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GLOBAL EXISTENCE FOR SECOND ORDER FUNCTIONAL SEMILINEAR INTEGRODIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we study the global existence of solutions for second order initial value problems for functional semilinear integrodifferential equations, where the linear operator in the differential equation is the infinitesimal generator of a strongly continuous cosine family in a Banach space X. Using the Leray-Schauder Alternative, we derive conditions under which a solution exists globally.

1. Introduction

In this paper we study the global existence of solutions for second order initial value problems (IVP for short) for semilinear functional integrodifferential equations of the form

$$x''(t) = Ax(t) + f\left(t, x_t, \int_0^t k(t,s)g\bigl(s, x_s, x'(s)\bigr)\,\mathrm{d}s,\, x'(t)\right), \tag{1.1}$$

$$\text{a.a. } t \in I := [0,b],$$

$$x_0 = \phi, \qquad x'(0) = \eta, \tag{1.2}$$

where A is a linear infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ in a Banach space X, and $f \colon I \times C \times X \times X \to X$, $g \colon I \times C \times X \to X$ and $k \colon I \times I \to \mathbb{R}$ are given functions.

Here $C = C\bigl([-r,0], X\bigr)$ is the Banach space of all continuous functions $u \colon [-r,0] \to X$ endowed with the sup-norm

$$\|u\| = \sup\bigl\{|u(\theta)| : -r \le \theta \le 0\bigr\}.$$

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Also for $x \in C([-r,b],X)$ we have $x_t \in C$ for $t \in [0,b]$, $x_t(\theta) = x(t+\theta)$ for $\theta \in [-r,0]$. By using topological degree arguments we prove the global existence of a solution of (1.1) – (1.2).

Recent results on global existence, for ordinary, functional, neutral or partial differential equations with the aid of the Topological Transversality method, may be found in the works listed in our references [3], [4], [5], [6], [7]. Our approach here is essentially an application of the Topological Transversality method to obtain global existence results for functional semilinear integrodifferential equations.

It is well known, see e.g [10] (for the case of ordinary differential equations), that only the continuity of f is not sufficient to assure local existence of solutions, even when X is a Hilbert space. Therefore, one has to restrict either the function f or the semi-group operator. Usually restrictions on f are imposed. The function f was assumed to be locally Lipschitz or monotone or completely continuous. Here we assume that $C(t)$ (defined below) is compact and the function f satisfies the following Caratheodory-type conditions, which do not imply that f is completely continuous:

(C_1) For each $t \in [0,b]$ the function $f(t,\cdot,\cdot,\cdot)\colon C \times X \times X \to X$ is continuous, and for each $(x,y,z) \in C \times X \times X$ the function $f(\cdot,x,y,z)\colon I \to X$ is strongly measurable.

(C_2) For every positive constant k there exists $h_k \in L^1(I)$ such that for a.a. $t \in I$

$$\sup_{\|x\|,|y|,|z| \le k} |f(t,x,y,z)| \le h_k(t)\,.$$

The consideration of this paper is based on the following fixed point result (cf. [2]).

Lemma 1.1 (Leray-Schauder Alternative). *Let S be a convex subset of a normed linear space E and assume $0 \in S$. Let $F\colon S \to S$ be a completely continuous operator, and let*

$$\mathcal{E}(F) = \{x \in S : \ x = \lambda F x \ \textit{for some} \ 0 < \lambda < 1\}\,.$$

Then either $\mathcal{E}(F)$ is unbounded or F has a fixed point.

2. Global existence

In this section we study the global existence of solutions for IVP (1.1)–(1.2).

In many cases it is advantageous to treat second order abstract differential equations directly rather than to convert them to first order systems. A useful machinery for the study of abstract second order equations is the theory of strongly continuous cosine families.

Given a Banach space X, we say that the family $\{C(t) : t \in \mathbb{R}\}$ in the space $L(X)$ of bounded linear operators on X is a *strongly continuous cosine family* if

(i) $C(0) = I$;
(ii) $C(t)x$ is strongly continuous in t on $\mathbb{R}$ for each fixed $x \in X$;
(iii) $C(t+s) + C(t-s) = 2C(t)C(s)$ for all $t, s \in \mathbb{R}$.

The *strongly continuous sine family* $\{S(t) : t \in \mathbb{R}\}$ is defined by

$$S(t)x = \int_0^t C(s)x \, \mathrm{d}s \,, \qquad x \in X \,, \ t \in \mathbb{R}.$$

The *infinitesimal generator* A of a cosine family $\{C(t) : t \in \mathbb{R}\}$ is defined by

$$Ax = \frac{\mathrm{d}^2}{\mathrm{d}t^2} C(t)x\Big|_{t=0} \,, \qquad x \in D(A) \,,$$

where

$$D(A) = \{x \in X : \ C(\cdot)x \in C^2(\mathbb{R}, X)\} \,.$$

Assume now that A is a linear infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ of bounded linear operators from X into itself. Moreover, we assume that the adjoint operator A^* is densely defined, i.e $\overline{D(A^*)} = X^*$. See [1].

It is known that if $C(t)$, $t \in \mathbb{R}$, is a strongly continuous cosine family with infinitesimal generator A then

$$x(t) = C(t)\phi(0) + S(t)\eta + \int_0^t S(t-s) f\left(s, x_s, \int_0^s k(s,\sigma) g\big(\sigma, x_\sigma, x'(\sigma)\big) \, \mathrm{d}\sigma, x'(s)\right) \mathrm{d}s \,, \qquad t \in I \,,$$

with $x_0 = \phi$. The above equation is more general than equation (1.1) and every solution of this is called *mild solution* of (1.1)–(1.2). Also this is easier to work with than (1.1) (1.2) because of the nice properties of the bounded operators $C(t)$, $t \in \mathbb{R}$ and $S(t)$, $t \in \mathbb{R}$, as opposed to the unbounded operator A in equation (1.1).

In the sequel we will use the following result which was proved in [8].

Lemma 2.1. *Let $C(t)$ (resp. $S(t)$), $t \in \mathbb{R}$, be a strongly continuous cosine (resp. sine) family on X. Then there exist constants $N \ge 1$ and $\omega \ge 0$ such* *that*

$$|C(t)| \leq N\,\mathrm{e}^{\omega|t|} \qquad \textit{for all} \quad t \in \mathbb{R},$$

$$|S(t_1) - S(t_2)| \leq N\left|\int_{t_1}^{t_2} \mathrm{e}^{\omega|s|}\,\mathrm{d}s\right| \qquad \textit{for all} \quad t_1, t_2 \in \mathbb{R}.$$

The global existence result for the IVP (1.1)–(1.2) is the following:

THEOREM 2.2. *Let* $f\colon I \times C \times X \times X \to X$ *be a function satisfying* (C_1), (C_2), $g\colon I \times C \times X \to X$ *be a continuous function,* $k\colon I \times I \to \mathbb{R}$ *a measurable function and* $C(t)$ *(resp.* $S(t)$*),* $t \in I$, *be a strongly continuous cosine (resp. sine) family on* X *with the infinitesimal generator* A *as defined above. Assume that:*

(Hg) *There exists a continuous function* $m\colon I \to [0,\infty)$ *such that*

$$|g(t,u,v)| \leq m(t)\Omega\big(\|u\| + |v|\big), \qquad 0 \leq t \leq b, \quad u \in C, \quad v \in X,$$

where $\Omega\colon [0,\infty) \to (0,\infty)$ *is a continuous nondecreasing function.*

(Hf) *There exists a continuous function* $p\colon I \to [0,\infty)$ *such that*

$$|f(t,u,v,w)| \leq p(t)\big(\|u\| + |v| + |w|\big),$$
$$0 \leq t \leq b, \quad u \in C, \quad v, w \in X.$$

(Hk) *There exists a constant* L *such that:*

$$|k(t,s)| \leq L \qquad \textit{for} \quad t \geq s \geq 0.$$

(HC) $C(t)$, $t > 0$, *is compact.*

Then if

$$\int_0^b \widehat{m}(s)\,\mathrm{d}s < \int_c^\infty \frac{\mathrm{d}s}{s + \Omega(s)},$$

where

$$\widehat{m}(t) = \max\{M(b+1)p(t), Lm(t)\},$$
$$M = \sup\{|C(t)| : t \in I\}, \qquad M' = \sup\{|C'(t)| : t \in I\},$$

and

$$c = (M + M')\|\phi\| + M(1+b)|\eta|,$$

the IVP (1.1)–(1.2) *has at least one mild solution on* $[-r,b]$.

P r o o f. In the space $B = C\big([-r,b],X\big) \cap C^1\big([0,b],X\big)$ consider the norm

$$\|x\|^* = \max\{\|x\|_r, \|x\|_1\}$$

where

$$\|x\|_r = \sup\{|x(t)| : \ -r \le t \le b\}, \qquad \|x\|_1 = \sup\{|x'(t)| : \ 0 \le t \le b\}.$$

To prove existence of a mild solution of the IVP (1.1) – (1.2) we apply Lemma 1.1. First we obtain the *a priori bounds* for the mild solutions of the IVP $(1.1)_\lambda$ – (1.2), $\lambda \in (0,1)$, where $(1.1)_\lambda$ stands for the equation

$$x''(t) = \lambda A x(t) + \lambda f\left(t, x_t, \int_0^t k(t,s) g\big(s, x_s, x'(s)\big)\, \mathrm{d}s,\ x'(t)\right), \qquad t \in I.$$

Let x be a mild solution of the IVP $(1.1)_\lambda$ – (1.2). From

$$\begin{aligned} x(t) &= \lambda\big[C(t)\phi(0) + S(t)\eta\big] \\ &\quad + \lambda \int_0^t S(t-s) f\left(s, x_s, \int_0^s k(s,\sigma) g\big(\sigma, x_\sigma, x'(\sigma)\big)\, \mathrm{d}\sigma,\ x'(s)\right) \mathrm{d}s, \qquad t \in I \end{aligned}$$

we have

$$\begin{aligned} |x(t)| &\le M\|\phi\| + Mb|\eta| \\ &\quad + Mb \int_0^t p(s)\left[\|x_s\| + L\int_0^s m(\sigma)\Omega\big(\|x_\sigma\| + |x'(\sigma)|\big)\, \mathrm{d}\sigma + |x'(s)|\right] \mathrm{d}s, \\ &\qquad t \in I. \end{aligned}$$

We consider the function μ given by

$$\mu(t) = \sup\{|x(s)| : \ -r \le s \le t\}, \qquad 0 \le t \le b.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |x(t^*)|$. If $t^* \in [0,t]$, by the previous inequality we have

$$\begin{aligned} \mu(t) &\le M\|\phi\| + Mb|\eta| \\ &\quad + Mb \int_0^{t^*} p(s)\left[\mu(s) + L\int_0^s m(\sigma)\Omega\big(\mu(\sigma) + |x'(\sigma)|\big)\, \mathrm{d}\sigma + |x'(s)|\right] \mathrm{d}s \\ &\le M\|\phi\| + Mb|\eta| \\ &\quad + Mb \int_0^t p(s)\left[\mu(s) + L\int_0^s m(\sigma)\Omega\big(\mu(\sigma) + |x'(\sigma)|\big)\, \mathrm{d}\sigma + |x'(s)|\right] \mathrm{d}s, \\ &\qquad t \in I. \end{aligned}$$

If $t^* \in [-r,0]$ then $\mu(t) = \|\phi\|$ and the previous inequality holds because $M \geq 1$, since $C(0) = I$.

Denoting by $u(t)$ the right-hand side of the above inequality we have

$$u(0) = M\|\phi\| + Mb|\eta|\,, \qquad \mu(t) \leq u(t)\,, \quad 0 \leq t \leq b\,,$$

and

$$\begin{aligned} u'(t) &= Mbp(t)\left[\mu(t) + |x'(t)| + L\int_0^t m(\sigma)\Omega\big(\mu(\sigma) + |x'(\sigma)|\big)\,\mathrm{d}\sigma\right] \\ &\leq Mbp(t)\left[u(t) + |x'(t)| + L\int_0^t m(\sigma)\Omega\big(u(\sigma) + |x'(\sigma)|\big)\,\mathrm{d}\sigma\right], \qquad t \in I\,. \end{aligned}$$

Therefore, if

$$v(t) = \sup\big\{|x'(s)| :\ s \in [0,t]\big\}\,, \qquad t \in I\,,$$

we obtain

$$u'(t) \leq Mbp(t)\left[u(t) + v(t) + L\int_0^t m(\sigma)\Omega\big(u(\sigma) + v(\sigma)\big)\,\mathrm{d}\sigma\right], \qquad t \in I\,.$$

But

$$\begin{aligned} x'(t) &= \lambda\big[C'(t)\phi(0) + S'(t)\eta\big] \\ &\quad + \lambda\int_0^t C(t-s)f\left(s, x_s, \int_0^s k(s,\sigma)g\big(\sigma, x_\sigma, x'(\sigma)\big)\,\mathrm{d}\sigma,\ x'(\tau)\right)\mathrm{d}s\,, \\ &\qquad t \in I\,. \end{aligned}$$

Thus if $t^* \in [0,t]$ is such that $v(t) = |x'(t^*)|$ we have

$$\begin{aligned} v(t) &= |x'(t^*)| \\ &\leq M'\|\phi\| + M|\eta| \\ &\quad + M\int_0^t p(s)\left[u(s) + v(s) + L\int_0^s m(\sigma)\Omega\big(u(\sigma) + v(\sigma)\big)\,\mathrm{d}\sigma\right]\mathrm{d}s\,, \\ &\qquad t \in I\,. \end{aligned}$$

Denoting by $r(t)$ the right-hand side in the above inequality we have

$$r(0) = M'\|\phi\| + M|\eta|\,, \qquad v(t) \leq r(t)\,, \quad t \in I\,,$$

and

$$
\begin{aligned}
r'(t) &= Mp(t)\left[u(t) + v(t) + L\int_0^t m(\sigma)\Omega\big(u(\sigma) + v(\sigma)\big)\,\mathrm{d}\sigma\right] \\
&\le Mp(t)\left[u(t) + r(t) + L\int_0^t m(\sigma)\Omega\big(u(\sigma) + r(\sigma)\big)\,\mathrm{d}\sigma\right], \qquad t \in I .
\end{aligned}
$$

Let

$$
w(t) = u(t) + r(t) + L\int_0^t m(\sigma)\Omega\big(u(\sigma) + r(\sigma)\big)\,\mathrm{d}\sigma\,, \qquad t \in I .
$$

Then

$$
w(0) = u(0) + r(0) = c\,, \qquad u(t) + r(t) \le w(t)\,, \quad t \in I\,,
$$

and

$$
\begin{aligned}
w'(t) &= u'(t) + r'(t) + Lm(t)\Omega\big(u(t) + r(t)\big) \\
&\le Mbp(t)w(t) + Mp(t)w(t) + Lm(t)\Omega\big(w(t)\big) \\
&\le \widehat{m}(t)\big[w(t) + \Omega\big(w(t)\big)\big]\,, \qquad t \in I .
\end{aligned}
$$

This implies

$$
\int_{w(0)}^{w(t)} \frac{\mathrm{d}s}{s + \Omega(s)} \le \int_0^b \widehat{m}(s)\,\mathrm{d}s < \int_c^\infty \frac{\mathrm{d}s}{s + \Omega(s)}\,, \qquad t \in I .
$$

This inequality implies that there is a constant K such that

$$
w(t) = u(t) + r(t) \le K\,, \qquad t \in I .
$$

Then

$$
\begin{aligned}
|x(t)| &\le \mu(t) \le u(t)\,, \qquad t \in I\,, \\
|x'(t)| &\le v(t) \le r(t)\,, \qquad t \in I\,,
\end{aligned}
$$

and hence

$$
\|x\|^* \le K\,,
$$

where K depends only on b and on the functions m, p and Ω.

In order to apply Lemma 1.1 we must prove that the operator $F\colon B \to B$ defined by

$$
(\Gamma y)(t) = \begin{cases} \phi(t)\,, & -r \le t \le 0\,, \\ C(t)\phi(0) + S(t)\eta + \int_0^t S(t-s)\,\cdot & \\ \cdot\, f\Big(s, y_s, \int_0^s k(s,\sigma)g\big(\sigma, y_\sigma, y'(\sigma)\big)\,\mathrm{d}\sigma,\, y'(s)\Big)\,\mathrm{d}s\,, & 0 \le t \le b \end{cases}
$$

is a completely continuous operator.

Let $B_k = \{y \in B : \|y\|^* \le k\}$ for some $k \ge 1$. We first show that F maps B_k into an equicontinuous family. Let $y \in B_k$ and $t_1, t_2 \in [0,b]$ and $\varepsilon > 0$. Then if $0 < \varepsilon < t_1 < t_2 \le b$

$$
\begin{aligned}
&|(Fy)(t_1) - (Fy)(t_2)| \\
&= \Big| C(t_1)\phi(0) - C(t_2)\phi(0) + S(t_1)\eta - S(t_2)\eta \\
&\quad + \int_0^{t_1} S(t_1 - s) f\Big(s, y_s, \int_0^s k(s,\sigma) g\big(\sigma, y_\sigma, y'(\sigma)\big)\, \mathrm{d}\sigma,\, y'(s)\Big)\, \mathrm{d}s \\
&\quad - \int_0^{t_2} S(t_2 - s) f\Big(s, y_s, \int_0^s k(s,\sigma) g\big(\sigma, y_\sigma, y'(\sigma)\big)\, \mathrm{d}\sigma,\, y'(s)\Big)\, \mathrm{d}s \Big| \\
&\le |C(t_1) - C(t_2)||\phi(0)| + |S(t_1) - S(t_2)||\eta| \\
&\quad + \Big| \int_0^{t_1-\varepsilon} [S(t_1 - s) - S(t_2 - s)] f\Big(s, y_s, \int_0^s k(s,\sigma) g\big(\sigma, y_\sigma, y'(\sigma)\big)\, \mathrm{d}\sigma,\, y'(s)\Big)\, \mathrm{d}s \Big| \\
&\quad + \Big| \int_{t_1-\varepsilon}^{t_1} [S(t_1 - s) - S(t_2 - s)] f\Big(s, y_s, \int_0^s k(s,\sigma) g\big(\sigma, y_\sigma, y'(\sigma)\big)\, \mathrm{d}\sigma,\, y'(s)\Big)\, \mathrm{d}s \Big| \\
&\quad + \Big| \int_{t_1}^{t_2} S(t_2 - s) f\Big(s, y_s, \int_0^s k(s,\sigma) g\big(\sigma, y_\sigma, y'(\sigma)\big)\, \mathrm{d}\sigma,\, y'(s)\Big)\, \mathrm{d}s \Big| \\
&\le |C(t_1) - C(t_2)||\phi(0)| + |S(t_1) - S(t_2)||\eta| \\
&\quad + \int_0^{t_1-\varepsilon} |S(t_1 - s) - S(t_2 - s)| h_{k'}(s)\, \mathrm{d}s \\
&\quad + \int_{t_1-\varepsilon}^{t_1} |S(t_1 - s) - S(t_2 - s)| h_{k'}(s)\, \mathrm{d}s + \int_{t_1}^{t_2} |S(t_2 - s)| h_{k'}(s)\, \mathrm{d}s\,,
\end{aligned}
$$

and

$$\begin{aligned}
&|(Fy)'(t_1) - (Fy)'(t_2)| \\
&= \Big| C'(t_1)\phi(0) - C'(t_2)\phi(0) + S'(t_1)\eta - S'(t_2)\eta \\
&\quad + \int_0^{t_1} C(t_1 - s) f\Big(s, y_s, \int_0^s k(s,\sigma) g\big(\sigma, y_\sigma, y'(\sigma)\big)\, \mathrm{d}\sigma,\, y'(s)\Big)\, \mathrm{d}s \\
&\quad - \int_0^{t_2} C(t_2 - s) f\Big(s, y_s, \int_0^s k(s,\sigma) g\big(\sigma, y_\sigma, y'(\sigma)\big)\, \mathrm{d}\sigma,\, y'(s)\Big)\, \mathrm{d}s \Big| \\
&\le |C'(t_1) - C'(t_2)||\phi(0)| + |S'(t_1) - S'(t_2)||\eta| \\
&\quad + \int_0^{t_1-\varepsilon} |C(t_1 - s) - C(t_2 - s)| h_{k'}(s)\, \mathrm{d}s \\
&\quad + \int_{t_1-\varepsilon}^{t_1} |C(t_1 - s) - C(t_2 - s)| h_{k'}(s)\, \mathrm{d}s + \int_{t_1}^{t_2} |C(t_2 - s)| h_{k'}(s)\, \mathrm{d}s\,,
\end{aligned}$$

where

$$k' = \max\{k, bLm_0\Omega(2k)\}\,, \qquad m_0 = \max\{m(t) :\ t \in I\}\,.$$

The right hand sides are independent of $y \in B_k$ and tend to zero as $t_2 - t_1 \to 0$ and ε sufficiently small, since $C(t)$, $S(t)$, $C'(t)$, $S'(t)$ are uniformly continuous for $t \in [0,b]$ and the compactness of $C(t)$, $S(t)$ for $t > 0$ imply the continuity in the uniform operator topology. (See also Lemma 2.1.) The compactness of $S(t)$ follows from that of $C(t)$, and [9; Lemma 2.1, Lemma 2.5].

Thus F maps B_k into an equicontinuous family of functions.

The equicontinuity for the cases $t_1 < t_2 \le 0$ and $t_1 \le 0 \le t_2$ follows from the uniform continuity of ϕ on $[-r, 0]$ and from the relation

$$|(Fy)(t_1) - (Fy)(t_2)| \le |\phi(t_1) - (Fy)(t_2)| \le |(Fy)(t_2) - (Fy)(0)| + |\phi(0) - \phi(t_1)|$$

respectively.

It is easy to see that the family FB_k is uniformly bounded.

Next, we show $\overline{FB_k}$ is compact. Since we have shown FB_k is an equicontinuous collection, it suffices by Arzela-Ascoli theorem to show that F maps B_k into a precompact set in X.

Let $0 < t \le b$ be fixed and ε a real number satisfying $0 < \varepsilon < t$. For $y \in B_k$ we define

$$\begin{aligned}
(F_\varepsilon y)(t) &= C(t)\phi(0) + S(t)\eta \\
&\quad + \int_0^{t-\varepsilon} S(t - s) f\Big(s, y_s, \int_0^s k(s,\sigma) g\big(\sigma, y_\sigma, y'(\sigma)\big)\, \mathrm{d}\sigma,\, y'(s)\Big)\, \mathrm{d}s\,.
\end{aligned}$$

Since $C(t)$, $S(t)$ are compact operators, the set $Y_\varepsilon(t) = \{(F_\varepsilon y)(t) :\ y \in B_k\}$ is precompact in X, for every ε, $0 < \varepsilon < t$. Moreover, for every $y \in B_k$ we have

$$\begin{aligned}
&|(Fy)(t) - (F_\varepsilon y)(t)| \\
&\leq \int_{t-\varepsilon}^{t} \left| S(t-s) f\left(s, y_s, \int_0^s k(s,\sigma) g(\sigma, y_\sigma, y'(\sigma))\ \mathrm{d}\sigma,\ y'(s)\right)\right| \mathrm{d}s \\
&\leq \int_{t-\varepsilon}^{t} |S(t-s)| h_{k'}(s)\ \mathrm{d}s\,,
\end{aligned}$$

$$\begin{aligned}
&|(Fy)'(t) - (F_\varepsilon y)'(t)| \\
&\leq \int_{t-\varepsilon}^{t} \left| C(t-s) f\left(s, y_s, \int_0^s k(s,\sigma) g(\sigma, y_\sigma, y'(\sigma))\ \mathrm{d}\sigma,\ y'(s)\right)\right| \mathrm{d}s \\
&\leq \int_{t-\varepsilon}^{t} |C(t-s)| h_{k'}(s)\ \mathrm{d}s\,.
\end{aligned}$$

Therefore there are precompact sets arbitrary close to the set $\{(Fy)(t)\ :\ y \in B_k\}$. Hence the set $\{(Fy)(t) :\ y \in B_k\}$ is precompact in X.

It remains to show that $F\colon B \to B$ is continuous. Let $\{u_n\}_0^\infty \subseteq B$ with $u_n \to u$ in B. Then there is a positive constant q such that $\|u_n\|^* \leq q$ for all n, and consequently $\|u\|^* \leq q$. By (C_1)

$$\begin{aligned}
&f\left(t, u_{nt}, \int_0^t k(t,s) g(s, u_{ns}, u_n'(s))\ \mathrm{d}s,\ u_n'(t)\right) \\
&\qquad\qquad \longrightarrow f\left(t, u_t, \int_0^t k(t,s) g(s, u_s, u'(s))\ \mathrm{d}s,\ u'(t)\right)
\end{aligned}$$

for each $t \in I$, and since

$$\begin{aligned}
&\left| f\left(t, u_{nt}, \int_0^t k(t,s) g(s, u_{ns}, u_n'(s))\ \mathrm{d}s,\ u_n'(t)\right)\right. \\
&\qquad \left. - f\left(t, u_t, \int_0^t k(t,s) g(s, u_s, u'(s))\ \mathrm{d}s,\ u'(t)\right)\right| \leq 2h_{q'}(t)
\end{aligned}$$

where $q' = \max\{q, bLm_0\Omega(2q)\}$, we have by dominated convergence

$$\begin{aligned}
|Fu_n - Fu| &= \sup_{t\in[0,b]} \bigg| \int_0^t S(t-s)\Big[f\Big(s, u_{ns}, \int_0^s k(s,\sigma)g(\sigma, u_{n\sigma}, u_n'(\sigma))\,\mathrm{d}\sigma,\ u_n'(s)\Big) \\
&\qquad - f\Big(s, u_s, \int_0^s k(s,\sigma)g(\sigma, u_\sigma, u'(\sigma))\,\mathrm{d}\sigma,\ u'(s)\Big)\Big]\,\mathrm{d}s \bigg| \\
&\le \int_0^b \bigg| S(t-s)\Big[f\Big(s, u_{ns}, \int_0^s k(s,\sigma)g(\sigma, u_{n\sigma}, u_n'(\sigma))\,\mathrm{d}\sigma,\ u_n'(s)\Big) \\
&\qquad - f\Big(s, u_s, \int_0^s k(s,\sigma)g(\sigma, u_\sigma, u'(\sigma))\,\mathrm{d}\sigma,\ u'(s)\Big)\Big]\bigg|\,\mathrm{d}s \longrightarrow 0
\end{aligned}$$

and

$$\begin{aligned}
&|(Fu_n)' - (Fu)'| \\
&= \sup_{t\in[0,b]} \bigg| \int_0^t C(t-s)\Big[f\Big(s, u_{ns}, \int_0^s k(s,\sigma)g(\sigma, u_{n\sigma}, u_n'(\sigma))\,\mathrm{d}\sigma,\ u_n'(s)\Big) \\
&\qquad - f\Big(s, u_s, \int_0^s k(\tau,\sigma)g(\sigma, u_\sigma, u'(\sigma))\,\mathrm{d}\sigma,\ u'(s)\Big)\Big]\,\mathrm{d}s \bigg| \\
&\le \int_0^b \bigg| C(t-s)\Big[f\Big(s, u_{ns}, \int_0^s k(s,\sigma)g(\sigma, u_{n\sigma}, u_n'(\sigma))\,\mathrm{d}\sigma,\ u_n'(s)\Big) \\
&\qquad - f\Big(s, u_s, \int_0^s k(s,\sigma)g(\sigma, u_\sigma, u'(\sigma))\,\mathrm{d}\sigma,\ u'(s)\Big)\Big]\bigg|\,\mathrm{d}s \longrightarrow 0\,.
\end{aligned}$$

Thus F is continuous. This completes the proof that F is completely continuous.

Finally, the set $\mathcal{E}(F) = \{y \in B : \ y = \lambda F y,\ \ \lambda \in (0,1)\}$ is bounded, as we proved in the first part. Consequently, by Lemma 1.1, the operator F has a fixed point in B. This means that the IVP (1.1)–(1.2) has a mild solution, completing the proof of the theorem. □

THEOREM 2.3. *Let $f\colon I \times C \times X \times X \to X$ be a function satisfying (C_1), (C_2), $g\colon I \times C \times X \to X$ be a continuous function, $k\colon I \times I \to \mathbb{R}$ a measurable function and $C(t)$ (resp. $S(t)$), $t \in I$ be a strongly continuous cosine (resp.*

sine) family on X with the infinitesimal generator A as defined above. Assume that (Hk) and (HC) hold and:

$(Hg-1)$ *There exists a continuous function $m\colon I \to [0,\infty)$ such that*

$$|g(t,u,v)| \le m(t)\big(\|u\| + |v|\big)\,, \qquad 0 \le t \le b\,,\ \ u \in C\,,\ \ v \in X\,.$$

$(Hf-1)$ *There exists a continuous function $p\colon I \to [0,\infty)$ and a continuous non-decreasing function $\Omega_1\colon [0,\infty) \to (0,\infty)$ such that*

$$|f(t,u,v,w)| \le p(t)\Omega_1\big(\|u\| + |v| + |w|\big)\,, \qquad 0 \le t \le b\,,\ \ u \in C\,,\ \ v,w \in X\,.$$

Then if

$$\int_0^b \widehat{p}(s)\,\mathrm{d}s < \int_c^\infty \frac{\mathrm{d}s}{s + 2\Omega_1(s)}$$

where

$$\widehat{p}(t) = \max\big\{M(b+1)p(t), Lm(t)\big\}\,,$$
$$M = \sup\big\{|C(t)| :\ t \in I\big\}\,, \qquad M' = \sup\big\{|C'(t)| :\ t \in I\big\}$$

and

$$c = (M + M')\|\phi\| + M(1+b)|\eta|\,,$$

the IVP (1.1) – (1.2) *has at least one mild solution on $[-r,b]$.*

P r o o f . As in the proof of the previous theorem it suffices to obtain the *a priori bounds* for the mild solutions of the IVP $(1.1)_\lambda$ – (1.2), $\lambda \in (0,1)$.

So, if x is such a solution, then for every $t \in I$ we have

$$|x(t)| \le M\|\phi\| + Mb|\eta| + Mb\int_0^t p(s)\Omega_1\left(\|x_s\| + L\int_0^s m(\sigma)\big[\|x_\sigma\| + |x'(\sigma)|\big]\,\mathrm{d}\sigma + |x'(s)|\right)\mathrm{d}s\,.$$

We consider the function μ given by

$$\mu(t) = \sup\big\{|x(s)| :\ -r \le s \le t\big\}\,, \qquad t \in I\,.$$

Let $t^* \in [-r,t]$ be such that $\mu(t) = |x(t^*)|$. If $t^* \in [0,t]$, by the previous inequality we have

$$\mu(t) \le M\|\phi\| + Mb|\eta| + Mb\int_0^t p(s)\Omega_1\left(\mu(s) + L\int_0^s m(\sigma)\big[\mu(\sigma) + |x'(\sigma)|\big]\,\mathrm{d}\sigma + |x'(s)|\right)\mathrm{d}s\,,$$
$$t \in I\,.$$

If $t^* \in [-r,0]$ then $\mu(t) = \|\phi\|$ and the previous inequality holds since $M \geq 1$.

Denoting by $u(t)$ the right-hand side of the above inequality we have

$$u(0) = M\|\phi\| + Mb|\eta|\,, \qquad \mu(t) \leq u(t)\,, \quad t \in I\,,$$

and

$$\begin{aligned} u'(t) &= Mbp(t)\Omega_1\left(\mu(t) + |x'(t)| + L\int_0^t m(\sigma)\big[\mu(\sigma) + |x'(\sigma)|\big]\,\mathrm{d}\sigma\right) \\ &\leq Mbp(t)\Omega_1\left(u(t) + |x'(t)| + L\int_0^t m(\sigma)\big[u(\sigma) + |x'(\sigma)|\big]\,\mathrm{d}\sigma\right), \qquad t \in I\,. \end{aligned}$$

We set

$$v(t) = \sup\big\{|x'(s)| :\ s \in [0,t]\big\}\,, \qquad t \in I\,.$$

Thus, as in the previous theorem, we obtain

$$\begin{aligned} |x'(t)| &\leq v(t) \\ &\leq M'\|\phi\| + M|\eta| \\ &\quad + M\int_0^t p(s)\Omega_1\left(u(s) + v(s) + L\int_0^s m(\sigma)\big[u(\sigma) + v(\sigma)\big]\,\mathrm{d}\sigma\right)\mathrm{d}s\,, \\ &\qquad t \in I\,. \end{aligned}$$

Denoting by $r(t)$ the right-hand side in the above inequality we have

$$r(0) = M'\|\phi\| + M|\eta|\,, \qquad v(t) \leq r(t)\,, \quad t \in I\,,$$

and

$$\begin{aligned} r'(t) &= Mp(t)\Omega_1\left(u(t) + v(t) + L\int_0^t m(\sigma)\big[u(\sigma) + v(\sigma)\big]\,\mathrm{d}\sigma\right) \\ &\leq Mp(s)\Omega_1\left(u(t) + r(t) + L\int_0^t m(\sigma)\big[u(\sigma) + r(\sigma)\big]\,\mathrm{d}\sigma\right), \qquad t \in I\,. \end{aligned}$$

Let

$$w(t) = u(t) + r(t) + L\int_0^t m(\sigma)\big[u(\sigma) + r(\sigma)\big]\,\mathrm{d}\sigma\,, \qquad t \in I\,.$$

Then

$$w(0) = u(0) + r(0) \equiv c\,,$$

and

$$\begin{aligned} w'(t) &= u'(t) + r'(t) + Lm(t)\big[u(t) + r(t)\big] \\ &\le Mbp(t)\Omega_1\big(w(t)\big) + Mp(t)\Omega_1\big(w(t)\big) + Lm(t)w(t) \\ &\le \widehat{p}(t)\big[w(t) + 2\Omega_1\big(w(t)\big)\big]\,, \qquad t \in I\,. \end{aligned}$$

This implies

$$\int\limits_{w(0)}^{w(t)} \frac{\mathrm{d}s}{s + 2\Omega_1(s)} \le \int\limits_0^b \widehat{p}(s)\,\mathrm{d}s < \int\limits_c^\infty \frac{\mathrm{d}s}{s + 2\Omega_1(s)}\,, \qquad t \in I\,.$$

This inequality implies that there is a constant K such that

$$w(t) \le K\,, \qquad t \in I\,.$$

Then

$$\begin{aligned} |x(t)| &\le \mu(t) \le u(t) \le w(t) \le K\,, \qquad t \in I\,, \\ |x'(t)| &\le v(t) \le r(t) \le w(t) \le K\,, \qquad t \in I\,, \end{aligned}$$

and hence

$$\|x\|^* \le K\,.$$

Therefore the desired a priori bounds are obtained.

The rest of the proof is similar to that of Theorem 2.2. □

By combining Theorems 2.2 and 2.3 we obtain the following more general result, which contains the above Theorems as special cases.

THEOREM 2.4. *Let $f\colon I \times C \times X \times X \to X$ be a function satisfying* (C_1), (C_2) *and* ($Hf - 1$), *$g\colon I \times C \times X \to X$ be a continuous function satisfying* (Hg), *$k\colon I \times I \to \mathbb{R}$ a measurable function satisfying* (Hk) *and $C(t)$ (resp. $S(t)$), $t \in I$ be a strongly continuous cosine (resp. sine) family on X with the infinitesimal generator A. Assume also that* (HC) *holds.*

Then if

$$\int\limits_0^b \widehat{m}(s)\,\mathrm{d}s < \int\limits_c^\infty \frac{\mathrm{d}s}{\Omega(s) + 2\Omega_1(s)}$$

with $\widehat{m}$ and c as defined in Theorem 2.2, *the IVP* (1.1)–(1.2) *has at least one mild solution on* $[-r, b]$.

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