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WHICH DIRECTED GRAPHS HAVE A SOLUTION?

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I. Introduction and Preliminaries

In this note, a *digraph* is a finite directed graph D with no loops or multiple arcs, as defined in the books [1], [5] and [6]. The *point set* and the *arc set* of D are denoted by $P(D)$ and $A(D)$. A subset S of $P(D)$ is *independent* in D if $a, b \in S$ implies that neither arc $\overrightarrow{ab}$ nor $\overrightarrow{ba}$ is in $A(D)$, while S is *dominant* in D if for each $a \in P(D) \setminus S$ there exists at least one $b \in S$ such that $\overrightarrow{ba} \in A(D)$. If S is both independent and dominant in D, then S is a *solution* of D. A subset S of $P(D)$ is absorbent in D if for each $a \in P(D) \setminus S$ there exists at least one $b \in S$ such that $\overrightarrow{ab} \in A(D)$. The set S is a *kernel* of D if it is both independent and absorbent in D. The notions not defined here can be found in [2, 5].

The concepts of solutions and kernels of digraphs are directional duals [4] and hence the Principle of Directional Duality (P.D.D.) applies [6, p. 38]. Both concepts stem from the work of von Neumann and Morgenstern [8, Ch. 12], where the mathematical aspects of p-person games have been investigated. There, it is proved in the language of relations, that an acyclic digraph has a solution.

Theorem A. *Every digraph with no directed cycles has a unique solution.*

By the P.D.D., every acyclic digraph has a unique kernel. Later Richardson in a series of papers [9, 10, 11] generalized the problem to arbitrary digraphs D (finite or infinite).

Theorem B. *Every digraph with no odd directed cycles has at least one solution.*

Harary and Richardson [7] posed the following questions:

Problem I. Characterize digraphs with a unique solution.

Problem II. Produce an algorithm for generating all the solutions of a given digraph if any exist.

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The concept of a kernel of a digraph has been studied rather intensively by many authors. (For a list of works on the topic and related subjects, see the references in Chapter 14 of Berge [3].) Although Problem I has not yet been settled, the existence of kernels, and hence of solutions, was established for some special classes of digraphs. The following observations are immediate and their proofs are omitted.

Proposition 1. Every symmetric digraph has a kernel.

Proposition 2. Every transitive digraph has a kernel. Moreover, all its kernels have the same cardinality.

While the concept of a kernel is the dual notion of a solution, there are digraphs with a kernel and no solutions. Of course the converse of such a digraph has a solution and no kernels, see Figure 1. On the other hand, Figure 2 shows a digraph with both a kernel and a solution and a digraph with neither a kernel nor a solution.

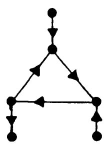

Fig. 1. A digraph with a solution but no kernels

a) b)

Fig. 2. One digraph with a kernel and a solution, and one with neither

There is no characterization of digraphs with solutions or kernels in terms of forbidden subdigraphs and it is not likely that one can be found. We intend to investigate some new necessary or sufficient conditions for the existence of solutions or kernels, and to study special classes of digraphs.

II. Conditions for a digraph to have a solution

Without loss of generality we consider in what follows finite nontrivial connected digraphs. While Theorems 1 and 2 do give necessary and sufficient conditions for a digraph to have a solution, they cannot be regarded as settling the problem of characterizing such digraphs because the conditions are very close to the definition and not generally applicable. Let $id(v, D)$ denote the indegree of point v in digraph D; similarly od denotes the outdegree.

Theorem 1. *A digraph D has a solution if and only if D does not have an induced subdigraph E with no solution such that the indegree of each point of E is the same in E as in D.*

Proof. If D has a solution and if D has an induced subdigraph E with $id\,(v, E) = id\,(v, D)$ for all $v \in P(E)$, then E must have a solution. Conversely, if D has no solutions, then $E = D$ is an induced subdigraph of D with no solutions in which the indegree condition holds trivially.

Corollary 1. *Let D' be the digraph obtained from a digraph D by removing all points v of D with $id\,(v, D) = 1$ and $od\,(v, D) = 0$. Then D has a solution if and only if D' has a solution.*

When we interchange $id\,(v)$ and $od\,(v)$ as well as solution and kernel in the theorem and its corollary and apply the P.D.D., we obtain the corresponding result for kernels.

Let a, b, c, d be points of a digraph D with b and c carriers so that $id\,(b) = od\,(b) = id(c) = od\,(c) = 1$ and let $\vec{ab}$, $\vec{bc}$, $\vec{cd} \in A(D)$. Before we state the next theorem we define an operation (α) on D involving these four points:

(α) If arc $\vec{ad} \notin A(D)$, remove points b, c, and arcs $\vec{ab}$, $\vec{bc}$, and $\vec{cd}$ from D, and add arc $\vec{ad}$; otherwise just remove the elements b, c, $\vec{ab}$, $\vec{bc}$, and $\vec{cd}$.

Theorem 2. *Let D' be a digraph obtained from a digraph D by repeated applications of the operation (a). Then D has a solution if and only if D' has a solution.*

Proof. Using mathematical induction we can confine ourselves to a single operation. Also, without loss of generality we assume that $\vec{ad} \in A(D)$. Let S be a solution of D. If $b \in S$, then $c \notin S$, and $S \setminus \{b\}$ is a solution for D'. If $b \notin S$, then $c \in S$, $a \in S$, and $d \notin S$. Hence $S \setminus \{c\}$ is a solution for D'. Conversely, let S' be a solution for D'. If $a \in S'$, then $d \notin S'$, and $S' \cup \{c\}$ is a solution for D, and if $a \notin S'$, then $S \cup \{b\}$ is a solution for D.

The following result provides a sufficient condition for the existence of a solution. It is extremal in nature because it shows that every digraph with "enough" arcs has a solution.

Theorem 3. *Let D be a digraph with $p \leqslant 3$ points and at least $p^2 - 2p + 1$* arcs. *Then D has a solution and a kernel. Moreover, there exists a digraph with $p^2 - 2p$* arcs *which has neither a solution nor a kernel, hence this bound is sharp.*

Proof. The digraph D has a point of outdegree $p - 1$, and a point of indegree $p - 1$, since otherwise the number of arcs is $q = \sum_{v \in P(D)} od\,(v) = \sum_{v \in P(D)} id\,(v) \leqslant$

$p(p-2)=p^2-2p$, which is a contradiction. Now, consider the complete symmetric digraph with $p \geq 3$ points $v_1, v_2, \ldots, v_p$, and remove the arcs of the Hamiltonian cycle $\overrightarrow{v_1v_2}, \overrightarrow{v_2v_3}, \ldots, \overrightarrow{v_{p-1}v_p}, \overrightarrow{v_pv_1}$ from it to obtain a digraph D. Although every two points are adjacent, D has neither a transmitter nor a receiver. Hence D has neither a solution nor a kernel.

Before we proceed in another direction, it is worth mentioning that if a digraph D consists of two odd cycles with a common directed path, then D has no solutions. Figure 2(b) provides one such example, where the common path has just one point.

In the light of Theorems A and B of Section I, it seems natural to investigate the existence of solutions for unicyclic digraphs, i.e., asymmetric digraphs whose underlying graphs contain exactly one cycle. If such a digraph contains no odd directed cycle, then it has a solution by Theorem B, and if it is an odd directed cycle, then it has no solutions. Hence in the next theorem we exclude these cases from the class of unicyclic digraphs.

Theorem 4. *Let D be a unicyclic digraph different from a directed cycle whose unique cycle is the odd directed cycle C: $v_1, v_2, \ldots, v_{2t+2}=v_1$, $t \geq 1$. Then D has a solution if and only if the solution of at least one of the components $D_1, D_2, \ldots, D_{2t+2}=D_1$, $v_i \in P(D_i)$ of $D \backslash A(C)$, say $S(D_1)$, does not contain v_1.*

Proof. Assume D has a solution $S(D)$. Since D_i is acyclic, $S(D_i)$ exists for each i and is unique. There exists two consecutive $v_i's$, say v_{2t+1} and v_1, neither of which are in $S(D)$. Hence $S(D) \cap P(D_1)=S(D_1)$ is a solution for D_1 with the desired property.

Conversely, define $S'(D_1)=S(D_1)$, and $S'(D_2)=S(D_2)$. For each value of i such that $3 \leq i \leq 2t+1$, let $S'(D_i)=S(D_i)$ if $v_i \notin S(D_i)$ or if $v_i \in S(D_i)$ but $v_{i-1} \notin S(D_{i-1})$, and let $S'(D_i)=S(D_i \backslash v_i)$ otherwise. Note that if $P(D_i)=\{v_i\}$, then $S(D_i \backslash v_i)=\emptyset$, and that $S(D_i \backslash v_i)$ exists otherwise. Then the set

$$S=\bigcup_{i=1}^{2t+1} S'(D_i)$$

is a solution for D.

There are several classes of digraphs, some but not all of which have solutions. These include tournaments and also digraphs which are flexible, eulerian, unipathic, stron, strictly unilateral, and strictky weak. Aside from tournaments, it is not yet explicitly known which digraphs in each such class have solutions.

It is clear that the number $s(D)$ of solutions of a disconnected digraph can be much larger than its order. Concerning connected digraphs we have the following theorem.

Theorem 5. *For each arbitrary positive integer m there exists a connected digraph D of order p such that $s(D)>mp$.*

Proof. Consider k copies of a symmetric complete digraph K of order n, where $n > 1$, and a digraph $K_{1,k}$with point set $\{a\} \cup \{b_1, \ldots, b_k\}$ such that $od\,(a) = k$, $id\,(a) = 0$, $od\,(b_i) = 0$, and $id\,(b_i) = 1$ for all $1 \leqslant i \leqslant k$. In a one-to-one manner, join a sink of $K_{1,k}$ by an arc to a point of a copy of K, and denote the resulting digraph by D. Then D has n^k solutions and has the order $p = kn + k + 1$. It is clear that $n^k/(kn + k + 1)$ approaches infinity as $k \to \infty$. Hence there exists an h such that $n^h > m(hn + h + 1)$, completing the proof.

Our final results provides a characterization of digraphs having the same set as their solution and kernel.

Theorem 6. *Let L be an independent set of points of a digraph D. Then L is both a solution and a kernel for D if and only if D contains a set $\mathscr{A}$ of directed paths or cycles containing all points not in L, which start and end with a point of L and alternately contain points of L.*

Proof. If $P(D) \backslash L = \emptyset$, then the implication is vacously true. Assume $a \in P(D) \backslash L$. Then there exists c, $b \in L$, not necessarily distinct, such that $\overrightarrow{ca}$ and $\overrightarrow{ab}$ are in $A(D)$. It is clear that all such pairs of arcs form the desired set ?. Conversely the set L is both dominant and absorbent as well as independent. Hence L is both a solution and a kernel for D.

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КОТОРЫЕ ОРИЕНТИРОВАННЫЕ ГРАФЫ ОБЛАДАЮТ РЕШЕНИЕМ?

Мехди Бехзад—Френк Харари

Резюме

В работе изучаются понятия решений и ядер ориентированных графов, происхождение которых связано с работой фон Ноймана и Моргезштерна о играх p лиц. Хотя эффективная характеризация ориентированных графов обладающих решением или ядром пока не известна, здесь установленно несколько необходимых и достаточных условий для существования решений в ориентированных графах и описано строение некоторых специальныхклассов графов имеющих решение.