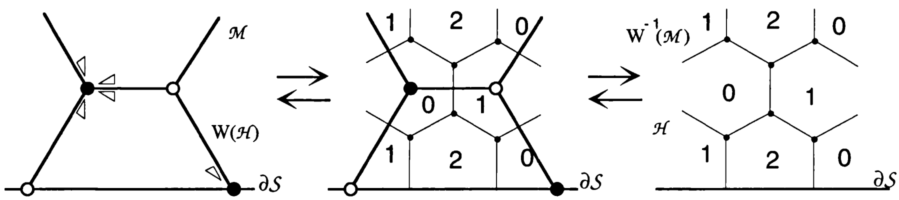

FIGURE 7. The Walsh bipartite map representation for maps and hypermaps with boundary.

When the subject matter is understood, these two hypermap representations will be equally used without mention.

1.6. Type and genus.

$\mathcal{H}$ has *type* (l, m, n) if l, m and n are, respectively, the least common multiples of the valencies of the hypervertices, hyperedges and hyperfaces. If $\mathcal{H}$ is reflexible, then l, m and n are the orders of r_1r_2, r_2r_0 and r_0r_1, and if $\mathcal{H}$ has boundary, then $r_i = 1$ for some $i = 0, 1, 2$. The number F_i of i-faces of a finite reflexible hypermap is given by the formula $F_i = \dfrac{\beta_i|G|}{2o_i}$, where $o_i = \mathrm{order}(r_jr_k)$, $\{i, j, k\} = \{0, 1, 2\}$ and

$$\beta_i = \begin{cases} 1 & \text{if } r_j \neq 1,\ r_k \neq 1, \\ 2 & \text{otherwise.} \end{cases}$$

The characteristic χ of a finite reflexible hypermap $\mathcal{H}$ without boundary (we mean the Euler characteristic of the underlying surface $\mathcal{S}$) is given by

$$\begin{aligned}\chi &= V + E + F - \frac{|G|}{2} \\ &= \frac{|G|}{2}\left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n} - 1\right),\end{aligned}$$

where V, E and F are the number of hypervertices, hyperedges and hyperfaces. The genus g is given by $g = \dfrac{2-\chi}{\eta}$, where $\eta = 2$ if $\mathcal{H}$ is orientable, and $\eta = 1$ otherwise.

1.7. The reflexible abelian hypermaps.

If $\mathcal{H}$ is a reflexible hypermap, the number of blades must be even. If $\mathcal{H}$ has boundary, then up to a permutation of the i-faces, $\mathcal{H}$ has type $(1,1,1)$, $(2,2,1)$ or $(n,2,2)$, thus $|G| = 1$, 2 or $2n$, and $\mathcal{H}$ has 1, 2 or n hyperfaces, 1 or n hyperedges and 1 hypervertex. If $\mathcal{H}$ has no boundary but $|G| \leq 4$, the number of hyperfaces $F \leq \frac{4}{2} = 2$. In both cases, if $\mathcal{H}$ is a reflexible hypermap with $|\Omega| \leq 4$, $\mathcal{H}$ has 1 or 2 hyperfaces; as the group $G = \operatorname{Aut}\mathcal{H} \cong 1$, C_2 or $C_2 \times C_2$, G is an abelian group, and thus $\mathcal{H}$ is one of the sixteen reflexible abelian hypermaps classified in [BJ1]. These sixteen abelian hypermaps have 1 or 2 hyperfaces, so we give here a brief review of them.

One blade. There is only one, the trivial hypermap $\mathcal{A}$, with boundary, defined by $r_0 = r_1 = r_2 = 1$ (Figure 8). It has hypermap subgroup Δ.

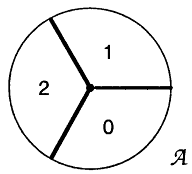

FIGURE 8. The unique reflexible abelian hypermap with one blade.

Two blades. There are seven of them: $\mathcal{B}^{\hat{i}}$, $\mathcal{B}^{i}$ and $\mathcal{B}^{+}$, for $i = 0,1,2$ (Figure 9), corresponding to the hypermap-subgroups $\Delta^{\hat{i}} = \langle R_j, R_k\rangle^{\Delta}$, $\Delta^{i} = \langle R_i, R_jR_k\rangle^{\Delta}$ and $\Delta^{+} = \langle R_1R_2, R_2R_0\rangle^{\Delta}$, where $\langle\,,\rangle^{\Delta}$ stands for the normal closure in Δ. The superscript $\hat{i}$ means that only the i-edges are not 'free' (i.e., they do not intersect the boundary) while the superscript i means that only the i-edges are 'free'.

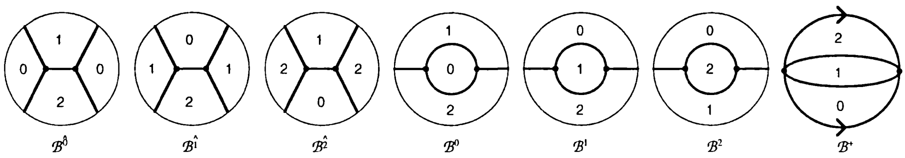

FIGURE 9. The reflexible abelian hypermaps with two blades.

Four blades. There are seven of them: $\mathcal{B}^{\hat{i}\hat{j}}$, $\mathcal{B}^{+\hat{i}}$ and $\mathcal{B}^{012}$, for $i = 0, 1, 2$ (Figure 10), with hypermap-subgroups $\Delta^{\hat{i}\hat{j}} = \Delta^{\hat{i}} \cap \Delta^{\hat{j}}$, $\Delta^{+\hat{i}} = \Delta^{+} \cap \Delta^{\hat{i}}$ and $\Delta^{012} = \Delta^{0} \cap \Delta^{1} \cap \Delta^{2}$. As these intersections suggest, the 4-blade hypermaps can be obtained by combining the 2-blade hypermaps $\mathcal{B}$ by products[2)].

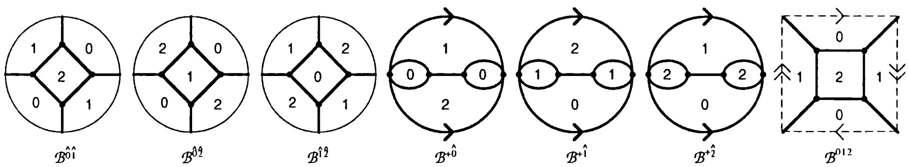

FIGURE 10. The reflexible abelian hypermaps with four blades.

The sixteenth reflexible abelian hypermap is the orientable hypermap $\mathcal{D}$ on the sphere with 8 blades corresponding to the derived subgroup $\Delta' = \Delta^{\hat{0}} \cap \Delta^{\hat{1}} \cap \Delta^{\hat{2}}$ (Figure 11).

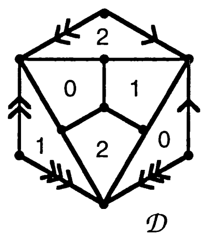

FIGURE 11. The unique reflexible abelian hypermaps with eight blades.

For a further reading on hypermaps with boundary, see [IS], and for maps with boundary, see [BS]. For a more general account of hypermaps without boundary, see [JS2], [LJ], [BJ1].

2) This is an extension to hypermaps [BJ1] of the concept of the parallel products for maps introduced by Wilson [SW2].

2. Finite reflexible hypermaps with one hyperface

Unless otherwise stated, hypermaps will be understood as finite and reflexible. Orientable hypermaps and non-orientable hypermaps will be understood without boundary. In general, $\mathcal{H}$ will denote a (reflexible) hypermap of type (l, m, n) with automorphism group G generated by involutions r_0, r_1, r_2, while the numbers $V = \frac{|G|}{2l}$, $E = \frac{|G|}{2m}$ and $F = \frac{|G|}{2n}$ will denote the number of hypervertices, hyperedges and hyperfaces of $\mathcal{H}$, respectively.

Let us first introduce some reflexible hypermaps with one hyperface (the pictures are placed within the proof of Theorem 1). By $\varepsilon_b^{1f}(n)$, we denote the hypermap with boundary ($r_2 = 1$) of type $(2, 2, n)$ defined by

$$\langle r_0, r_1, r_2 \mid r_0^2 = r_1^2 = r_2 = (r_0 r_1)^n = 1 \rangle .$$

The superscript "$1f$" stands for "one face (or hyperface)" while the subscript "b" means "with boundary". For even values of n we denote by $\varepsilon_-^{1f}(n)$ the non-orientable hypermap of type $(2, 2, n)$ and characteristic $\chi = 1$ defined by

$$\langle r_0, r_1, r_2 \mid r_0^2 = r_1^2 = (r_0 r_1)^n = r_2 (r_0 r_1)^{\frac{n}{2}} = 1 \rangle .$$

The subscript "$-$" means "non-orientable". For $k = 0, 1, \ldots, n-1$ we denote by $\varepsilon_k^{1f}(n)$ the orientable hypermap of type $\left(\frac{n}{(n,k+1)}, \frac{n}{(n,k)}, n\right)$ and characteristic $\chi = (n, k+1) + (n, k) + 1 - n$, where $(,)$ stands for the greatest common divisor, defined by

$$\langle r_0, r_1, r_2 \mid r_0^2 = r_1^2 = r_2^2 = (r_0 r_1)^n = r_2 (r_0 r_1)^k r_0 = 1 \rangle .$$

Let $\mathcal{H}$ be a hypermap with one hyperface of valency n. Then $r_2 \in \langle r_0, r_1 \rangle$, and G is generated by r_0 and r_1, so G is a dihedral group D_n of order $2n$, a cyclic group C_2, or the trivial group $\{1\}$.

THEOREM 1. *The reflexible hypermaps with one hyperface of valency n ($n \in \mathbb{N}$) are:*

i) *For $n = 1$: $\mathcal{A}$, $\varepsilon_b^{1f}(1)$, and $\varepsilon_0^{1f}(1)$.*
ii) *For $n = 2$: $\mathcal{B}^0$, $\mathcal{B}^{\hat{0}}$, $\mathcal{B}^1$, $\mathcal{B}^{\hat{1}}$, $\varepsilon_b^{1f}(2)$, $\varepsilon_-^{1f}(2)$, and $\varepsilon_k^{1f}(2)$ $(k = 0, 1)$.*
iii) *For n (> 2) even: $\varepsilon_b^{1f}(n)$, $\varepsilon_-^{1f}(n)$, and $\varepsilon_k^{1f}(n)$ $(k = 0, 1, \ldots, n-1)$.*
iv) *For n (> 2) odd: $\varepsilon_b^{1f}(n)$, and $\varepsilon_k^{1f}(n)$ $(k = 0, 1, \ldots, n-1)$.*

Proof.
i) $r_2 = 1$. Then $\mathcal{H}$ has boundary.
If $r_0 = r_1 = 1$, then $n = 1$, $|G| = 1$, and $\mathcal{H}$ is the one blade hypermap $\mathcal{A}$.
If only one of r_0 or r_1 is the identity, then $n = 2$, $|G| = 2$, and $\mathcal{H}$ is the two blade hypermap $\mathcal{B}^{\hat{0}}$ or $\mathcal{B}^{\hat{1}}$, according as $r_1 = 1$ or $r_0 = 1$.

If $r_0 \neq 1$ and $r_1 \neq 1$, then $r_0^2 = r_1^2 = r_2 = (r_0 r_1)^n = 1$ defines the map with boundary $\varepsilon_b^{1f}(n)$, of type $(2,2,n)$ consisting of a single face surrounded by n vertices and n edges lying on the boundary (Figure 12). For $n = 1$ we have the 2-blade map $\varepsilon_b^{1f}(1) = \mathcal{B}^2$ while for $n = 2$ we have the 4-blade map $\varepsilon_b^{1f}(2) = \mathcal{B}^{\hat{0}\hat{1}}$.

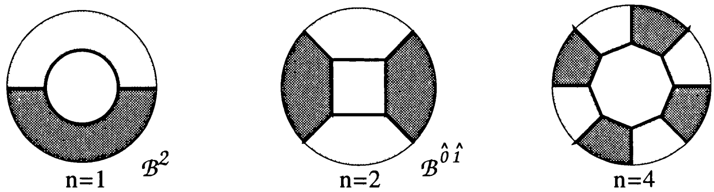

FIGURE 12. The hypermap with boundary $\varepsilon_b^{1f}(n)$.

ii) $r_2 \neq 1$.

a) r_2 is in the cyclic group C generated by $a = r_0 r_1$. Then n must be even and $r_2 = a^{\frac{n}{2}}$.

If $\mathcal{H}$ has boundary, then $r_i = 1$ for some $i \in \{0,1\}$ and $r_0 \neq r_1$. Then $r_2 = r_j$ ($\{i,j\} = \{0,1\}$), $n = 2$, and $\mathcal{H}$ is the map $\mathcal{B}^i$.

If $\mathcal{H}$ has no boundary (i.e., $r_0 \neq 1$ and $r_1 \neq 1$), then $r_0^2 = r_1^2 = (r_0 r_1)^n = r_2 (r_0 r_1)^{\frac{n}{2}} = 1$ defines the projective (i.e., non-orientable with characteristic 1) map $\varepsilon_-^{1f}(n)$ of type $(2,2,n)$ (Figure 13).

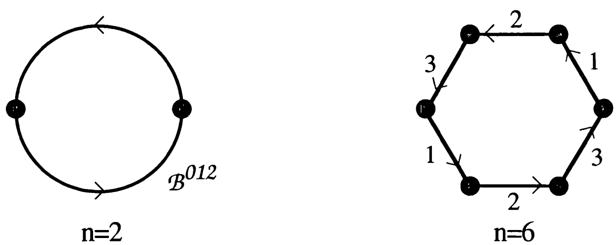

FIGURE 13. The non-orientable hypermap $\varepsilon_-^{1f}(n)$ for $n = 2$ and 6.

b) $r_2 \notin C$, say $r_2 = a^k r_0$ for some $k \in \{0, 1, \ldots, n-1\}$. Then $\mathcal{H}$ is orientable. Fixing r_2, we have $|\operatorname{Aut} D_n|$ choices of pairs (r_0, r_1) generating D_n such that $\operatorname{order}(r_0 r_1) = n$, all these choices giving isomorphic hypermaps. Thus we have n distinct hypermaps $\varepsilon_k^{1f}(n)$ (Figure 14) corresponding to different choices of r_2, each of type $\left(\frac{n}{(n,k+1)}, \frac{n}{(n,k)}, n\right)$ and characteristic $\chi = (n,k+1) + (n,k) + 1 - n$. The hypermap $\varepsilon_k^{1f}(n)$ can be obtained from a polygon with $2n$ sides and with its $2n$ vertices "bipartitioned" into n white vertices (representing the hyperedges) and n black vertices (representing the hypervertices) by identifying the edges according to the rule dictated by the equation $r_2 a^k r_0 = 1$: an edge i is identified with the $(2k+1)$th edge counting from edge i (excluding this edge) and following the arrow tail direction (arrows pointing always to black vertices).

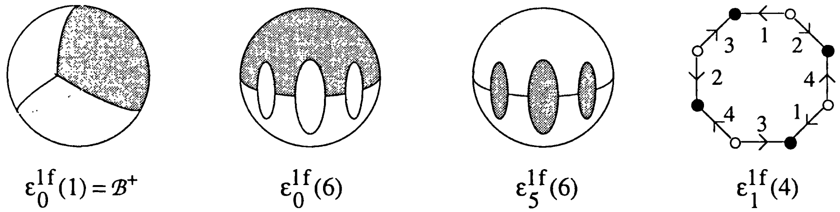

FIGURE 14. The orientable hypermaps $\varepsilon_k^{1f}(n)$.

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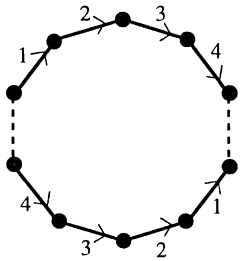

FIGURE 15. The orientable hypermaps $\varepsilon_{\frac{n}{2}}^{1f}(n)$.

3. A brief commentary

The hypermap with boundary $\varepsilon_b^{1f}(n)$ is the map $\mathcal{D}_n^\circ$ in [BJ2] while the non-orientable hypermap $\varepsilon_-^{1f}(n)$ is the map called δ_n in [SW]. The hypermap $\varepsilon_0^{1f}(n)$ of type $(n,1,n)$ with genus 0 is the dihedral hypermap $\mathcal{D}_n^*$ in [BJ2]. When $n = 1$, we have the 2-blade hypermap $\mathcal{B}^+$ in [BJ1]. Its associate $\varepsilon_0^{1f}(n)^{(01)}$ obtained by applying the hypermap operation (0 1) which transpose hypervertices and hyperedges, is the hypermap $\varepsilon_{n-1}^{1f}(n)$ of type $(1,n,n)$ on the sphere. If n is even, we may take $k = \frac{n}{2}$. The resulting orientable hypermap $\varepsilon_{k=\frac{n}{2}}^{1f}(n)$ (see Figure 15) of type $(n,2,n)$ or $(k,2,n)$, according as k is even or odd, and characteristic $\chi = 2-k$ or $3-k$, is the dihedral map $\mathcal{D}_n^{\diamond}$ in [BJ2] and the map called M_k in [SW].

The hypermap $\varepsilon_1^{1f}(n)$ has type $\left(\frac{n}{2}, n, n\right)$ or (n,n,n), according as n is even or odd, and characteristic $\chi = 4-n$ or $3-n$. For n prime, the hypermap $\varepsilon_k^{1f}(n)$ is the $2n$-blade hypermap $\mathcal{K}_n^{1,0}$, $\mathcal{K}_n^{0,1}$ or $\mathcal{K}_n^{1,\frac{1}{m}}$ in [B], where $m = -\frac{k+1}{k} \pmod n$, according as $k=0$, $k=n-1$, or otherwise.

4. Finite reflexible hypermaps with two hyperfaces

If a 2-edge e joins the same hyperface F of a reflexible hypermap $\mathcal{H}$ (Figure 16), then r_2 belongs to the dihedral group D_n generated by r_0 and r_1, $G = D_n$, and $\mathcal{H}$ has one hyperface only.

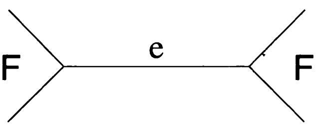

FIGURE 16.

If $\mathcal{H}$ has two hyperfaces, any 2-edge must then join distinct hyperfaces. We must have $r_2 \neq 1$ and $r_2 \neq r_i$ for $i = 0,1$. Besides, G has $4n$ elements, unless $\mathcal{H}$ has boundary, in which case, G is a dihedral group of order $2n$. The dihedral group D generated by r_0 and r_1 of order $2n$ is a normal subgroup of index 1 or 2 in G, according as $\mathcal{H}$ has boundary or not. Hence D contains any conjugate of r_i for $i = 0,1$ and $G = D \vee \langle r_2 \rangle$. Their two hyperfaces must be arranged in the way pictured by Figure 17.

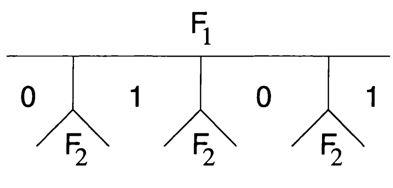

FIGURE 17.

From this diagram, $\mathcal{H}$ must have type (l,m,n) with l and m even. Conversely, if $r_i^{r_2} \in D$ for $i = 0,1$, then the number of hyperfaces of a reflexible hypermap $\mathcal{H}$ is less or equal than 2.

Write $a = r_0 r_1$.

For $k = 0, 1$, denote by $\varepsilon_-^{2f}(k)$, the non-orientable hypermap of characteristic $\chi = 1$ and type $(2,4,2)$ or $(4,2,2)$, respectively, defined by the following presentation of the dihedral group D_4,

$$\langle r_0, r_1, r_2 \mid r_0^2 = r_1^2 = r_2^2 = a^n = (r_0^k r_1^{k+1} r_2)^2 = (r_0^{k+1} r_1^k)^{r_2} a = 1 \rangle .$$

If $k = 0$, this presentation reduces to $\langle r_0, r_1, r_2 \mid r_i^2 = (r_0 r_2)^4 = 1,\ r_1 = (r_0 r_2)^2 \rangle$, while if $k = 1$, it reduces to $\langle r_0, r_1, r_2 \mid r_i^2 = (r_1 r_2)^4 = 1,\ r_0 = (r_1 r_2)^2 \rangle$. Notice that applying the operation $(0\ 1)$, that transposes hypervertices with hyperedges, to $\varepsilon_-^{2f}(0)$ we get $\varepsilon_-^{2f}(1)$, and vice-versa.

For each $k_0 \in \{0, \ldots, n-2\}$ and $k_1 \in \{1, \ldots, n-1\}$ such that $k_0(1 - \triangle) \equiv 0 \pmod{n}$ and $\triangle^2 \equiv 1 \pmod{n}$, where $\triangle = k_1 - k_0$, we denote by $\varepsilon_n^{2f}(k_0, k_1)$ the orientable hypermap (without boundary) of type $\left(\frac{2n}{(n,k_1+1)}, \frac{2n}{(n,k_0)}, n\right)$, induced by the presentation

$$\langle r_0, r_1, r_2 \mid r_0^2 = r_1^2 = r_2^2 = a^n = r_0^{r_2} a^{k_0} r_0 = r_1^{r_2} a^{k_1} r_0 = 1 \rangle .$$

As we shall see later, this hypermap has $4n$ blades.

Theorem 2. *Let $\mathcal{H}$ be a reflexible hypermap with two hyperfaces of valency n. Then*

i) *if $\mathcal{H}$ has boundary, $\mathcal{H} = \mathcal{B}^{\hat{2}}$, $\mathcal{B}^{\hat{1}\hat{2}}$ or $\mathcal{B}^{\hat{0}\hat{2}}$;*

ii) *if $\mathcal{H}$ is non-orientable, $\mathcal{H}$ is the projective hypermap $\varepsilon_-^{2f}(k)$ for some $k \in \{0,1\}$;*

iii) *if $\mathcal{H}$ is orientable, $\mathcal{H} = \mathcal{B}^{+\hat{2}}$, or $\varepsilon_n^{2f}(k_0, k_1)$ for some $k_0 \in \{0, \ldots, n-2\}$ and $k_1 \in \{1, \ldots, n-1\}$.*

P r o o f. If $r_0 = r_1$, then $n = 1$ and $G = \mathrm{gp}\langle r_1, r_2 \rangle = C_2$ or V_4, depending on whether $r_1 = 1$ or not, that is, whether $\mathcal{H}$ has boundary or not. We have in this case $\mathcal{H} = \mathcal{B}^{\hat{2}}$ or $\mathcal{B}^{+\hat{2}}$, both of type $(2,2,1)$.

If $r_0 \neq r_1$ but one of them is the identity, say r_i, then $\mathcal{H}$ has boundary and $r_j \rightleftharpoons r_2$, where $\{i,j\} = \{0,1\}$. Consequently, $l = 2$, $m = 2$, $n = 2$, $G = V_4$, and $\mathcal{H} = \mathcal{B}^{\hat{1}\hat{2}}$ or $\mathcal{B}^{\hat{0}\hat{2}}$, according as $r_0 = 1$ or $r_1 = 1$.

Let now $r_0 \neq r_1$ and $r_i \neq 1$ for $i = 0, 1$.

In these circumstances, $\mathcal{H}$ has no boundary, and $l \geq 2$, $m \geq 2$, and $n \geq 2$. As $\langle r_2 \rangle \cap D = \{1\}$, G is a split extension of D with complement $\langle r_2 \rangle$. Let C be the cyclic group of order n generated by $a = r_0 r_1$, and let G^+ be the subgroup generated by the "even" words a and $r_1 r_2$. As ${r_i}^{r_2}$ is in D, we have three choices:

i) Both ${r_0}^{r_2}$ and ${r_1}^{r_2}$ belong to C. Then n must be even and ${r_0}^{r_2} = {r_1}^{r_2} = a^{\frac{n}{2}}$. This implies that $r_0 = r_1$, and so ${r_0}^{r_2}$ and ${r_1}^{r_2}$ cannot both belong to C.

ii) For $i = 0, 1$, one of ${r_i}^{r_2}$ belongs to C, say ${r_1}^{r_2} \in C$. Then ${r_0}^{r_2} \in D \setminus C$, n is even, and $l = 4$ since $(r_2 r_1)^2 \in D \setminus C$. Furthermore, $a^{r_2} = {r_0}^{r_2} {r_1}^{r_2} \in D \setminus C$, so $n = 2$, $r_0 \rightleftharpoons r_1$, and $a = {r_1}^{r_2}$. Then we have ${r_0}^{r_2} = r_0$ or ${r_0}^{r_2} = r_1 = a r_0$. If ${r_0}^{r_2} = r_1$, then $r_0 = {r_1}^{r_2} = a$, and this implies that $r_1 = 1$. Thus we must have ${r_0}^{r_2} = r_0$. In this case, $m = 2$ and $\mathcal{H}$ is the non-orientable map $\varepsilon_-^{2f}(1)$ of type $(4, 2, 2)$ (Figure 18). Its automorphism group is the dihedral group D_4 with presentation $\langle r_0, r_1, r_2 \mid r_0^2 = r_1^2 = r_2^2 = a^2 = r_0^{r_2} r_0 = r_1^{r_2} a = 1 \rangle = \langle r_0, r_1, r_2 \mid r_i^2 = (r_1 r_2)^4 = 1\,,\ r_0 = (r_1 r_2)^2 \rangle$.

FIGURE 18. The map $\varepsilon_-^{2f}(1)$ and the hypermap $\varepsilon_-^{2f}(0)$.

iii) Both ${r_0}^{r_2}$ and ${r_1}^{r_2}$ belong to $D \setminus C$. Write $r_i^{r_2} = a^{k_i} r_0$, $i = 0, 1$, where $k_0 \in \{0, \ldots, n-2\}$ and $k_1 \in \{1, \ldots, n-1\}$. Then we have $a^{k_0} = (r_2 r_0)^2$ and $a^{k_1} = r_2 r_1 r_2 r_0$. From the first equality, we have $m = \dfrac{2n}{(n, k_0)}$, and from the second, $l = \dfrac{2n}{(n, k_1 + 1)}$. Furthermore, $a^{-\triangle} = a^{r_2}$, so $(n, \triangle) = 1$; $a^{k_0 + 1} = a^{\triangle k_1}$, so $(k_1 + 1)(1 - \triangle) \equiv 0 \pmod{n}$, and since $(k_1 + 1)(1 - \triangle) \equiv k_0(1 - \triangle) \pmod{n}$, $k_0(1 - \triangle) \equiv 0 \pmod{n}$; finally, $a^{-k_0} = r_2 a^{k_0} r_2$, thus $a^{k_0} = a^{\triangle k_0}$, and hence $a^{\triangle k_1} = a^{\triangle k_0 + 1}$, so $\triangle^2 \equiv 1 \pmod{n}$. Notice that the equation $(n, \triangle) = 1$ is unnecessary since this can be obtained from $\triangle^2 \equiv 1 \pmod{n}$. As G splits over D, G has presentation

$$\langle r_0, r_1, r_2 \mid r_0^2 = r_1^2 = r_2^2 = a^n = r_0^{r_2} a^{k_0} r_0 = r_1^{r_2} a^{k_1} r_0 = 1 \rangle\,,$$

and so $\mathcal{H}$ is the orientable hypermap $\varepsilon_n^{2f}(k_0, k_1)$.

Conversely, let $G = G_n(k_0, k_1)$ be the group with the above presentation, where k_0 and k_1 are as above. So, k_0 and k_1 satisfy:

$$(n, \triangle) = 1\,, \tag{1}$$

$$(k_1 + 1)(1 - \triangle) \equiv 0 \pmod{n}\,, \tag{2}$$

$$\triangle^2 \equiv 1 \pmod{n}\,, \tag{3}$$

where $\triangle = k_1 - k_0$. By the Reidemaster-Schreier process, we see that $|G| = 4 \cdot \operatorname{order}(a)$. To show that the relations $r_0^{r_2} a^{k_0} r_0 = 1$ and $r_1^{r_2} a^{k_1} r_0 = 1$ do not force $\operatorname{order}(a)$ to "collapse", that is, to be less than n, take two bipartite $2n$-gons

F_1 and F_2 with their edges labelled $0, 1, \ldots, 2n-1$ as shown in Figure 19a. Join the two 0-edges of F_1 and F_2 so that vertex-colors match (Figure 19b) and denote by $\mathcal{R}$ the union $F_1 \cup F_2$.

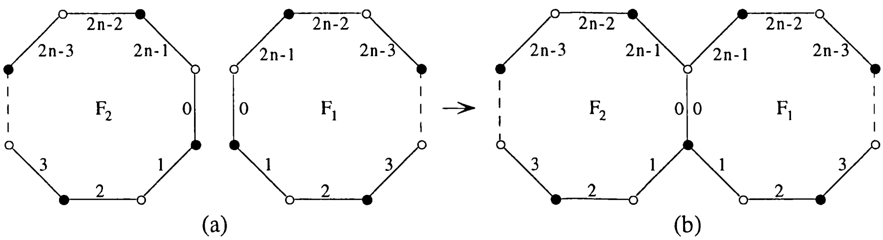

FIGURE 19. The two $2n$-gons representing the two n-hyperfaces.

The relator $r_1^{r_2} a^{k_1} r_0$ applied to the edge $1 \in F_1$ leads to an edge $\overline{1} \in F_2$ and hence to an edge-identification, or pairing, $1 \sim \overline{1}$. Acting next the relator $r_0^{r_2} a^{k_0} r_0$ to the edge $2 \in F_1$ we get an edge $\overline{2} \in F_2$ and an edge-identification $2 \sim \overline{2}$. So acting alternately the relators $r_1^{r_2} a^{k_1} r_0$ and $r_0^{r_2} a^{k_0} r_0$ to the odd and even edges (respectively) we get new labels (identification-labels) $\overline{0} = 0, \overline{1}, \overline{2}, \ldots, \overline{n}$ on the edges of F_2 and a pairing $1 \sim \overline{1}$, $2 \sim \overline{2}$, $\ldots$, $n \sim \overline{n}$ (Figure 20).

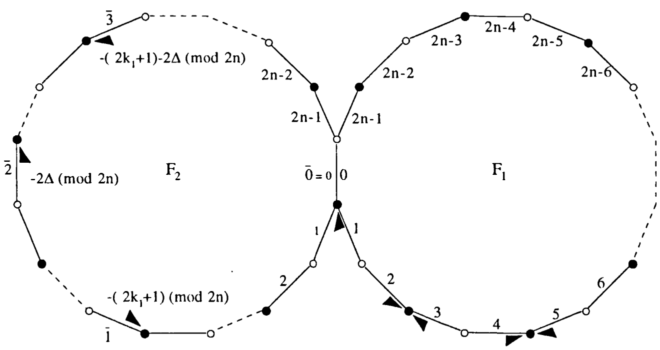

FIGURE 20. The identification-labels $\bar{\imath}$ in $\mathcal{R}$.

It is important to note that at this point, we are looking at r_0, r_1 and r_2 as permutations of the set of blades Ω rather then as reflections. We can then write,

$$\begin{aligned}\overline{1} &= 1.r_1^{r_2} a^{k_1} r_0 = 0.a^{k_1} r_0\,,\\ \overline{2} &= 2.r_0^{r_2} a^{k_0} r_0 = \overline{1}.a^{k_0} r_0 = 0.a^{\triangle}\,.\end{aligned}$$

So, by induction,

$$\overline{m} = 0.a^{\frac{m}{2}\triangle} \qquad \text{if } m \text{ is even,}$$

and

$$\overline{m} = 0.a^{\frac{m-1}{2}\triangle+k_1}r_0 \qquad \text{if } m \text{ is odd.}$$

Relatively to the initial labelling in F_2, we have $\overline{m} = -m\triangle \pmod{2n}$ if m is even and $\overline{m} = -(2k_1+1)-(m-1)\triangle \pmod{2n}$ if m is odd. The difference $\overline{m} - \overline{m-2}$ is constant and equal to $-2\triangle \pmod{2n}$ no matter if m is even or odd, so starting from zero and counting successively (clockwise) $-2\triangle \pmod{2n}$ (or $2\triangle \pmod{2n}$ counterclockwise) we get the even identification-labels $\overline{2}, \overline{4}, \overline{6}, \ldots$, while starting from $\overline{1}$ and counting successively (clockwise) $-2\triangle \pmod{2n}$ we get the odd identification-labels $\overline{3}, \overline{5}, \overline{7}, \ldots$ (see the example given in Figure 21). Since $(n,\triangle) = 1$, this shows that all the $2n$ edges of F_2 get an identification-label.

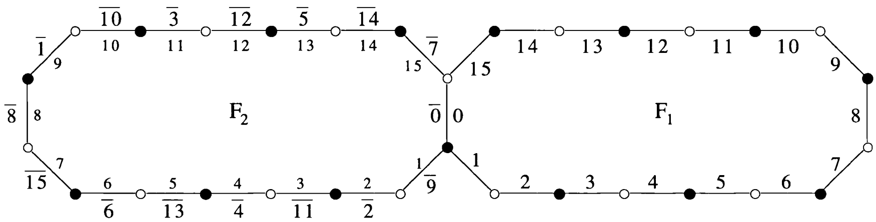

FIGURE 21. The hypermap $\varepsilon_8^{2f}(4,3)$ of type $(4,4,8)$.

Let us make it more general. If k is an edge in one "hyperface", denote by $\overline{k}$ the identification-edge in the other "hyperface". Then, relatively to the initial labelling, we have $\overline{0} = 0$, $\overline{1} = -2k_1-1 \pmod{2n}$, $\overline{2} = -2\triangle \pmod{2n}$, $\overline{3} = -(2k_1+1)-2\triangle \pmod{2n}$, $\ldots$, $\overline{m} = -m\triangle \pmod{2n}$ if m is even, and $\overline{m} = -(2k_1+1)-(m-1)\triangle \pmod{2n}$ if m is odd. Moreover, $\overline{\overline{i}} = i$, for any $i \in \{0,1,\ldots,2n-1,\overline{0},\overline{1},\ldots,\overline{2n-1}\}$.

We observe that we could have labelled the edges of F_2 counterclockwise instead to prevent writing $\overline{m}$ in the non-palatable "negative" form. Yet we would lose the symmetry of the labelling and consequently, in return, we would get a "negative" form for the last formula, namely: $\overline{-\overline{i}} = -i$.

Now we need to check whether these identifications under the action of G do or do not collapse[3] $\mathcal{R}$. If R_0, R_1 and R_2 (seeing them as reflections) do not collapse $\mathcal{R}$, then, by induction, any word in R_0, R_1 and R_2 does not collapse $\mathcal{R}$: in fact, suppose that any word $w \in F(R_0,R_1,R_2)$ with length$(w) \leq n-1$ does not collapse $\mathcal{R}$, and let w be a word with length n. Put $iw = j$, where

[3] A word w does not collapse $\mathcal{R}$ if $iw = j$ implies that $\overline{i}w = \overline{j}$ for any $i,j \in \{0,1,\ldots,2n-1,\overline{0},\overline{1},\ldots,\overline{2n-1}\}$.

$i,j \in \{0,1,\ldots,2n-1,\overline{0},\overline{1},\ldots,\overline{2n-1}\}$. We may write $w = R_s w'$ for some $s \in \{0,1,2\}$ and w' of length $n-1$. Let $k = iR_s$. Then $kw' = j$, and so $\overline{k}w' = \overline{j}$. As $\overline{k} = \overline{i}R_s$, we also have $\overline{i}w = \overline{j}$. Thus, any word in R_0, R_1 and R_2 does not collapse $\mathcal{R}$. In other words, this means that $\mathcal{R}$ may be seen as a fundamental region for some subgroup (the hypermap subgroup) of the triangle group Δ.

The counterclockwise rotation σ one step around F_1 is a clockwise rotation $\overline{\sigma}$ by $-2\Delta \pmod{2n}$ steps around F_2, and this rotation has order n if $\mathcal{R}$ does not collapse. Now take the following three reflections (see Figure 22): the reflection R_2 fixing the 0-edge, which is given by the permutation (m, m), when referring to the initial labelling, which can be written as $(m, \overline{\overline{m}}) = (m, \overline{-m\Delta}) \pmod{2n}$ if m is even, and $(m, \overline{-(2k_1+1)-(m-1)\Delta}) \pmod{2n}$ if m is odd; the reflection R_1 permuting the edges labelled 0 and 1 in F_1, which is given by the permutation $(m, 1-m) \pmod{2n}$ in F_1 and the permutation $(m, \overline{1-\overline{m}}) = (m, -(2k_1+1)-m) \pmod{2n}$ in F_2; the reflection R_0 permuting the edges labelled 0 and $2n-1$ in F_1, which is given by the permutation $(m, -m-1) \pmod{2n}$ in F_1 and the permutation $(m, \overline{-\overline{m}-1}) = (m, -(2k_0+1)-m) \pmod{2n}$ in F_2. Using the relations (2) and (3) we may also write $R_1 = (m-k_1, -m-k_1-1) \pmod{2n}$ and $R_0 = (m-k_0, -m-k_0-1) \pmod{2n}$, in F_2. Taking $m = 0$, R_1 and R_2 correspond to the reflections $(k_1, -k_1-1) \pmod{2n}$ and $(-k_0, -k_0-1) \pmod{2n}$, respectively.

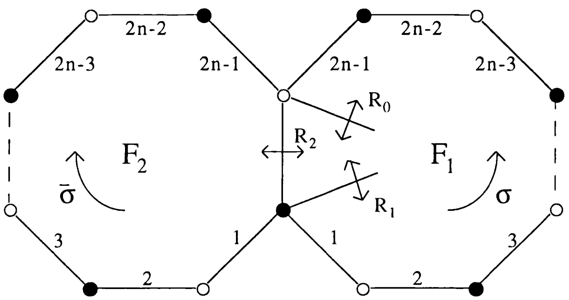

FIGURE 22. The reflections R_2, R_1 and R_0.

R_2 is an involution since $\overline{\overline{m}} = m$. Using the relations (2) and (3) we see that R_2 does not collapse $\mathcal{R}$. This is equivalent to checking that $\overline{-\overline{m}\Delta} = -m\Delta$ and $\overline{-(2k_1+1)-(\overline{m}-1)\Delta} = -(2k_1+1)-(m-1)\Delta$. R_1 and R_0 are involutions and do not collapse $\mathcal{R}$, in fact, for any m, $\overline{1-m} = -(2k_1+1)-\overline{m}$ and $\overline{-m-1} = -(2k_0+1)-\overline{m}$. So, by induction, any word in R_0, R_1 and R_2 does not collapse $\mathcal{R}$. Hence $|G| = 4n$. □

5. A brief note

Taking $k_0 = 0$ in $\varepsilon_n^{2f}(k_0, k_1)$ we get the 2-face map $\varepsilon_n^{2f}(0, k_1)$ called M'_{n,k_1} in [SW]. In particular, $k_1 = 1$ gives $M'_n = \varepsilon_n^{2f}(0,1)$, following Wilson's notation [SW], while $k_1 = n-1$ gives the 'ring' spherical map $\varepsilon_n = \varepsilon_n^{2f}(0, n-1)$ ([SW]) with automorphism group a direct product $G = D_n \times C_2$. If in addition n is odd, then $G \cong D_{2n}$, and we get the dihedral map $\varepsilon_n^{2f}(0, n-1)$ called $\mathcal{D}_{2n}^{\ominus}$ in [BJ2]. For $n = 2$ we have the abelian hypermap $\mathcal{D} = \varepsilon_2^{2f}(0,1)$.

6. Reflexible hypermaps of characteristic -2

A rotary hypermap $\mathcal{H}$ (without boundary) with $\chi = -2$ has genus $g = 2$ or 4 depending on whether $\mathcal{H}$ is orientable or not. The orientable case is classified in [BJ2]. According to that paper, there are 43 rotary hypermaps, all reflexible, of genus 2 of which 30 are maps and 13 are hypermaps (see Table 1). We may associate with each (finite) reflexible hypermap $\mathcal{H}$ (without boundary) with automorphism group G a sequence of numbers $s = \big(l, m, n, V, E, F, |G|\big)$, where (l, m, n) is the hypermap-type, V, E and F are the number of hypervertices, hyperedges and hyperfaces, respectively, and $|G|$ is the order of G. If $\mathcal{H}$ has characteristic χ, we say that s has characteristic χ. More generally, if $s = \big(l, m, n, V, E, F, |G|\big)$ is such that $2lV = 2mE = 2nF = |G|$, then we say that s is a hypermap-sequence, for short a H-sequence. Each hypermap $\mathcal{H}$ is associated with a H-sequence, but the converse is false. The number $\chi = \frac{|G|}{2}\big(\frac{1}{l} + \frac{1}{m} + \frac{1}{n} - 1\big)$ is the s-characteristic of the H-sequence s. The characteristic χ of a hypermap $\mathcal{H}$ is the s-characteristic of its associate H-sequence. If $\mathcal{H}$ has boundary, we may also associate a H-sequence $\big(l, m, n, V, E, F, |G|\big)$, but this time we have $lV = |G|$ or $2lV = |G|$ according as the hypervertex meets or not the boundary. Similarly for the other numbers. We notice that if $|G| > 4$, then the only reflexible hypermap with boundary up to an operation permuting hyperfaces is $\varepsilon_b^{1f}(n)$, the map corresponding to the dihedral group $G = D_n = \langle r_0, r_1 \rangle$; this has H-sequence $(2, 2, n, n, n, 1, 2n)$.

Table 1 lists the H-sequences with s-characteristic -2 and shows the number of orientable and non-orientable hypermaps associated with them.

TABLE 1. H-sequences for $\chi = -2$.

#	l	m	n	V	E	F	$\lvert G\rvert$	#Orientable	#Non-orientable
1	5	5	5	1	1	1	10	3	0
2	3	6	6	2	1	1	12	1	0
3	4	4	4	2	2	2	16	1	0
4	2	8	8	4	1	1	16	1	0
5	3	3	9	3	3	1	18	0	0
6	2	5	10	5	2	1	20	1	0
7	3	4	4	4	3	3	24	1	1
8	3	3	6	4	4	2	24	0	0
9	2	6	6	6	2	2	24	1	0
10	2	4	12	6	3	1	24	0	0
11	3	3	5	5	5	3	30	0	0
12	2	4	8	8	4	2	32	1	0
13	2	3	18	9	6	1	36	0	0
14	2	5	5	10	4	4	40	0	0
15	3	3	4	8	8	6	48	1	0
16	2	4	6	12	6	4	48	1	2
17	2	3	12	12	8	2	48	0	0
18	2	3	10	15	10	3	60	0	0
19	2	3	9	18	12	4	72	0	0
20	2	4	5	20	10	8	80	0	0
21	2	3	8	24	16	6	96	1	0
22	2	3	7	42	28	12	168	0	0

Most of the H-sequences in this list do not correspond to non-orientable hypermaps. Concerning non-orientability, Theorem 1 and 2 withdraw from this table all the hypermaps with one and two hyperfaces. The elimination of H-sequences follows the same basic argument. Keeping in mind that Sylow 5-subgroups in item 11 are cyclic groups generated by rotations around hyperfaces, the number n_5 of Sylow 5-subgroups must divide F, so there is only one Sylow 5-subgroup in G. Factoring it out would leave a hypermap of type $(3,3,1)$ with one hypervertex. But according to Theorem 1 (after transposing hyperfaces and hypervertices) this H-sequence cannot correspond to a non-orientable hypermap. Similar argument for items 14, 18, 20 and 22. In item 15, there are one or four Sylow 3-subgroups. It cannot be one because factoring it out would contradict Theorem 2. Being four, we have a permutation representation of degree 4. Any rotation r_v around a hypervertex must fix one Sylow 3-subgroup and permute the other three. This leads to the Schreier graphical representation of type $(3,3,2)$ of S_4 (Figure 23), and so $\mathcal{H}$ must be a covering of $\mathcal{T}^{(12)}$, an orientable hypermap obtained from the tetrahedron $\mathcal{T}$ by an operation transposing hyperedges and hyperfaces, so it must be itself orientable.

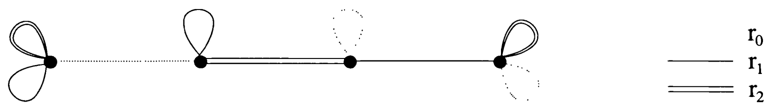

FIGURE 23.

Similar argument for item 19. In item 21, any hyperface rotation r_f must fix two hyperfaces (otherwise, it would fix all the hyperfaces, generating in this way a normal subgroup that factored out would leave a 2-hyperedge hypermap not covered by Theorem 2). The square ${r_f}^2$, of order 4, cannot fix all the hyperfaces, otherwise it would generate a normal subgroup and factoring it out would give a hypermap (without boundary) with s-characteristic $\chi > 2$, so ${r_f}^2$ must also fix two hyperfaces. Then the permutation representation of degree 6 induced by the action of G on the six hyperfaces leads to a Schreier graphical representation of type $(2,3,4)$ of S_4 (Figure 24), and so $\mathcal{H}$ must be the orientable hypermap $\mathcal{C}^{(01)}$, an associate of the Cube $\mathcal{C}$.

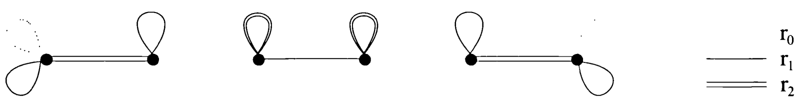

FIGURE 24.

In item 16, replace the H-sequence $(2,4,6,12,6,4,48)$ by $(4,2,6,6,12,4,48)$. Then the action of G on the four hyperfaces gives a permutation representation of degree 4 with Schreier graphical representation of type $(4,2,3)$ of S_4 (Figure 25).

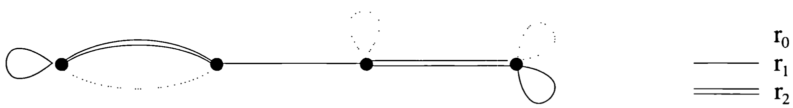

FIGURE 25.

Thus G is a double covering of S_4. By [BJ4], we must have $G = S_4 \times C_2$, $GL_2(3)$, $\tilde{S}_4$ or B, where B is the group with presentation

$$\langle x, y \mid \; x^4 = y^6 = (xy)^4 = 1\,, \;\; x^2 = y^3 \rangle\,.$$

The binary group $\tilde{S}_4$ has only one involution, which is central, so it cannot be the automorphism group of a reflexible hypermap. The group B has a faithful permutation representation of degree 16 given by

$$\begin{aligned} X &= (1\ 2\ 3\ 5\ 6\ 7)(4\ 8)(9\ 10\ 11\ 13\ 14\ 15)(12\ 16) \qquad \text{and} \\ Y &= (1\ 16\ 7\ 10)(2\ 13\ 8\ 11)(3\ 14\ 5\ 12)(4\ 15\ 6\ 9)\,, \end{aligned}$$

which projects over S_4 by $X \to X \pmod 4$ and $Y \to Y \pmod 4$. One infers from this projection that B has only four involutions, one central and three projecting over the even involutions of S_4 so, as with the binary group, B cannot be generated by involutions, and so B cannot be the automorphism group of a reflexible hypermap either. The linear group $GL(2,3)$ can be generated by involutions, but, contrary to the group B, its involutions (with the exception of the central involution) project over the odd involutions of S_4 via the projection $GL(2,3) \to PGL(2,3) \cong S_4$. As G is generated by two odd involutions and one even involution, it cannot be $GL(2,3)$ either. Hence G must be $S_4 \times C_2$, and thus $\mathcal{H}$ must be a double covering of the Projective Octahedron $\mathcal{PO}$. From [BJ3], there are only two double coverings of this map, the maps $\mathcal{PO}^{\hat{1}}$ and $\mathcal{PO}^{\hat{0}}$ (Table 2).

TABLE 2.

Map	Notation in [CM]	Hypermap type	#	V E F	$\mathrm{Aut}^+\mathcal{M} = \mathrm{Aut}\,\mathcal{M}$
$\mathcal{PO}^{\hat{1}}$	$\{6,4\}_3$	4 2 6	6	6 12 4	$S_4 \times C_2$
$\mathcal{PO}^{\hat{0}}$	$\{6,4\}_6$	4 2 6	6	6 12 4	$S_4 \times C_2$

The map $\mathcal{PO}^{\hat{1}}$ (Figure 26), in the Coxeter notation $\{6,4\}_3$ ([CM]), is the non-bipartite map opp($\mathcal{C}$) in [SW]. The second map $\mathcal{PO}^{\hat{0}}$ (Figure 26) is the bipartite map called Gamma(2) in [SW]. Following Coxeter's notation, we should write $\mathcal{PO}^{\hat{0}} = \{6,4\}_6$.

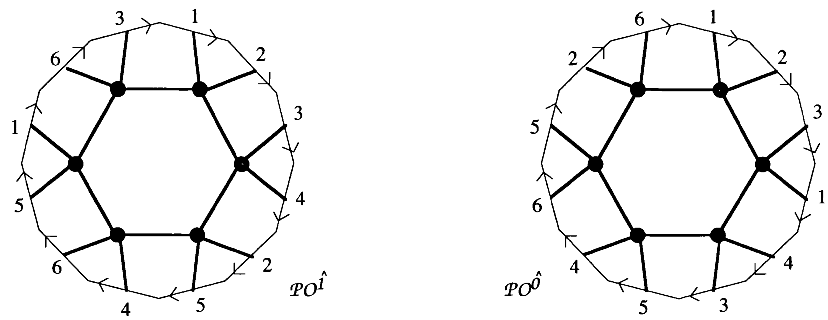

FIGURE 26. The maps $\mathcal{PO}^{\hat{1}}$ and $\mathcal{PO}^{\hat{0}}$.

The action of G on the hypervertices in item 7 induces a faithful permutation representation of degree 4 that gives S_4 (a rotation r_v around a hypervertex fixes one hypervertex and permutes the other three), so G must be itself S_4. According to [BJ3], there is only one hypermap and this is $\mathcal{H} = \mathcal{W}^{-1}(\mathcal{PO}^{\hat{0}})$ (Table 3).

TABLE 3.

Hypermap	Hypermap type	#	V E F	$\mathrm{Aut}^+\mathcal{H} = \mathrm{Aut}\,\mathcal{H}$
$\mathcal{W}^{-1}(\mathcal{PO}^{\hat{0}})$	4 4 3	3	3 3 4	S_4

This hypermap $\mathcal{W}^{-1}(\mathcal{PO}^{\hat{0}})$ can be constructed from the bipartite map $\mathcal{PO}^{\hat{0}}$ obtained before (Figure 26) by regarding it as the Walsh bipartite map representation of $\mathcal{H}$ (Figure 27).

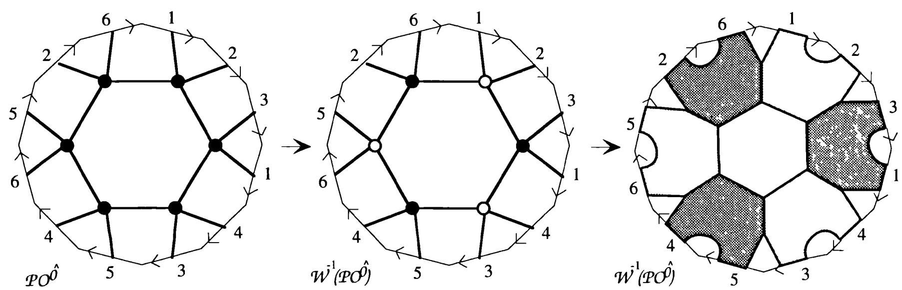

FIGURE 27. The hypermap $\mathcal{W}^{-1}(\mathcal{PO}^{\hat{0}})$.

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