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EXISTENCE OF POSITIVE SOLUTIONS TO VECTOR BOUNDARY VALUE PROBLEMS II

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ABSTRACT. We show that the question about the existence of a positive solution to certain n-dimensional differential system of second order with Dirichlet boundary condition can be answered by multiple (step-by-step) solving of differential equations of the first order.

1. Introduction

In [2] M. Fečkan has dealt with the existence of a solution of the problem:

$$\begin{aligned} -u'' &= \big(f_a(x) + g(u)\big) \cdot u - s(u) \cdot v\,, \\ -v'' &= \big(a + r(u)\big) \cdot v - v^2\,, \\ u(0) &= u(\pi) = v(0) = v(\pi) = 0\,, \\ u(x) > 0\,, &\quad v(x) > 0 \qquad \text{for all} \quad x \in (0,\pi)\,, \end{aligned} \tag{1.0.1}$$

where the functions f, g, r, s fulfil the following conditions:

$$\begin{aligned} &f_{(\cdot)}(\cdot) \in C^1(\mathbb{R}\times\mathbb{R},\mathbb{R})\,, \qquad g,s,r \in C^1(\mathbb{R},\mathbb{R})\,, \\ &\frac{\partial}{\partial a} f_a(\cdot) > 0\,, \qquad f_a(\cdot) \geq 2\,, \\ &g(0) = g'(0) = 0\,, \qquad g'(u) < 0 \quad \text{for } u > 0\,, \\ &r(0) = r'(0) = 0\,, \qquad s(0) = s'(0) = 0\,, \\ &r/\langle 0,\infty) \leq 1\,, \qquad r'/(0,\infty) > 0\,, \qquad s/(0,\infty) \geq 0\,, \\ &\lim g = -\infty \qquad \text{for} \quad x \to \infty\,. \end{aligned}$$

Using the bifurcation method he found a necessary and sufficient condition for the parameter a that problem (1.0.1) may have at least one positive solution u, v.

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This is the second part of the paper, in which we investigate the existence of a positive solution depending on definition intervals which are determined by the Dirichlet boundary conditions for single components of the solution. We consider the second-order n-dimensional vector differential system, $n \geq 2$ (see (3.0.1)). Attention to similar problems has been paid in papers [7], [4] where solutions in a cone have been studied. Another problems with similar formulation or with similar method of solution (degree theory) were studied in papers [3], [1]. In this whole paper the question about the existence of solution to n-dimensional differential system can be answered by multiple (step-by-step) solving of differential equations of the first order. This can be considered as the contribution of this paper. In the first part [8] some auxiliary lemmas were stated which will be proved in this part of the paper. Using them Theorem 6.1 (in [8]) was proved which gives a sufficient condition for definition intervals that guarantee the existence of a positive solution to problem (3.0.1). In this part of the paper we shall also introduce and prove the second main result – Theorem 5.3 which gives a necessary condition on definition intervals for the existence of a positive solution to problem (3.0.1) under some assumptions on the form of the right sides of that problem. The last main result in this part is Theorem 6.1 which gives simple conditions on the right sides of problem (3.0.1). This result gives a necessary and sufficient condition for the existence of a positive solution to our problem.

2. Auxiliary lemmas

In this section auxiliary lemmas are stated and proved which were necessary for previous part of this work [8].

LEMMA 2.1. *Let the function $u \in C^1_{\mathrm{Abs}}\langle a, b\rangle$ fulfil the following conditions:*

(1)
$$u(x) \geq 0 \qquad \textit{for all} \quad x \in \langle a, b\rangle\,. \tag{2.1.1}$$

(2)
$$u(a) = 0\,, \qquad u'(a) = 0\,. \tag{2.1.2}$$

(3) *Let for almost every $x \in \langle a, b\rangle$ the inequality*
$$u''(x) \leq M \cdot \big(|u'(x)| + u(x)\big) \tag{2.1.3}$$
hold, where $M \geq 0$ is a suitable fixed constant.

Then the following assertion is true:
$$u \equiv 0 \qquad \textit{on the interval} \quad \langle a, b\rangle\,. \tag{2.1.4}$$

Proof. By contradiction. Hence, let $y \in (a, b\rangle$ exist such that $u(y) > 0$. Let x_0 be the first zero point of function u on the left to y. Let ε be chosen so small, that the following condition holds

$$\begin{gathered} a \le x_0 < x_0 + \varepsilon \le y \le b\,, \\ u(x_0) = 0 = u'(x_0)\,, \\ \forall\, x \in (x_0, x_0 + \varepsilon\rangle \qquad u(x) > 0\,. \end{gathered} \tag{2.1.5}$$

Let us now define auxiliary function g on the interval $(x_0, x_0 + \varepsilon\rangle$

$$g(x) \overset{\text{def}}{=} \max_{t \in \langle x_0, x\rangle} \big(u'(t)\big)\,. \tag{2.1.6}$$

The following assertion is evident:

(2.1.7) If we put $g(x_0) = 0$, then function g is continuous and nondecreasing.

Now we shall also try to prove absolute continuity of g. Let us choose $\eta > 0$ arbitrary small. Using the fact that $u'(\cdot)$ is absolute continuous, we can choose $\delta > 0$ such that it holds:

(2.1.8) For arbitrary choice of δ-partition $x_0 \le c_1 \le d_1 \le c_2 \cdots \le c_n \le d_n \le x_0 + \varepsilon$ such that $\sum\limits_{i=1}^{n} (d_i - c_i) < \delta$ we have $\sum\limits_{i=1}^{n} |u'(d_i) - u'(c_i)| < \eta$.

Let us verify that this δ fulfils the condition analogous to (2.1.8) where $u'(\cdot)$ is replaced by function $g(\cdot)$. Let us choose fixed partition $(c_1, d_1, \dots, c_n, d_n)$ and $i \in \{1, 2, \dots, n\}$. According to (2.1.7) only two possibilities can now be true:

(1) $g(d_i) > g(c_i)$.
(2) $g(d_i) = g(c_i)$.

In both cases we can find c_i^1, d_i^1 such that

$$\begin{gathered} c_i \le c_i^1 \le d_i^1 \le d_i\,, \\ |g(d_i) - g(c_i)| \le |u'(d_i^1) - u'(c_i^1)|\,. \end{gathered} \tag{2.1.9}$$

In fact, let us verify:

(1) If $g(d_i) > g(c_i)$, then from nondecreaseness of g it follows $d_i > c_i$. Let us put $c_i^1 = \max\{t\,;\ t \in \langle c_i, d_i\rangle\,,\ g(t) = g(c_i)\}$.

If we at first suppose that $u'(c_i^1) < g(c_i^1)$, then from continuity of $u'(\cdot)$ it would follow for sufficiently small $\varepsilon_1 > 0$ that $u'(t) < g(c_i^1) = g(c_i)$ for all t from interval $\langle c_i^1, c_i^1 + \varepsilon_1)$, what would contradict our definition of c_i^1.

So, the case $u'(c_i^1) \ge g(c_i^1) = g(c_i)$ must be true, from what with (2.1.6) we get

$$u'(c_i^1) = g(c_i) = g(c_i^1)\,. \tag{2.1.10}$$

Because we assume $g(d_i) > g(c_i)$, according to (2.1.6), $u'(\cdot)$ must reach, on the interval $(c_i, d_i\rangle$, the value $g(d_i)$ at some point d_i^1. Then the following estimations must hold $u'(d_i^1) = g(d_i) \overset{(2.1.7)}{\geq} g(d_i^1) \overset{(2.1.6)}{\geq} u'(d_i^1)$ from what we get

$$u'(d_i^1) = g(d_i) = g(d_i^1)\,. \tag{2.1.11}$$

So, inequality (2.1.9) follows from (2.1.10), (2.1.11) if we are assuming $g(d_i) > g(c_i)$ and nondecreaseness of g.

(2) If $g(d_i) = g(c_i)$, then choosing $c_i^1 = c_i$ and $d_i^1 = d_i$ we see that (2.1.9) is fulfilled.

From (2.1.9) for $i = 1, 2, \dots, n$, when we realize that $(c_1^1, d_1^1, \dots, c_n^1, d_n^1)$ is a δ-partition in (2.1.8), we obtain

$$\sum_{i=1}^{n} |g(d_i) - g(c_i)| \leq \sum_{i=1}^{n} |u'(d_i^1) - u'(c_i^1)| < \eta \tag{2.1.12}$$

and by it the absolute continuity of g is verified.

Therefore $g'(x)$, $u''(x)$ exist almost everywhere in the interval $\langle x_0, x_0 + \varepsilon)$. Now we can define

$$f(x) \overset{\text{def}}{=} \begin{cases} g'(x) & \text{for all } x \in \langle x_0, x_0 + \varepsilon) \text{ such that} \\ & g'(x)\,,\ u''(x) \text{ exist and (2.1.3) holds,} \\ 0 & \text{for remaining } x \in \langle x_0, x_0 + \varepsilon\rangle\,. \end{cases} \tag{2.1.13}$$

In particular we can write $f(x) = g'(x)$ almost everywhere, and so from (2.1.7) and (2.1.13) it follows that $f(x) \geq 0$. Now we shall prove that on the interval $\langle x_0, x_0 + \varepsilon\rangle$ it holds that

$$f(x_1) \leq M \cdot (1 + \varepsilon) \cdot g(x_1)\,. \tag{2.1.14}$$

So, let x_1 be arbitrary but fixed. Two following cases are now possible

(1) $f(x_1) = 0$, then (2.1.14) holds evidently.

(2) $f(x_1) > 0$, then from (2.1.13), x_1 simultaneously fulfils

$$u''(x_1) \leq M \cdot \big(|u'(x_1)| + u(x_1)\big) \qquad \text{and} \qquad g'(x_1) = f(x_1) > 0\,. \tag{2.1.15}$$

(a) Let us now exclude the case $u'(x_1) < g(x_1)$. By contradiction: If it were true, then from continuity of $u'(\cdot)$ we would obtain that $u'(x) < g(x_1)$ is true in some neighbourhood of x_1, exactly for all $x \in \langle x_1, x_1 + \varepsilon_2)$ and then (2.1.6) would imply that on the same interval $g(\cdot) \equiv g(x_1)$, what implies the contradiction $g'(x_1) = 0$.

(b) So, (2.1.6) implies

$$u'(x_1) = g(x_1) \geq 0\,. \tag{2.1.16}$$

Let $x \in (x_1, x_0 + \varepsilon)$ be chosen arbitrarily. From definition of $g(\cdot)$ in (2.1.6) we know that $t_x \leq x$ exists such that $u'(t_x) = g(x)$. Let us exclude the case $t_x \leq x_1$. If it were true, we could obtain $g(x) = u'(t_x) \overset{(2.1.6)}{\leq} g(t_x) \overset{(2.1.7)}{\leq} g(x_1) \overset{(2.1.7)}{\leq} g(x)$, what implies $g(x_1) = g(x)$ and then from (2.1.7) we obtain that $g(\cdot)$ is constant on the interval $\langle x_1, x\rangle$, from what it follows that $g'(x_1) = 0$, and by (2.1.15) we obtain the contradiction.

So, $x_1 < t_x \leq x$ holds, and we can write

$$\frac{g(x) - g(x_1)}{x - x_1} = \frac{u'(t_x) - g(x_1)}{x - x_1} \overset{(2.1.16)}{=} \frac{u'(t_x) - u'(x_1)}{x - x_1} \leq \frac{u'(t_x) - u'(x_1)}{t_x - x_1}\,.$$

If $x \to x_1^+$, then evidently $t_x \to x_1^+$ and so $g'(x_1) \leq u''(x_1)$. From this we obtain

$$\begin{aligned} f(x_1) &\overset{(2.1.15)}{=} g'(x_1) \leq u''(x_1) \overset{(2.1.15)}{\leq} M\cdot\big(|u'(x_1)| + u(x_1)\big)\\ &\overset{(2.1.16)}{=} M\cdot\big(g(x_1) + u(x_1)\big) \overset{(2.1.5)}{=} M\cdot\left(g(x_1) + \int_{x_0}^{x_1} u'(t)\ \mathrm{d}t\right)\\ &\overset{(2.1.6)}{\leq} M\cdot\left(g(x_1) + \int_{x_0}^{x_1} g(t)\ \mathrm{d}t\right) \overset{(2.1.7)}{\leq} M\cdot\big(g(x_1) + (x_1 - x_0)\cdot g(x_1)\big)\\ &\leq\ M\cdot(1+\varepsilon)\cdot g(x_1) \end{aligned}$$

and so we verified (2.1.14).

From absolute continuity of $g(\cdot)$ and from (2.1.14) it follows that for almost every $x \in \langle x_0, x_0 + \varepsilon)$

$$\Big(g(x)\cdot e^{-M\cdot(1+\varepsilon)\cdot(x-x_0)}\Big)' = e^{-M\cdot(1+\varepsilon)\cdot(x-x_0)}\cdot\Big(g'(x) - M\cdot(1+\varepsilon)\cdot g(x)\Big) \leq 0$$

where we have also used (2.1.13). By integration from x_0 to x for x from the above interval we obtain

$$g(x)\cdot e^{-M\cdot(1+\varepsilon)\cdot(x-x_0)} \leq g(x_0) \overset{(2.1.7)}{=} 0\,.$$

Because $g(\cdot)$ is non-negative, we obtain $g(\cdot) \equiv 0$ on interval $\langle x_0, x_0 + \varepsilon)$. This together with (2.1.6) implies $u(x) - u(x_0) = \int_{x_0}^{x} u'(t)\ \mathrm{d}t \leq 0$. And finally

$$0 \overset{(2.1.1)}{\leq} u(x) \leq u(x_0) \overset{(2.1.5)}{=} 0\,,$$

what implies $u(\cdot) \equiv 0$ on interval $\langle x_0, x_0 + \varepsilon)$ in contradiction with (2.1.5). □

LEMMA 2.2. *Let the function $u \in C^1_{\mathrm{Abs}}\langle a,b\rangle$ fulfil the following assumptions:*

(1)
$$u(x) \geq 0 \qquad \text{for all} \quad x \in \langle a,b\rangle\,. \tag{2.2.1}$$

(2)
$$\exists\, x_1 \in \langle a,b\rangle \qquad u(x_1) = 0 \ \ \&\ \ u'(x_1) = 0\,. \tag{2.2.2}$$

(3) *Let for almost every $x \in \langle a,b\rangle$ (in the meaning of the Lebesgue measure), the inequality*
$$u''(x) \leq M \cdot \big(|u'(x)| + u(x)\big) \tag{2.2.3}$$
holds, where $M \geq 0$ is a fixed constant.

Then the following identity holds:
$$u \equiv 0 \qquad \text{on} \quad \langle a,b\rangle\,. \tag{2.2.4}$$

P r o o f . It can be done by an analogous method to that used in the proof of Lemma 2.1. □

LEMMA 2.3. *Let the functions $f(x,u_1,u_2)$, $g(x,v_1,v_2)$ satisfy locally Carathéodory's conditions on the set $\big(\langle a,b\rangle \times \mathbb{R}_0^+ \times \mathbb{R}\big)$ and the conditions:*

(1)
$$f(x,0,0) \equiv 0 \qquad \text{for all} \quad x \in \langle a,b\rangle\,. \tag{2.3.1}$$

(2)
$$\begin{gathered} f(x,\alpha\cdot u_1,\alpha\cdot u_2) \geq \alpha\cdot f(x,u_1,u_2) \\ \text{for all} \quad (x,u_1,u_2) \in \big(\langle a,b\rangle \times \mathbb{R}_0^+ \times \mathbb{R}\big) \qquad \text{and for all} \quad \alpha \geq 1\,. \end{gathered} \tag{2.3.2}$$

(3) *The function f satisfies locally Lipschitz's condition*
$$|f(x,u_1,u_2) - f(x,v_1,v_2)| \leq L_{\mathrm{loc}} \cdot \big(|u_1 - v_1| + |u_2 - v_2|\big)\,. \tag{2.3.3}$$

(4)
$$\begin{gathered} g(x,u_1,u_2) \geq f(x,u_1,u_2) \\ \text{for all} \quad (x,u_1,u_2) \in \big(\langle a,b\rangle \times \mathbb{R}_0^+ \times \mathbb{R}\big)\,. \end{gathered} \tag{2.3.4}$$

Let the functions $u(\cdot), v(\cdot) \in AC^1\big(\langle a,b\rangle, \mathbb{R}_0^+\big)$ be solutions of the equations
$$\begin{aligned} u''(x) &= f\big(x,u(x),u'(x)\big) \\ v''(x) &= g\big(x,v(x),v'(x)\big) \end{aligned} \qquad \text{for almost all} \quad x \in \langle a,b\rangle \tag{2.3.5}$$
which satisfy
$$u(x) > 0 \qquad \text{for all} \quad x \in (a,b)\,, \tag{2.3.6}$$
$$v(a) \leq u(a)\,, \qquad v(b) \leq u(b)\,. \tag{2.3.7}$$

Then at least one of two assertions is true:

$$\textit{Simultaneously} \begin{cases} v(x) < u(x) & \textit{for all} \quad x \in (a,b)\,, \\ \big(|v(a)-u(a)| + |v'(a)-u'(a)|\big) > 0\,, \\ \big(|v(b)-u(b)| + |v'(b)-u'(b)|\big) > 0\,. \end{cases} \tag{2.3.8}$$

$$\exists\, \alpha \geq 1 \qquad v(x) = \alpha \cdot u(x) \quad \textit{for all } x \in \langle a,b\rangle\,. \tag{2.3.9}$$

P r o o f . If case (2.3.8) is false, then we shall show that case (2.3.9) must be true. It shall be useful for us to define the following auxiliary functions φ_u and φ_v on the interval (a,b).

$$\begin{aligned} \varphi_u(x) &\stackrel{\text{def}}{=} \frac{u(x)}{\left(s_a + (1-s_a)\cdot\frac{x-a}{b-a}\right)\cdot\left(s_b + (1-s_b)\cdot\frac{b-x}{b-a}\right)} \\ \varphi_v(x) &\stackrel{\text{def}}{=} \frac{v(x)}{\left(s_a + (1-s_a)\cdot\frac{x-a}{b-a}\right)\cdot\left(s_b + (1-s_b)\cdot\frac{b-x}{b-a}\right)} \end{aligned} \tag{2.3.10}$$

where

$$s_a = \begin{cases} 0 & \text{if } u(a) = 0\,, \\ 1 & \text{if } u(a) > 0\,, \end{cases} \qquad \text{and} \qquad s_b = \begin{cases} 0 & \text{if } u(b) = 0\,, \\ 1 & \text{if } u(b) > 0\,. \end{cases}$$

Let us define the values of functions φ_u, φ_v also at points a, b so that these functions be continuous at boundary points a, b. (The existence of boundary limits in these cases, when at least one number from s_a, s_b is equal to zero, follows from inequalities (2.3.7).)

From assumption $u > 0$ on (a,b), it follows that also $\varphi_u > 0$ on (a,b). If the case $\varphi_u(a) = 0$ were true then from the definition it would follow $u(a) = 0$, $u'(a) = 0$ what according to (2.3.1) and (2.3.3) implies that u is identically equal to zero and this would be a contradiction with (2.3.6). Therefore $\varphi_u(a) > 0$ and similarly $\varphi_u(b) > 0$ must be true. From continuity of φ_u there exists a suitable small $\varepsilon > 0$ such that:

$$\forall\, x \in \langle a,b\rangle \qquad \varphi_u(x) \geq \varepsilon > 0\,. \tag{2.3.11}$$

Since (2.3.8) is supposed not to be true, by (2.3.10) it easily follows

$$\min_{x\in\langle a,b\rangle} \big(\varphi_u(x) - \varphi_v(x)\big) \leq 0\,. \tag{2.3.12}$$

For $\alpha \geq 1$ let us define the function

$$\psi(\alpha) \stackrel{\text{def}}{=} \min_{x\in\langle a,b\rangle} \big(\alpha\cdot\varphi_u(x) - \varphi_v(x)\big)\,. \tag{2.3.13}$$

According to (2.3.11) function ψ is increasing, continuous and for a suitable great α also positive. From (2.3.12) it follows that $\psi(1) \leq 0$ and therefore there exists a unique number $\alpha_0 \geq 1$ (which we shall denote in the sequel only by α) such that:

$$\psi(\alpha) = 0\,. \tag{2.3.14}$$

Let us consider two possible cases:

$$\begin{aligned} &\exists\, x_0 \in (a,b) \qquad \alpha \cdot \varphi_u(x_0) - \varphi_v(x_0) = 0\,. \\ &\forall\, x \in (a,b) \qquad \alpha \cdot \varphi_u(x) - \varphi_v(x) > 0\,. \end{aligned} \tag{2.3.15}$$

Now we shall show that on the interval $\langle a,b\rangle$ there exists a point x_1 with the following property:

$$\alpha \cdot u(x_1) - v(x_1) = 0 \qquad \text{and} \qquad \alpha \cdot u'(x_1) - v'(x_1) = 0\,. \tag{2.3.16}$$

If the former case is true, then by (2.3.10), (2.3.13) and (2.3.14), statement (2.3.16) follows. So let us assume that the latter case is true. Then according to (2.3.14) without loss of generality we can assume that $\alpha \cdot \varphi_u(a) - \varphi_v(a) = 0$ (because in the other case the same would be true at the point b). If we now consider (2.3.10), then we obtain:

(1) If $s_a = 0$ is true, then $u(a) = v(a) = \alpha \cdot u'(a) - v'(a) = 0$.
(2) If $s_a = 1$ is true, then $\alpha \cdot u(a) = v(a) > 0$.

In the first case we obtain the validity of (2.3.16) for $x_1 = a$. Only the second case is remaining:

$$0 < u(a) \overset{1\leq\alpha}{\leq} \alpha \cdot u(a) = v(a) \overset{(2.3.7)}{\leq} u(a)\,.$$

This implies $\alpha = 1$, and therefore (2.3.15) is transformed to the statement for all $x \in (a,b)$, $u(x) > v(x)$ is true. Because we assume that (2.3.8) is not valid, at least at one of the points a or b (without lose of generality let it be a) it must hold: $u(a) - v(a) = u'(a) - v'(a) = 0$, what immediately implies (2.3.16), because $\alpha = 1$.

Now by (2.3.16) and (2.3.14), if we define

$$w(x) \overset{\text{def}}{=} \alpha \cdot u(x) - v(x)\,, \tag{2.3.17}$$

we obtain that

$$w(x_1) = w'(x_1) = 0\,, \qquad w(x) \geq 0 \quad \text{on } \langle a,b\rangle\,. \tag{2.3.18}$$

Let us verify the remaining assumption of Lemma 2.2. Almost everywhere it holds:

$$\begin{aligned} w''(x) &\overset{(2.3.17)}{=} \alpha \cdot u''(x) - v''(x) \\ &\overset{(2.3.5)}{=} \alpha \cdot f\big(x, u(x), u'(x)\big) - g\big(x, v(x), v'(x)\big) \\ &\overset{(2.3.2),(2.3.4)}{\leq} f\big(x, \alpha \cdot u(x), \alpha \cdot u'(x)\big) - f\big(x, v(x), v'(x)\big) \\ &\leq L_K\big(|\alpha u(x) - v(x)| + |\alpha u'(x) - v'(x)|\big) \\ &\overset{(2.3.17)}{=} L_K\big(|w(x)| + |w'(x)|\big)\,. \end{aligned}$$

In the last inequality we have used that by (2.3.3) there exists a Lipschitz constant on the compact set $K = \langle a,b\rangle \times \langle 0,M\rangle \times \langle -M,M\rangle$ where $M = \max\limits_{x\in\langle a,b\rangle} \big\{|\alpha \cdot u(x)|,\, |\alpha \cdot u'(x)|,\, |v(x)|, |v'(x)|\big\}$. By this, together with (2.3.18), we have verified all assumptions of Lemma 2.2, from which it follows that $w(\cdot) \equiv 0$ on $\langle a,b\rangle$ what implies validity of (2.3.9). □

LEMMA 2.4. *Let the functions $f(x,u)$, $g(x,v)$ satisfy locally Carathéodory's conditions on the set $\big(\langle 0,a\rangle \times \mathbb{R}_0^+\big)$ and all assumptions* (2.3.1), (2.3.2), (2.3.3) *and* (2.3.4) *from Lemma* 2.3, *where f, g do not depend on arguments u_2, v_2. Let now $v(\cdot) \in AC^1\langle 0,a\rangle$ be a solution of the equation*

$$\begin{gathered} v''(x) = g\big(x, v(x)\big)\,, \\ v(x) > 0 \ \text{for all}\ x \in (0,a)\,, \quad v(0)=0\,, \quad v'(0)>0\,, \quad v(a)=0\,. \end{gathered} \tag{2.4.1}$$

Then the solution $u(\cdot)$ of the equation

$$u''(x) = f\big(x,u(x)\big)\,, \qquad u(0)=0\,, \quad u'(0)=v'(0) \tag{2.4.2}$$

has a further zero in the interval $(0,a\rangle$.

Proof. By contradiction. If the assertion were false, then u should be in the interval $(0,a\rangle$ positive, what together with (2.4.1) and (2.4.2) implies that assumptions of Lemma 2.3 are fulfilled. When we use it, we obtain that at least one assertion of (2.3.8), (2.3.9) must be true. The first assertion cannot be true, because by (2.4.1), (2.4.2) it follows that

$$\big(|u(0)-v(0)| + |u'(0)-v'(0)|\big) = 0\,.$$

Also the second one cannot be true, because it is in contradiction with our assumption:

$$u(a) > 0 \overset{(2.4.1)}{=} v(a)\,.$$

Hence, proof is done. □

3. Preliminaries

Throughout the paper we shall use the following notations

(1) $E_n \overset{\text{def}}{=} \underbrace{(0,\infty) \times (0,\infty) \times \cdots \times (0,\infty)}_{n \text{ times}}$.

(2) E_n^* is defined as the compactification of topological space E_n by adding point ∞ and defining its base of neighbourhoods $O_k \overset{\text{def}}{=} \{\vec{x} \in E_n\,;\ \|\vec{x}\| > k\} \cup \{\infty\}$.

(3) $E_{n,o} \overset{\text{def}}{=} \{\vec{x} \in E_n\,;\ x_i = 0 \text{ for some } i \in \{1,2,\dots,n\}\}$.

(4) $E_{n,+} \overset{\text{def}}{=} E_n \setminus E_{n,o}$.

(5) $E^*_{n,o} \overset{\text{def}}{=} E_{n,o} \cup \{\infty\}$.

(6) $E^*_{n,+} \overset{\text{def}}{=} E_{n,+} \cup \{\infty\}$.

In the paper we will study the problem

$$\vec{u}\,''(x) \overset{\text{a.e.}}{=} \vec{F}\big(x, \vec{u}(x)\big) \overset{\text{def}}{\iff} \begin{cases} u_1''(x) \overset{\text{a.e.}}{=} F_1\big(x, u_1(x), u_2(x), \dots, u_n(x)\big)\,, \\ u_2''(x) \overset{\text{a.e.}}{=} F_2\big(x, u_1(x), u_2(x), \dots, u_n(x)\big)\,, \\ \quad\vdots \\ u_n''(x) \overset{\text{a.e.}}{=} F_n\big(x, u_1(x), u_2(x), \dots, u_n(x)\big)\,, \end{cases} \tag{3.0.1}$$

with the boundary conditions

$$\begin{matrix} u_i(0) = u_i(T_i) = 0 \\ \forall\, x \in (0, T_i) \qquad u_i(x) > 0 \end{matrix} \qquad \text{for } i = 1, 2, \dots, n\,.$$

In the sequel we shall assume some of assumptions:

$$\begin{aligned} &\forall\, k \in \{1,2,\dots,n\} \quad \forall\, x \in \mathbb{R}_0^+ \quad \forall\, u_i \in \mathbb{R} \\ &\qquad F_k(x, u_1, \dots, u_{k-1}, 0, u_{k+1}, \dots, u_n) = 0\,. \end{aligned} \tag{3.0.2}$$

$$\begin{aligned} &\forall\, k \in \{1,2,\dots,n\} \quad \forall\, x \in \mathbb{R}_0^+ \quad \forall\, u_i \in \mathbb{R} \\ &\qquad F_k(x, u_1, \dots, u_n) = F_k\Big(x, \tfrac{u_1 + |u_1|}{2}, \tfrac{u_2 + |u_2|}{2}, \dots, \tfrac{u_n + |u_n|}{2}\Big) \end{aligned} \tag{3.0.3}$$

$$\begin{aligned} &\forall\, k \in \{1,2,\dots,n-2\} \quad \forall\, x \in \mathbb{R}_0^+ \quad \forall\, u_i \in \mathbb{R},\ u_{k+1} \cdot u_{k+2} \cdots u_n = 0 \\ &\qquad F_k(x, u_1, u_2, \dots, u_n) = F_k(x, u_1, \dots, u_k, \underbrace{0, 0, \dots, 0}_{n-k \text{ times}})\,. \end{aligned} \tag{3.0.4}$$

$$\begin{aligned} &\forall\, k \in \{1,2,\dots,n-1\} \quad \forall\, x \in \mathbb{R}_0^+ \quad \forall\, u_i \in \mathbb{R} \\ &\qquad \frac{\partial F_k}{\partial u_k^+}(x, u_1, \dots, u_{k-1}, 0, u_{k+1}, \dots, u_n) = \frac{\partial F_k}{\partial u_k^+}(x, u_1, \dots, u_{k-1}, 0, \underbrace{0, \dots, 0}_{n-k \text{ times}})\,. \end{aligned} \tag{3.0.5}$$

(3.0.6) The functions $F_k(x, u_1, u_2, \dots, u_n)$ and $\frac{\partial F_k}{\partial u_i^+}(x, u_1, u_2, \dots, u_n)$ are continuous in $(u_1, u_2, \dots, u_n)$ on the set E_n for any fixed $x \in \mathbb{R}_0$, for all $k, i \in \{1, 2, \dots, n\}$ and $F_k(x, u_1, u_2, \dots, u_n)$ are measurable in $x \in \langle 0, \infty)$ for each fixed $(u_1, \dots, u_n) \in E_n$ and for all $k \in \{1, 2, \dots, n\}$.

(3.0.7) $\frac{\partial F_k}{\partial u_i^+}(x, u_1, u_2, \dots, u_n)$ is locally bounded on the set $\mathbb{R}_0^+ \times E_n$ for all $k, i \in \{1, 2, \dots, n\}$.

(3.0.8) For all $T > 0$ there exist continuous functions $c_1(\lambda), \dots, c_n(\lambda)$; $c_i(\cdot)\colon \langle 0,\infty) \to (0,\infty)$ such that $\lim\limits_{\lambda\to\infty} c_i(\lambda) = \infty$ for all $i \in \{1,2,\dots,n\}$ and
$\forall k \in \{1,2,\dots,n\}$ $\forall x \in \langle 0,T\rangle$ $\forall \lambda \ge 0$
$\forall \vec{u} \in \{\vec{u};\ u_k = c_k(\lambda)$ and $0 \le u_i \le c_i(\lambda)$ for all $i \in \{1,\dots,n\}$, $i \ne k\}$

$$F_k(x,u_1,u_2,\dots,u_n) \ge 0\,.$$

$\forall k \in \{1,2,\dots,n-1\}$ $\forall x \in \mathbb{R}_0^+$ $\forall u_i \in \mathbb{R}_0^+$ $\forall u_k > 0$

$$\frac{\partial F_k}{\partial u_k^+}(x,u_1,\dots,u_k,\underbrace{0,\dots,0}_{n-k\ \text{times}}) > \frac{1}{u_k} F_k(x,u_1,\dots,u_k,\underbrace{0,\dots,0}_{n-k\ \text{times}})\,. \tag{3.0.9}$$

$\forall x \in \mathbb{R}_0^+$ $\forall u_1,u_2,\dots,u_{n-1} \in \mathbb{R}_0^+$ $\forall u_n > 0$

$$\frac{\partial F_n}{\partial u_n^+}(x,u_1,u_2,\dots,u_{n-1},0) < \frac{1}{u_n} F_n(x,u_1,\dots,u_{n-1},u_n)\,. \tag{3.0.10}$$

$\forall k,\ 2 \le k \le n-1$ $\forall i,\ 1 \le i \le k-1$ $\forall u_1,u_2,\dots,u_k \in \mathbb{R}_0^+$ $\forall x \in \mathbb{R}_0^+$

$$\frac{\partial F_k}{\partial u_i^+}(x,u_1,u_2,\dots,u_k,\underbrace{0,0,\dots,0}_{n-k\ \text{times}}) \le 0\,. \tag{3.0.11}$$

(3.0.12) $\frac{\partial F_n}{\partial u_n^+}(x,u_1,\dots,u_{i-1},u,u_{i+1},\dots,u_{n-1},0)$ is nonincreasing in u in the interval $\langle 0,\infty)$ for all i, $1 \le i \le n-1$, for all $u_1,\dots,u_{i-1},u_{i+1},\dots,u_{n-1} \in \mathbb{R}_0^+$, and for all $x \in \mathbb{R}_0^+$.

$\forall k,\ 1 \le k \le n-1$ $\forall x \in \mathbb{R}_0^+$ $\forall u_1,u_2,\dots,u_n \in \mathbb{R}_0^+$

$$F_k(x,u_1,\dots,u_k,u_{k+1},\dots,u_n) \ge F_k(x,u_1,\dots,u_k,\underbrace{0,0,\dots,0}_{n-k\ \text{times}})\,. \tag{3.0.13}$$

We will study the question when problem (3.0.1) has at least one positive solution. We shall apply the shooting method and therefore the following definition of the mapping $\vec{T}(\vec{\alpha})$ will be of use.

DEFINITION 3.1. Let $\vec{\alpha} \overset{\text{def}}{=} (\alpha_1,\dots,\alpha_n) \in E_{n,+}$. Let $\vec{u}$ be the solution of the following problem

$$\begin{gathered} \vec{u}\,''(x) \overset{\text{a.e.}}{=} \vec{F}\big(x,\vec{u}(x)\big)\,, \\ \vec{u}(0) = \vec{0}\,, \qquad \vec{u}\,'(0) = \vec{\alpha}\,. \end{gathered} \tag{3.1.1}$$

If for each component u_i, $i = 1,2,\dots,n$, of solution $\vec{u}$ there exists a point T_i such that

$$0 < T_i < \infty\,, \qquad u_i(T_i) = 0\,, \qquad u_i(x) > 0 \quad \text{for all } x\,,\ 0 < x < T_i\,,$$

then we define $\vec{T}(\vec{\alpha}) \overset{\text{def}}{=} (T_1, T_2, \dots, T_n)$.

In the case that at least one component $u_i(\cdot)$ is positive on the whole interval $(0, T_{\max}(\vec{\alpha}))$, where $\langle 0, T_{\max}(\vec{\alpha}))$ is the maximal interval where $\vec{u}$ is defined, then we put $\vec{T}(\vec{\alpha}) \overset{\text{def}}{=} \infty \in E_n^*$.

In the following definition the domain of $\vec{T}$ will be extended from E_n^+ to E_n. For this purpose we use the functions $\hat{u}_i(x) \overset{\text{def}}{=} \frac{u_i(x)}{\alpha_i} = \frac{u_i(x)}{u_i'(0)}$.

DEFINITION 3.2. Put

$$G_i(x, \hat{u}_1, \dots, \hat{u}_n, \alpha_1, \dots, \alpha_n) \overset{\text{def}}{=} \frac{1}{\alpha_i} F_i(x, \hat{u}_1 \cdot \alpha_1, \dots, \hat{u}_n \cdot \alpha_n) \qquad \text{for all} \quad \alpha_i > 0 \tag{3.2.1}$$

and

$$G_i(x, \hat{u}_1, \dots, \hat{u}_n, \alpha_1, \dots, \alpha_n) \overset{\text{def}}{=} \frac{\partial F_i}{\partial u_i^+}\big(x, \hat{u}_1 \cdot \alpha_1, \dots, \hat{u}_{i-1} \cdot \alpha_{i-1}, 0, \hat{u}_{i+1} \cdot \alpha_{i+1}, \dots, \hat{u}_n \cdot \alpha_n\big) \cdot \frac{\hat{u}_i + |\hat{u}_i|}{2} \qquad \text{for all} \quad \alpha_i = 0\,.$$

Let now $\vec{\alpha} \in E_n^*$.

(1) If $\vec{\alpha} = \infty$, then we define $\vec{T}(\vec{\alpha}) = \infty$.

(2) If $\vec{\alpha} \neq \infty$, then we shall consider the solution $\hat{\vec{u}}$ of the following problem

$$\hat{\vec{u}}''(x) \overset{\text{a.e.}}{=} \vec{G}\big(x, \hat{\vec{u}}(x), \vec{\alpha}\big)\,, \qquad \hat{\vec{u}}(0) = \vec{0}\,, \qquad \hat{\vec{u}}'(0) = \underbrace{(1, 1, \dots, 1)}_{n \text{ times}}\,. \tag{3.2.2}$$

Let $\langle 0, \hat{T}_{\max}(\vec{\alpha}))$ be the maximal interval where the solution $\hat{\vec{u}}$ can be defined.

(a) If there exists $i \in \{1, \dots, n\}$ such that $\hat{u}_i(x) > 0$ for all $x \in (0, \hat{T}_{\max}(\vec{\alpha}))$, then we define

$$\vec{T}(\vec{\alpha}) \overset{\text{def}}{=} \infty\,.$$

(b) Otherwise let T_i be the zero point of $\hat{u}_i$ for $i \in \{1, 2, \dots, n\}$ such that

$$\hat{u}_i(T_i) = 0\,, \quad T_i \in \big(0, \hat{T}_{\max}(\alpha)\big)\,,$$

and

$$\hat{u}_i(x) > 0 \qquad \text{for all} \quad x \in (0, T_i)\,.$$

Then we define

$$\vec{T}(\vec{\alpha}) \overset{\text{def}}{=} (T_1, T_2, \dots, T_n)\,.$$

In the following lemmas we shall show the correctness of the previous definitions as well as the relation between $\vec{u}$, $T_{\max}(\vec{\alpha})$ and $\hat{\vec{u}}$, $\hat{T}_{\max}(\vec{\alpha})$.

Lemma 3.3. *Let $\vec{F}$ fulfil* (3.0.2), (3.0.3), (3.0.6) *and* (3.0.7). *Then $\vec{G}(x, \hat{\vec{u}}, \vec{\alpha})$ defined in Definition* 3.2 *satisfies Carathéodory's conditions.*

Proof. From the second part of assumption (3.0.6) it follows that G_i is for fixed $\hat{\vec{u}}$, $\vec{\alpha}$ measurable in x in the interval $\langle 0, \infty)$.

Let us show its continuity in $\hat{\vec{u}}$, $\vec{\alpha}$. From assumption (3.0.3) we have that

$$\vec{G}\left(x, \tfrac{\hat{u}_1+|\hat{u}_1|}{2}, \ldots, \tfrac{\hat{u}_n+|\hat{u}_n|}{2}, \alpha_1, \ldots, \alpha_n\right) = \vec{G}(x, \hat{u}_1, \ldots, \hat{u}_n, \alpha_1, \ldots, \alpha_n).$$

Therefore it is sufficient for us to prove the continuity only for $(\hat{\vec{u}}, \vec{\alpha}) \in (E_n \times E_n)$. From (3.0.2), (3.0.6) it follows

$$G_i(x, \hat{\vec{u}}, \vec{\alpha}) = \hat{u}_i \cdot \int_0^1 \frac{\partial F_i}{\partial u_i^+}(x, \alpha_1 \cdot \hat{u}_1, \ldots, p \cdot \alpha_i \cdot \hat{u}_i, \ldots, \alpha_n \cdot \hat{u}_n)\, \mathrm{d}p. \tag{3.3.1}$$

Continuity of the right-hand expression follows from continuity of

$$\int_0^1 \frac{\partial F_i}{\partial u_i^+}(x, v_1, \ldots, v_{i-1}, p \cdot v_i, v_{i+1}, \ldots, v_n)\, \mathrm{d}p$$

in $\vec{v} \in E_n$ for a fixed x and this easily follows from assumption (3.0.6) (from continuity of $\frac{\partial F_i}{\partial u_i^+}$). The last condition what we need to prove is the existence of integrable local majorant of function G_i, and this follows from (3.3.1) and from assumption (3.0.7). Hence, function $\vec{G}(x, \hat{\vec{u}}, \vec{\alpha})$ fulfils locally Carathéodory's conditions. □

Lemma 3.4. *Let $\vec{F}$ fulfils* (3.0.2), (3.0.3), (3.0.6) *and* (3.0.7). *Then the problem* (3.2.2) *has the property of global uniqueness.*

Proof. Let us choose $\vec{\alpha} \in E_n$ arbitrary, but fixed. Let $\vec{u}(\cdot)$ and $\hat{\vec{u}}(\cdot)$ be two solutions of problem (3.2.2), defined in the intervals $\langle 0, T_{\max})$ and $\langle 0, \hat{T}_{\max})$ respectively; we shall show that they are identical on the intersection $\langle 0, T_p)$ of these two intervals.

Let

$$M^o \overset{\text{def}}{=} \{i;\ i \in \{1, 2, \ldots, n\},\ \alpha_i = 0\},$$

$$M^+ \overset{\text{def}}{=} \{i;\ i \in \{1, 2, \ldots, n\},\ \alpha_i > 0\}.$$

Now we define new functions in the interval $\langle 0, T_p)$ for $i \in \{1, 2, \ldots, n\}$ by

$$v_i(x) \overset{\text{def}}{=} \alpha_i \cdot u_i(x); \qquad \hat{v}_i(x) \overset{\text{def}}{=} \alpha_i \cdot \hat{u}_i(x).$$

Assuming Definition 3.2 we obtain that these functions fulfil:

$$\begin{aligned} v_i''(x) &\overset{\text{a.e.}}{=} F_i\big(x, v_1(x), v_2(x), \dots, v_n(x)\big) \qquad \text{for all} \quad i \in M^+ ; \\ v_i(0) &= 0\,, \qquad v_i' = \alpha_i \end{aligned} \tag{3.4.1}$$

$$v_i(x) \equiv 0 \qquad \text{for all} \quad i \in M^o . \tag{3.4.2}$$

By (3.4.2) we see that in problem (3.4.1) only functions $v_i(\cdot)$ for $i \in M^+$ are entering. From assumption (3.0.7) we get that F_i fulfil locally Lipschitz's condition and this implies the property of local and global uniqueness. Therefore for all $i \in M^+$ we have

$$v_i(x) \equiv \hat{v}_i(x) \quad \text{for all } x \in \langle 0, T_p) \implies u_i(x) \equiv \hat{u}_i(x) \quad \text{for all } x \in \langle 0, T_p) .$$

Now it remains to verify that $u_i(x) \equiv \hat{u}_i(x)$ for all $x \in \langle 0, T_p)$ and for $i \in M^o$. When we use Definition 3.2, we can write for all $i \in M^o$

$$\begin{aligned} u_i''(x) &\overset{\text{a.e.}}{=} p_i(x) \cdot \frac{u_i(x) + |u_i(x)|}{2}\,, \qquad u_i(0) = 0\,, \quad u_i'(0) = 1\,, \\ \hat{u}_i{}''(x) &\overset{\text{a.e.}}{=} p_i(x) \cdot \frac{\hat{u}_i(x) + |\hat{u}_i(x)|}{2}\,, \qquad \hat{u}_i(0) = 0\,, \quad \hat{u}_i{}'(0) = 1\,, \end{aligned} \tag{3.4.3}$$

where

$$p_i(x) \overset{\text{def}}{=} \frac{\partial F_i}{\partial u_i^+}\big(x, v_1(x), \dots, v_n(x)\big) \equiv \frac{\partial F_i}{\partial u_i^+}\big(x, \hat{v}_1(x), \dots, \hat{v}_n(x)\big) .$$

From assumption (3.0.7) it follows that $p_i(x)$ is locally bounded, therefore right-hand sides of (3.4.3) fulfil locally Carathéodory's, and also locally Lipschitz's conditions, and from uniqueness of solutions of (3.4.3) it follows: $u_i(x) = \hat{u}_i(x)$ for all $x \in \langle 0, T_p)$. Hence, we have proved $\vec{u}(x) = \hat{\vec{u}}(x)$ for all $x \in \langle 0, T_p)$. □

Note. From Lemmas 3.3, 3.4 correctness and uniqueness of Definition 3.2 follows.

The following lemma is useful for us to estimate some solutions of problem (3.1.1).

Lemma 3.5. *Let $\vec{F}$ fulfil conditions* (3.0.2), (3.0.3), (3.0.6), (3.0.7) *and* (3.0.8). *Let $T > 0$ be a fixed number. Let $c_1(\lambda), \dots, c_n(\lambda)$ be some functions satisfying* (3.0.8). *Then the following assertion is true:*

Let $\vec{\alpha} \in E_n$ be fixed. Let $\vec{u}(\cdot)$ be the maximal solution of problem (3.1.1) *which is defined on the interval $\big\langle 0, T_{\max}(\vec{\alpha})\big)$, that is*

$$\begin{aligned} \vec{u}\,''(x) &\overset{\text{a.e.}}{=} \vec{F}\big(x, \vec{u}(x)\big)\,, \\ \vec{u}(0) &= \vec{0}\,, \qquad \vec{u}\,'(0) = \vec{\alpha}\,. \end{aligned} \tag{3.5.1}$$

If

$$T < T_{\max}(\vec{\alpha}) \tag{3.5.2}$$

and

$$u_i(T) \le 0 \qquad \textit{for all} \quad i \in \{1,2,\dots,n\}\,, \tag{3.5.3}$$

then $u_i(x) < c_i(0)$ *for all* $i \in \{1,2,\dots,n\}$, *for all* $x \in \langle 0, T_{\max}(\vec{\alpha}))$.

P r o o f. By contradiction. Let all assumptions be fulfilled and let the assertion be not true. So, for suitable $\vec{\alpha} \in E_n$ solution $\vec{u}(\cdot)$ of problem (3.5.1) would fulfil

$$\exists\, i_1 \in \{1,2,\dots,n\} \quad \exists\, x_1 \in \langle 0, T_{\max}(\vec{\alpha})) \qquad u_{i_1}(x_1) \ge c_{i_1}(0) > 0\,. \tag{3.5.4}$$

Assuming (3.5.3) we obtain that if some u_i fulfils $u_i(T) \le 0$, then there is a point $x_0 \le T$ such that $u_i(x_0) = 0$, $u_i'(x_0) \le 0$ and this by assumptions (3.0.2), (3.0.3) and also by (3.0.7) (which mean locally Lipschitz conditions for F_i), implies that $u_i(x) = u_i'(x_0)\cdot(x - x_0)$ for all $x \in \langle x_0, T_{\max}(\vec{\alpha}))$ and therefore u_i is not positive here. By (3.5.4) this gives

$$\exists\, i_1 \in \{1,2,\dots,n\} \quad \exists\, x_1 \in (0,T) \qquad u_{i_1}(x_1) \ge c_{i_1}(0) > 0\,. \tag{3.5.5}$$

Let us define auxiliary function ψ_λ

$$\psi_\lambda \stackrel{\text{def}}{=} \max\Big\{ \tfrac{u_i(x)}{c_i(\lambda)}\,;\ 1 \le i \le n\,,\ 0 \le x \le T \Big\}\,. \tag{3.5.6}$$

Evidently this is a continuous function and if we also assume $\lim\limits_{\lambda\to\infty} c_i(\lambda) = \infty$ (by (3.0.8)), then we get $\lim\limits_{\lambda\to\infty} \psi_\lambda = 0$. From (3.5.5) we obtain $\psi_0 \ge 1$, so the following definition is correct

$$\lambda_0 \stackrel{\text{def}}{=} \max\{\lambda :\ \psi_\lambda \ge 1\}\,. \tag{3.5.7}$$

From this definition of λ_0 we immediately have:

$$\forall\,\lambda > \lambda_0 \quad \forall\, x \in \langle 0,T\rangle \quad \forall\, i \in \{1,2,\dots,n\} \qquad u_i(x) < c_i(\lambda)\,, \tag{3.5.8}$$

$$\forall\,\lambda \ge \lambda_0 \quad \forall\, x \in \langle 0,T\rangle \quad \forall\, i \in \{1,2,\dots,n\} \qquad u_i(x) \le c_i(\lambda)\,. \tag{3.5.9}$$

When we assume $\psi_{\lambda_0} = 1$ and also definition (3.5.6), we have $\exists x_2 \in \langle 0,T\rangle$ such that $\exists i_2 \in \{1,2,\dots,n\}$ for which $u_{i_2}(x_2) = c_{i_2}(\lambda_0)$. Let $x_0 \in (0,T)$ be the smallest x_2 with the introduced property, so according to definition we have

$$\forall\, i \in \{1,2,\dots,n\} \quad \forall\, x \in \langle 0, x_0) \qquad u_i(x) < c_i(\lambda_0)$$

and (3.5.10)

$$\exists\, i_0 \in \{1,2,\dots,n\} \qquad u_{i_0}(x_0) = c_{i_0}(\lambda_0)\,.$$

Let us consider i_0 from (3.5.10). According to $u_{i_0}(x_0) = c_{i_0}(\lambda_0)$ and (3.5.9) we have $u'_{i_0}(x_0) = 0$. Let us define function $v(\cdot)$ in some neighbourhood of x_0 as the solution of the following problem:

$$
\begin{gathered}
v''(x) \overset{\text{a.e.}}{=} p\big(x, v(x)\big)\,, \\
v(x_0) = u_{i_0}(x_0) = c_{i_0}(\lambda_0)\,, \qquad v'(x_0) = u'_{i_0}(x_0) = 0\,,
\end{gathered} \tag{3.5.11}
$$

where

$$
p(x,v) \overset{\text{def}}{=} F_{i_0}\Big(x, u_1(x), \dots, u_{i_0-1}(x), \max\{v, c_{i_0}(\lambda_0)\}, \dots, u_n(x)\Big)\,.
$$

Because function $p(x,v)$ fulfils locally Carathéodory conditions, there exists a solution of this problem defined in an interval $(x_0 - \delta, x_0 + \delta)$, where $\delta > 0$ is so small that the following conditions also hold:

$$
\begin{gathered}
\forall\, x \in (x_0 - \delta, x_0\rangle \qquad v(x) \le c_{i_0}(\lambda_0 + 1) \\
\big(\, c_{i_0}(\lambda_0 + 1) \overset{(3.5.8)}{>} u_{i_0}(x_0) \overset{(3.5.10)}{=} c_{i_0}(\lambda_0) \overset{(3.5.11)}{=} v(x_0)\,\big)\,.
\end{gathered} \tag{3.5.12}
$$

Hence, if we put for arbitrary $x \in (x_0 - \delta, x_0\rangle$, $v_x \overset{\text{def}}{=} \max\{v(x), c_{i_0}(\lambda_0)\}$, then $v_x \in \big\langle c_{i_0}(\lambda_0), c_{i_0}(\lambda_0 + 1)\big)\big\rangle$ and so there exists some $\lambda \in \langle \lambda_0, \lambda_0 + 1\rangle$ such that $v_x = c_{i_0}(\lambda)$. Then from (3.0.8), (3.5.11) and (3.5.9) we get for all $x \in (x_0 - \delta, x_0\rangle$, $p\big(x, v(x)\big) \ge 0$. When we use this, we obtain by integration

$$
v'(x) \overset{(3.5.11)}{=} v'(x_0) - \int\limits_x^{x_0} p\big(z, v(z)\big)\, \mathrm{d}z \le v'(x_0) \overset{(3.5.11)}{=} 0\,.
$$

Hence, function $v(\cdot)$ is on the interval $(x_0 - \delta, x_0\rangle$ nonincreasing, and therefore

$$
\forall\, x \in (x_0 - \delta, x_0\rangle \qquad v(x) \ge v(x_0) \overset{(3.5.11)}{=} c_{i_0}(\lambda_0)\,. \tag{3.5.13}
$$

Thus, using definition (3.5.11) of function $p(x,v)$ we obtain that function v is in the interval $(x_0 - \delta, x_0\rangle$ a solution of the following problem:

$$
\begin{gathered}
v''(x) = F_{i_0}\big(x, u_1(x), \dots, u_{i_0-1}(x), v(x), u_{i_0+1}(x), \dots, u_n(x)\big)\,, \\
v(x_0) = u_{i_0}(x_0)\,, \qquad v'(x_0) = u'_{i_0}(x_0)\,.
\end{gathered}
$$

Because F_{i_0} fulfils locally Lipschitz conditions, from uniqueness of this problem we get, when we realize that u_{i_0} is also solution of this problem, that:

$$
\forall\, x \in (x_0 - \delta, x_0\rangle \qquad u_{i_0}(x) = v(x)\,,
$$

what together with (3.5.13) and (3.5.10) gives us the contradiction. □

The following lemma deals with the relation between solutions $\vec{u}$ and $\hat{\vec{u}}$ of problems (3.1.1) and (3.2.2), respectively.

Lemma 3.6. *Let the function $\vec{F}$ fulfil (3.0.2), (3.0.3), (3.0.6) and (3.0.7). Let $\vec{\alpha} \in E_n$. Let $\vec{u}(\cdot)$ and $\hat{\vec{u}}(\cdot)$ be the maximal solutions of problems (3.1.1) and (3.2.2) respectively, which are defined on the intervals $\langle 0,T)$ and $\langle 0,\hat{T})$, respectively.*

Then $T = \hat{T}$ and $u_i(\cdot) \equiv \alpha_i \cdot \hat{u}_i(\cdot)$ for all $i \in \{1,\dots,n\}$.

Proof. Validity of $u_i(x) = \alpha_i \cdot \hat{u}_i(x)$ for $x \in \langle 0,T) \cap \langle 0,\hat{T})$ is evident, when we realize that functions $\alpha_i \cdot \hat{u}_i(\cdot)$ are components of the solution of problem (3.1.1) and when we assume also its uniqueness (which follows from (3.0.7)). Hence, it only remains to show that $T = \hat{T}$. From the above procedure it follows that $\hat{T} \leq T$. Let us show also inverse inequality.

Let

$$M^+ \overset{\text{def}}{=} \{i \in \{1,2,\dots,n\};\ \alpha_i > 0\},$$
$$M^o \overset{\text{def}}{=} \{i \in \{1,2,\dots,n\};\ \alpha_i = 0\}.$$

Let $\vec{u}(\cdot)$ be the solution of (3.1.1) defined in the interval $\langle 0,T)$. By means of it we shall construct a solution $\vec{v}(\cdot)$ of problem (3.2.2), which will be also defined at least in the interval $\langle 0,T)$ and then the proof will be done if we use the uniqueness of this solution (Lemma 3.4), from what we have $\hat{\vec{u}}(\cdot) \equiv \vec{v}(\cdot)$ on $\langle 0,T)$. Let us put:

$$v_i(x) \overset{\text{def}}{=} \frac{u_i(x)}{\alpha_i}, \qquad x \in \langle 0,T),\ \ i \in M^+. \tag{3.6.1}$$

Because $\alpha_i = 0$ for $i \in M^o$, if we later define the other $v_i(\cdot)$ arbitrarily, then this will be always true:

$$v_i(0) = 0, \quad v_i'(0) = 1 \qquad \text{for all} \quad i \in M^+$$

and

$$v_i''(x) \overset{\text{a.e.}}{=} \frac{F_i\bigl(x, \alpha_1 \cdot v_1(x), \dots, \alpha_n \cdot v_n(x)\bigr)}{\alpha_i}, \qquad x \in \langle 0,T). \tag{3.6.2}$$

Let us choose $i_0 \in M^o$ arbitrary. We define $v_{i_0}(\cdot)$ as the solution of the following problem:

$$\begin{gathered} v_{i_0}''(x) \overset{\text{a.e.}}{=} \frac{\partial F_{i_0}}{\partial u_{i_0}^+}\bigl(x, \alpha_1 \cdot v_1(x), \dots, \alpha_n \cdot v_n(x)\bigr) \cdot \frac{v_{i_0}(x) + |v_{i_0}(x)|}{2}, \\ v_{i_0}(0) = 0, \qquad v_{i_0}'(0) = 1. \end{gathered} \tag{3.6.3}$$

We see that the other $v_i(\cdot)$ for $i \in M^o$, $i \neq i_0$, do not appear directly in this equation and these are defined by formula (3.6.1). Function $p(x, v_{i_0})$ (in the right-hand side of (3.6.3)) fulfils in v_{i_0} locally Lipschitz condition in the interval of its definition $\langle 0, T)$, so we have guaranteed existence of the maximal solution and also its uniqueness. Moreover from the following estimate

$$|p(x, v_{i_0})| \leq \left| \frac{\partial F_{i_0}}{\partial u^+_{i_0}}(x, \alpha_1 \cdot v_1(x), \dots, \alpha_n \cdot v_n(x)) \right| \cdot |v_{i_0}|$$

and from assumption (3.0.7) we would get the estimation of $v_{i_0}(\cdot)$ on every compact interval, and this enables to prolong its definition interval up to $\langle 0, T)$. Hence, $\vec{v}(\cdot)$ is defined in the whole interval $\langle 0, T)$ and from (3.6.2), (3.6.3) and Definition 3.2 it follows that v is solution of problem (3.2.2). So the proof is complete. □

4. Continuity of the mapping $\vec{T}$

In this section we shall prove the continuity of mapping $\vec{T}$.

THEOREM 4.1. *Let $\vec{F}$ fulfil* (3.0.2.), (3.0.3), (3.0.6), (3.0.7) *and* (3.0.8). *Then the mapping $\vec{T}\colon E^*_n \to E^*_n$ defined in Definition* 3.2 *is continuous.*

P r o o f . We shall prove the continuity at every point $\hat{\vec{\alpha}} \in E^*_n$; the proof of this will be divided into several parts and the proof of each of them will be done separately in the following lemmas.

(1) $\hat{\vec{\alpha}} \in E_n$ and simultaneously $\hat{\vec{T}} \overset{\text{def}}{=} \vec{T}(\hat{\vec{\alpha}}) \in E_{n,+}$.
The proof is done in Lemma 4.3.

(2) $\hat{\vec{\alpha}} \in E_n$ and simultaneously $\vec{T}(\hat{\vec{\alpha}}) = \infty$, $T_{\max}(\hat{\vec{\alpha}}) = \infty$.
The proof is done in Lemma 4.4.

(3) $\hat{\vec{\alpha}} \in E_n$ and simultaneously $\vec{T}(\hat{\vec{\alpha}}) = \infty$, $T_{\max}(\hat{\vec{\alpha}}) < \infty$.
The proof is done in Lemma 4.5.

(4) $\hat{\vec{\alpha}} = \infty$ what implies $\vec{T}(\hat{\vec{\alpha}}) = \infty$.
The proof is done in Lemma 4.6.

□

LEMMA 4.2. *Let $\vec{F}$ fulfil assumptions of Theorem* 4.1. *Then the mapping $\vec{T}$ ·) from Definition* 3.2 *is correctly defined and problem* (3.2.2) *has the property of continuous dependence of its solution $\hat{\vec{u}}$ on parameter $\vec{}$.*

P r o o f . Correctness of the definition follows from Lemmas 3.3, 3.4 and by the well-known theorems on ordinary differential equations they also imply the continuous dependence of solution on parameter $\vec{\alpha}$ (for example [5; Lemma 6.1] can be used). □

Lemma 4.3. *Let $\vec{F}$ fulfil assumptions of Theorem 4.1. Let $\hat{\vec{\alpha}} \in E_n^*$ be chosen so that:*

$$\hat{\vec{\alpha}} \in E_n \qquad \textit{and simultaneously} \qquad \hat{\vec{T}} \stackrel{\text{def}}{=} \vec{T}(\hat{\vec{\alpha}}) \in E_{n,+}\,.$$

Then the mapping $\vec{T}(\cdot)$ from Definition 3.2 is continuous at the point $\hat{\vec{\alpha}}$.

Proof. Let $\hat{\vec{u}}(\cdot)$ be the solution of problem (3.2.2), where we put $\vec{\alpha} = \hat{\vec{\alpha}}$. We shall prove the following statements:

$$\forall i \in \{1,2,\dots,n\} \qquad \begin{aligned} &\hat{u}_i(\hat{T}_i) = 0\,, \\ &\hat{u}_i'(\hat{T}_i) < 0\,. \end{aligned} \tag{4.3.1}$$

The first part is evident. The second part will be proved by contradiction: If the second part were not true, according to definition of $\hat{\vec{T}}$ we would get that

$$\exists\, k \in \{1,2,\dots,n\} \qquad \hat{u}_k'(\hat{T}_k) = 0\,.$$

Hence, the component $\hat{u}_k(\cdot)$ would be a solution of the following problem:

$$\begin{gathered} \hat{u}_k''(x) \stackrel{\text{a.e.}}{=} p\big(x, \hat{u}_k(x)\big)\,, \qquad x \in \langle 0, \hat{T}_k\rangle\,, \\ \hat{u}_k(\hat{T}_k) = 0\,, \qquad \hat{u}_k'(\hat{T}_k) = 0\,, \end{gathered}$$

where (4.3.2)

$$p(x,u) \stackrel{\text{def}}{=} G_k\big(x, \hat{u}_1(x), \dots, \hat{u}_{i-1}(x), u, \hat{u}_{i+1}(x), \dots, \hat{u}_n(x), \hat{\alpha}_1, \dots, \hat{\alpha}_n\big)\,.$$

From Definition 3.2 it easily follows that $p(x,u)$ fulfils locally Lipschitz condition in u and so from uniqueness of solution it would follow that $\hat{u}_k \equiv 0$ in the interval $\langle 0, \hat{T}_k\rangle$, what would give us the contradiction, because according to (3.2.2), $\hat{u}_k'(0) = 1$ holds. Hence, (4.3.1) is proved. Now we shall prove the continuity which we need. Let $\varepsilon > 0$ be chosen arbitrarily small. From assumptions of this lemma it follows

$$0 < \hat{T}_i < T_{\max}(\hat{\vec{\alpha}}) \qquad \text{for all} \quad i \in \{1,\dots,n\}\,,$$

therefore when we use (4.3.1), we get that for suitable small $\varepsilon_1 > 0$ the following assumptions hold:

$$0 < \varepsilon_1 < \varepsilon\,, \qquad 0 < \hat{T}_i - \varepsilon_1 < \hat{T}_i + \varepsilon_1 < T_{\max}(\hat{\vec{\alpha}}) \quad \text{for all } i \in \{1,\dots,n\} \tag{4.3.3}$$

$$\hat{u}_i(\hat{T}_i - \varepsilon_1) > 0 > \hat{u}_i(\hat{T}_i + \varepsilon_1) \qquad \text{for all} \quad i \in \{1,2,\dots,n\}\,. \tag{4.3.4}$$

If we use continuity of solutions of problem (3.2.2) on parameter $\vec{\alpha}$ (proved in Lemma 4.2), then we can choose a small δ such that the following property holds: For all $\vec{\alpha} \in E_n$ with $\|\vec{\alpha} - \hat{\vec{\alpha}}\| < \delta$ we have: If $\vec{u}(\cdot)$ is a solution of problem (3.2.2) for the chosen $\vec{\alpha}$, then $\vec{u}(\cdot)$ will be defined at least in the interval $\left\langle 0, \max_{1\le i\le n} (\hat{T}_i + \varepsilon_1)\right\rangle$ and will fulfil the following inequality:

$$u_i(\hat{T}_i - \varepsilon_1) > 0 > u_i(\hat{T}_i + \varepsilon_1) \qquad \text{for all} \quad i \in \{1, 2, \ldots, n\}. \tag{4.3.5}$$

If we use the property (which follows from (3.0.2), (3.0.3) and (3.2.2)) that every solution u_i can have at most one zero point in the interval $\langle 0, T_{\max}(\vec{\alpha})\rangle$ (because when it reaches the first zero point then it is only linearly decreasing), then using (4.3.5) we obtain

$$\hat{T}_i - \varepsilon_1 < T_i < \hat{T}_i + \varepsilon_1 \qquad \text{for all} \quad i \in \{1, 2, \ldots, n\},$$

where $T_i > 0$ are the zero points of solutions $u_i(\cdot)$, precisely $(T_1, T_2, \ldots, T_n) = \vec{T} = \vec{T}(\vec{\alpha})$ and that is why $\|\vec{T}(\vec{\alpha}) - \vec{T}(\hat{\vec{\alpha}})\| \le \sqrt{n} \cdot \varepsilon_1 \overset{(4.3.3)}{\le} \sqrt{n} \cdot \varepsilon$ holds for all $\vec{\alpha}$, δ – near to $\hat{\vec{\alpha}}$. □

LEMMA 4.4. *Let $\vec{F}$ fulfil all assumptions of Theorem 4.1. Let $\hat{\vec{\alpha}} \in E_n^*$ be chosen in such a way, that*

$$\hat{\vec{\alpha}} \in E_n \qquad \textit{and simultaneously} \qquad \vec{T}(\hat{\vec{\alpha}}) = \infty, \quad T_{\max}(\hat{\vec{\alpha}}) = \infty.$$

Then the mapping $\vec{T}(\cdot)$ from Definition 3.2 is continuous at the point $\hat{\vec{\alpha}}$.

Proof. Let $\hat{\vec{u}}(\cdot)$ be a solution of problem (3.2.2) (where we put $\vec{\alpha} = \hat{\vec{\alpha}}$) defined in the interval $\langle 0, \infty)$. From our assumption by Definition 3.2 we have that there exists $k \in \{1, 2, \ldots, n\}$ such that $\hat{u}_k(x) > 0$ for all $x \in (0, \infty)$. Let us choose sufficiently small neighbourhood of point ∞ in space E_n^*; for example basic neighbourhood $\{\vec{T};\ \|\vec{T}\| > R\} \cup \{\infty\}$. Because $\hat{u}_k'(0) = 1$ and $\hat{u}_k(x) > 0$ for all $x \in (0, R+1\rangle$, we can choose suitable $d \in (0, R+1)$, $\varepsilon > 0$ such that:

$$\hat{u}_k'(x) \ge 1/2 \quad \text{for all } x \in \langle 0, d\rangle; \qquad \hat{u}_k(x) \ge \varepsilon \quad \text{for all } x \in \langle d, R+1\rangle. \tag{4.4.1}$$

Then from continuous dependence of solution (3.2.2) on parameter $\vec{\alpha}$ (which follows from Lemma 4.2) we obtain the existence of such a suitably small δ neighbourhood of point $\hat{\vec{\alpha}}$, that it holds:

For all $\vec{\alpha} \in E_n$ such that $\|\vec{\alpha} - \hat{\vec{\alpha}}\| < \delta$ it holds: The solution $\vec{u}(\cdot)$ of problem (3.2.2) with this new $\vec{\alpha}$ will be defined at least in the interval $\langle 0, R+1\rangle$ and simultaneously:

$$\|\vec{u}(\cdot) - \hat{\vec{u}}(\cdot)\|_{C^1\langle 0, R+1\rangle} < \min\{\tfrac{1}{4}, \tfrac{\varepsilon}{2}\}$$

holds. From this together with (4.4.1) it follows

$$u_k'(x) \ge \tfrac{1}{4} \quad \text{for all } x \in \langle 0, d\rangle \qquad \text{and} \qquad u_k(x) \ge \frac{\varepsilon}{2} \quad \text{for all } x \in \langle d, R+1\rangle$$

from what it is evident that function $u_k(\cdot)$ has no zero point in the interval $(0, R+1\rangle$. So, either the case $\vec{T}(\vec{\alpha}) = \infty$ will be true, or in the opposite case we shall have $\|\vec{T}(\vec{\alpha})\| \ge R+1 > R$. Hence, continuity of $\vec{T}$ at the point $\hat{\vec{\alpha}}$ is proved. □

LEMMA 4.5. *Let $\vec{F}$ fulfil assumptions of Theorem 4.1. Let $\hat{\vec{\alpha}} \in E_n^*$ be chosen such that*

$$\hat{\vec{\alpha}} \in E_n \qquad \textit{and simultaneously} \qquad \vec{T}(\hat{\vec{\alpha}}) = \infty\,, \quad T_{\max}(\hat{\vec{\alpha}}) < \infty\,.$$

Then the mapping $\vec{T}(\cdot)$ from Definition 3.2 is continuous at the point $\hat{\vec{\alpha}}$.

Proof. Let $\hat{\vec{u}}(\cdot)$ be the maximal solution of problem (3.2.2) for the above $\hat{\vec{\alpha}}$ defined in the interval $\langle 0, \hat{T})$ where $\hat{T} \overset{\text{def}}{=} T_{\max}(\hat{\vec{\alpha}})$. By Lemma 3.6 we obtain that $\hat{\vec{v}}(\cdot)$ which is defined as

$$\hat{v}_i(x) \overset{\text{def}}{=} \alpha_i \cdot \hat{u}_i(x)\,, \qquad x \in \langle 0, \hat{T})\,,\ i \in \{1, \dots, n\}$$

is a maximal solution of problem (3.1.1). Let us choose $R > 0$, which will determine arbitrarily small basic neighbourhood of the point $\infty \in E_n^*$ of the form

$$O_\infty = \{\vec{T};\ \|\vec{T}\| > R\} \cup \{\infty\}\,. \tag{4.5.1}$$

Let $c_1(\cdot), \dots, c_n(\cdot)$ be function whose existence is guaranteed by assumption (3.0.8) for $T = R$. Let us define:

$$\text{Min} \overset{\text{def}}{=} \inf\{\hat{v}_i(x);\ x \in \langle 0, \hat{T})\,,\ i \in \{1, 2, \dots, n\}\}\,. \tag{4.5.2}$$

From assumptions (3.0.2) and (3.0.3) it follows that if some $\hat{v}_i(\cdot)$ (solution of problem (3.1.1)) reaches negative value, then it is further only linearly decreasing and that is why we have $0 \geq \text{Min} > -\infty$. Hence, the following set will be compact:

$$K \overset{\text{def}}{=} \{(x, \vec{u}) \in \mathbb{R}_0^+ \times \mathbb{R}^n\,;\ 0 \leq x \leq \hat{T}\,,\ \text{Min} \leq u_i \leq c_i(0) \text{ for } i \in \{1, \dots, n\}\}\,.$$

We know that $\hat{\vec{v}}(\cdot)$ is a maximal solution of problem (3.1.1) (which fulfils the condition of local existence and uniqueness of its solution), and it is defined according to our assumption in a bounded interval $\langle 0, \hat{T})$. By the known theorem on behaviour of solutions of ordinary differential equation at both ends of maximal existence interval ([5; Theorem 5.4], [6; Theorem 2.1]) we obtain that the whole graph of this solution must not be contained in compact K. According to its definition using definition (4.5.2) of constant Min we get:

$$\exists\, k \in \{1, 2, \dots, n\} \quad \exists\, x_0 \in (0, \hat{T}) \qquad \hat{v}_k(x_0) > c_k(0)\,.$$

When we use continuous dependence of solution $\vec{v}$ of problem (3.1.1) on parameter $\vec{\alpha}$ (which can be obtained formally from the already proved continuous dependence of solutions of a problem (3.2.2) (see Lemma 4.2) and from Lemma 3.6), we get that there exists a suitable small δ such that

for all $\vec{\alpha} \in E_n$, $\|\vec{\alpha} - \hat{\vec{\alpha}}\| < \delta$, the above property is true, this means that the solution $\vec{v}$ of problem (3.1.1) for such $\vec{\alpha}$ is defined at least in the interval $\langle 0, x_0\rangle$ and the following inequality is true

$$v_k(x_0) > c_k(0)\,. \tag{4.5.3}$$

We assert that from this it already follows

$$\vec{T}(\vec{\alpha}) \in O_\infty \qquad (\text{see } (4.5.1)) \tag{4.5.4}$$

and this gives us the statement of our lemma. Statement (4.5.4) can be proved by contradiction. If it were not true, then we would consider solution $\vec{u}(\cdot)$ of problem (3.2.2) and according to Definition 3.2 every $u_i(\cdot)$ reaches the zero value at the point T_i, $0 < T_i \le R$, where $(T_1, \dots, T_n) = \vec{T}(\vec{\alpha})$. Because components $u_i(\cdot)$ of that solution will be on the right to these points linearly decreasing, we get $R \le T_{\max}(\vec{\alpha}) = \infty$ and $u_i(R) \le 0$ for all $i \in \{1, 2, \dots, n\}$. From Lemma 3.6 it follows that solution $\vec{v}(\cdot)$ of problem (3.1.1) also fulfils the same inequalities and it is also defined in the interval $\langle 0, \infty)$ what implies that all assumptions of Lemma 3.5 are fulfilled. From this lemma it follows

$$v_i(x) < c_i(0) \qquad \text{for all } x \in \langle 0, \infty)\,, \ \text{ for all } i \in \{1, \dots, n\}\,,$$

what gives us contradiction to (4.5.3). □

LEMMA 4.6. *Let $\vec{F}$ fulfil assumptions of Theorem 4.1. Let $\hat{\vec{\alpha}} \in E_n^*$ be such that $\hat{\vec{\alpha}} = \infty$ and thus $\vec{T}(\hat{\vec{\alpha}}) = \infty$. Then mapping $\vec{T}(\cdot)$ from Definition 3.2 is continuous at the point $\hat{\vec{\alpha}}$.*

Proof. By contradiction. If it were not continuous, then there would exist a suitable sequence $\{\vec{\alpha}_l\}_{l=1}^{\infty} \subset E_n$ and $R > 0$ such that:

$$\|\vec{\alpha}_l\| \to \infty \quad \text{for } l \to \infty\,, \qquad \text{but} \qquad \|\vec{T}(\vec{\alpha}_l)\| \le R \quad \text{for all } l \in \mathbb{N}\,. \tag{4.6.1}$$

Let $c_1(\cdot), \dots, c_n(\cdot)$ be functions whose existence is guaranteed by assumption (3.0.8) for $T = R$. Let us define the set $\tilde{K}$:

$$\tilde{K} \overset{\text{def}}{=} \big\{(x, \vec{u}) \in \mathbb{R}_0^+ \times \mathbb{R}^n\,;\ 0 \le x \le 1\,,\ 0 \le u_i \le c_i(0) \text{ for all } i \in \{1, \dots, n\}\big\}\,. \tag{4.6.2}$$

Evidently $\tilde{K}$ is a compact set. By assumption (3.0.2) and (3.0.7), we obtain that $F_i(x, \vec{u})$ are locally bounded in $\mathbb{R}_0^+ \times E_n$ and therefore also in every compact set. So, if we define

$$M \overset{\text{def}}{=} \sup\big\{|F_i(x, \vec{u})|\,;\ (x, \vec{u}) \in \tilde{K} \text{ for } i \in \{1, \dots, n\}\big\} \tag{4.6.3}$$

then we get $M < \infty$. Let us put:

$$\alpha_{\max} \overset{\text{def}}{=} M + \sum_{i=1}^{n} c_i(0) > 0\,. \tag{4.6.4}$$

According to (4.6.1), there exists l such that $\|\vec{\alpha}_l\| > \sqrt{n} \cdot \alpha_{\max}$ and therefore if we put $\vec{\alpha} \stackrel{\text{def}}{=} \vec{\alpha}_l$, then there exists $k \in \{1, 2, \dots, n\}$ such that there holds:

$$\alpha_k > \alpha_{\max}, \qquad \|\vec{T}(\vec{\alpha})\| \le R. \tag{4.6.5}$$

For this chosen $\vec{\alpha}$ we define $\vec{u}(\cdot)$ as the maximal solution of problem (3.2.2) which is defined in the interval $\langle 0, T_{\max}\rangle$. From the second part of (4.6.5) we would obtain from (3.0.2) and (3.0.3) by standard way that:

$$T_{\max} = \infty, \qquad u_i(R) \le 0 \quad \text{for all } i \in \{1, 2, \dots, n\}. \tag{4.6.6}$$

If we define a function $\vec{v}$ as $v_i(\cdot) \stackrel{\text{def}}{=} \alpha_i \cdot u_i(\cdot)$ for all $i \in \{1, \dots, n\}$, then from Lemma 3.6 it follows that $\vec{v}(\cdot)$ is solution of problem (3.1.1) and by (4.6.6) we obtain:

$$v_i(R) \le 0 \qquad \text{for all} \quad i \in \{1, 2, \dots, n\}. \tag{4.6.7}$$

This by Lemma 3.5 implies

$$\forall x \in \langle 0, \infty) \quad \forall i \in \{1, 2, \dots, n\} \qquad v_i(x) < c_i(0), \tag{4.6.8}$$

and therefore the following estimations are true for $x \in \langle 0, 1\rangle$:

$$\begin{aligned} |v_k'(x) - v_k'(0)| &= \left| \int_0^x v_k''(t)\, \mathrm{d}t \right| \\ &\stackrel{(3.1.1)}{=} \left| \int_0^x F_k\big(t, v_1(t), \dots, v_n(t)\big)\, \mathrm{d}t \right| \\ &\stackrel{(3.0.3)}{=} \left| \int_0^x F_k\Big(t, \tfrac{v_1(t) + |v_1(t)|}{2}, \dots, \tfrac{v_n(t) + |v_n(t)|}{2}\Big)\, \mathrm{d}t \right| \\ &\stackrel{(4.6.3),(4.6.2),(4.6.8)}{\le} \int_0^x M\, \mathrm{d}t \le M. \end{aligned}$$

Using this we obtain for all $x \in \langle 0, 1\rangle$:

$$\begin{aligned} v_k'(x) &\ge v_k'(0) - |v_k'(x) - v_k'(0)| \\ &\ge v_k'(0) - M \stackrel{(3.1.1),(4.6.5)}{>} \alpha_{\max} - M \\ &\stackrel{(4.6.4)}{=} \sum_{i=1}^n c_i(0) > c_k(0). \end{aligned}$$

Hence, we have $v_k'(x) > c_k(0)$ in the interval $\langle 0, 1\rangle$ from what we obtain

$$v_k(1) \stackrel{(3.1.1)}{=} \int_0^1 v_k'(x)\, \mathrm{d}x > 1 \cdot c_k(0)$$

and this gives us contradiction to (4.6.8). □

5. Necessary condition for the existence of a solution

In the previous part of this paper ([8; Chapter 5]) we have defined set Ω_n^o. Then we have proved ([8; Chapter 6]) that if $(T_1, \dots, T_n) \in \Omega_n^o$, then problem (3.0.1) has at least one positive solution.

In this part we shall prove inverse theorem which gives us that if problem (3.0.1) has a positive solution for some $(T_1, \dots, T_n)$, then $(T_1, \dots, T_n) \in \Omega_n^o$ must be true. At first we shall prove two auxiliary lemmas:

LEMMA 5.1. *Let $F_1, \dots, F_n$ fulfil assumptions* (3.0.2), (3.0.6) *and* (3.0.9). *Then they also fulfil the following assumption:*

(5.1.1) *For all $k \in \{1, \dots, n-1\}$, for all $x \in \mathbb{R}_0^+$ for all $u_1, \dots, u_{k-1} \in \mathbb{R}_0^+$, function $\frac{1}{u} \cdot F_k(x, u_1, \dots, u_{k-1}, u, \underbrace{0, \dots, 0}_{n-k \text{ times}})$ is increasing in u in the interval $(0, \infty)$.*

P r o o f. So, let $\hat{u} > u > 0$ be arbitrary, but fixed.

$$
\begin{aligned}
&\frac{F_k(x, u_1, \dots, u_{k-1}, \hat{u}, \overbrace{0, \dots, 0}^{n-k \text{ times}})}{\hat{u}} \\
&\overset{(3.0.2),(3.0.6)}{=} \frac{\partial F_k}{\partial u_k^+}(x, u_1, \dots, u_{k-1}, 0, \overbrace{0, \dots, 0}^{n-k \text{ times}}) \\
&\qquad + \int_0^{\hat{u}} \left(\frac{\partial F_k}{\partial u_k^+}(x, u_1, \dots, u_{k-1}, \beta, \overbrace{0, \dots, 0}^{n-k \text{ times}}) \right. \\
&\qquad\qquad \left. - \frac{F_k(x, u_1, \dots, u_{k-1}, \beta, \overbrace{0, \dots, 0}^{n-k \text{ times}})}{\beta} \right) \frac{\mathrm{d}\beta}{\beta} \\
&\overset{(3.0.9)}{>} \frac{\partial F_k}{\partial u_k^+}(x, u_1, \dots, u_{k-1}, 0, \overbrace{0, \dots, 0}^{n-k \text{ times}}) \\
&\qquad + \int_0^{u} \left(\frac{\partial F_k}{\partial u_k^+}(x, u_1, \dots, u_{k-1}, \beta, \overbrace{0, \dots, 0}^{n-k \text{ times}}) \right. \\
&\qquad\qquad \left. - \frac{F_k(x, u_1, \dots, u_{k-1}, \beta, \overbrace{0, \dots, 0}^{n-k \text{ times}})}{\beta} \right) \frac{\mathrm{d}\beta}{\beta} \\
&\overset{(3.0.9),(3.0.6)}{=} \frac{F_k(x, u_1, \dots, u_{k-1}, u, \overbrace{0, \dots, 0}^{n-k \text{ times}})}{u}
\end{aligned}
$$

what we had to prove. □

LEMMA 5.2. *Let $F_1,\dots,F_n$ fulfil assumptions* (3.0.2), (3.0.6) *and* (3.0.11). *Then they also fulfil the following assumption:*

(5.2.1) *For all $k \in \{1,\dots,n-1\}$ for all i, $1 \le i \le k-1$, for all $u_j \in \mathbb{R}_0^+$, $1 \le j \le k-1$, $j \ne i$, for all $x \in \mathbb{R}_0^+$,*

$$\frac{\partial F_k}{\partial u_k^+}(x,u_1,\dots,u_{i-1},u,u_{i+1},\dots,u_{k-1},0,\underbrace{0,\dots,0}_{n-k\ \text{times}})$$

is a nonincreasing function in u in the interval $\langle 0,\infty)$.

Proof. Let us choose $x \in \mathbb{R}_0^+$, $u_j \in \mathbb{R}_0^+$ for $1 \le j \le k-1$, $j \ne i$, $u \in \mathrm{R}_0^+$ and $\alpha > 0$ arbitrarily. Then for all $u_k > 0$ by (3.0.11) we obtain

$$\frac{F_k(x,u_1,\dots,u_{i-1},u,u_{i+1},\dots,u_{k-1},u_k,\overbrace{0,\dots,0}^{n-k\ \text{times}})}{u_k}$$
$$\ge \frac{F_k(x,u_1,\dots,u_{i-1},u+\alpha,u_{i+1},\dots,u_{k-1},u_k,\overbrace{0,\dots,0}^{n-k\ \text{times}})}{u_k}.$$

When $u_k \to 0^+$, we obtain:

$$\frac{\partial F_k}{\partial u_k^+}(x,u_1,\dots,u_{i-1},u,u_{i+1},\dots,u_{k-1},0,\overbrace{0,\dots,0}^{n-k\ \text{times}})$$
$$\ge \frac{\partial F_k}{\partial u_k^+}(x,u_1,\dots,u_{i-1},u+\alpha,u_{i+1},\dots,u_{k-1},0,\overbrace{0,\dots,0}^{n-k\ \text{times}})$$

what we had to prove. □

THEOREM 5.3[1]**.** *Let $F_1,\dots,F_n$ fulfil assumptions* (3.0.2), (3.0.3) *and* (3.0.6) – (3.0.13). *Let Ω_n^o be defined according to* [8; Definition 5.12]. *If problem* (3.0.1) *has a positive solution for some $(T_1,\dots,T_n) \in E_{n,+}$, then there holds $(T_1,\dots,T_n) \in \Omega_n^o$.*

Proof. Let functions $u_1(\cdot),\dots,u_n(\cdot)$ be defined at least in the interval $\left\langle 0, \max\limits_{1\le i\le n} T_i \right\rangle$ and let they fulfil (3.0.1). By induction we shall prove the following assertion step-by-step for $k = 0,1,\dots,n$.

[1]In [8] mentioned as Theorem 7.3.

Assertion. There exist k functions $v_1(\cdot),\dots,v_k(\cdot)$, which fulfil the following three conditions:

(1) These functions fulfil in the interval $\left\langle 0, \max\limits_{1\le i\le n}(T_i)\right\rangle$ the following system:

$$
\begin{aligned}
&v_1''(x) \overset{\text{a.e.}}{=} F_1\big(x, v_1(x), \overbrace{0,\dots,0}^{n-1 \text{ times}}\big)\,,\\
&v_1(0) = v_1(T_1)\,, \qquad v_1(x) > 0 \quad \text{for all } x \in (0,T_1)\,,\\
&v_2''(x) \overset{\text{a.e.}}{=} F_2\big(x, v_1(x), v_2(x), \overbrace{0,\dots,0}^{n-2 \text{ times}}\big)\,,\\
&v_2(0) = v_2(T_2)\,, \qquad v_2(x) > 0 \quad \text{for all } x \in (0,T_2)\,,\\
&\qquad\vdots\\
&v_k''(x) \overset{\text{a.e.}}{=} F_k\big(x, v_1(x),\dots, v_k(x), \overbrace{0,\dots,0}^{n-k \text{ times}}\big)\,,\\
&v_k(0) = v_k(T_k)\,, \qquad v_k(x) > 0 \quad \text{for all } x \in (0,T_k)\,.
\end{aligned}
\tag{5.3.1}
$$

(2)

$$
\forall i,\ 1 \le i \le \min\{k, n-1\} \quad \forall x \in \langle 0, T_i\rangle \qquad v_i(x) \ge u_i(x)\,. \tag{5.3.2}
$$

(3) If $a_1, a_2(\cdot),\dots,a_k(\cdot)$ are defined according to [8; Definitions 5.6, 5.9], then it holds:

$$
\begin{aligned}
&T_1 > a_1\,,\\
&T_2 > a_2(T_1)\,,\\
&\quad\vdots\\
&T_k > a_k(T_1,\dots,T_{k-1})\,.
\end{aligned}
\tag{5.3.3}
$$

This assertion is evidently true for $k = 0$. Let now according to induction assumption the assertion be true for $k-1$. So, we have defined functions $v_1(\cdot),\dots,v_{k-1}(\cdot)$. Let us define function $w(\cdot)$ as a solution of the following problem:

$$
\begin{aligned}
&w''(x) \overset{\text{a.e.}}{=} \frac{\partial F_k}{\partial u_k^+}\big(x, v_1(x),\dots,v_{k-1}(x), 0, \overbrace{0,\dots,0}^{n-k \text{ times}}\big)\cdot \frac{w(x) + |w(x)|}{2}\,,\\
&\qquad w(0) = 0\,, \qquad w'(0) = u_k'(0) > 0\,.
\end{aligned}
\tag{5.3.4}
$$

Now we shall prove that $w(\cdot)$ has the zero point in interval $(0,T_k)$. By contradiction. Let $w(x) > 0$ for all $x \in (0,T_k)$. No we can use Lemma 2.3, in which we put $\langle a,b\rangle = \langle 0,T_k\rangle$, $u(\cdot) = w(\cdot)$, $v(\cdot) = u_k(\cdot)$, $f(x,\hat{u}_1,\hat{u}_2) = \frac{\partial F_k}{\partial u_k^+}\big(x, v_1(x),\dots,v_{k-1}(x), 0, \overbrace{0,\dots,0}^{n-k \text{ times}}\big) \cdot \frac{\hat{u}_1+|\hat{u}_1|}{2}$, $g(x,\hat{v}_1,\hat{v}_2) = F_k\big(x,u_1(x),\dots \dots, u_{k-1}(x), \hat{v}_1, u_{k+1}(x),\dots,u_n(x)\big)$. Assumptions (2.3.1), (2.3.2), (2.3.5), (2.3.6) and (2.3.7) easily follow from our assumptions. Lipschitz condition (2.3.3) easily follows from (3.0.7). Now it only remains to verify (2.3.4). So, let $x \in \langle 0,T_k\rangle$, and let $u > 0$. Then it holds:

$$F_k\big(x,u_1(x),\dots,u_{k-1}(x),u,u_{k+1}(x),\dots,u_n(x)\big)$$

$$\overset{(3.0.13),(3.0.3)}{\geq} u \cdot \frac{F_k\big(x,u_1(x),\dots,u_{k-1}(x),u,\overbrace{0,\dots,0}^{n-k \text{ times}}\big)}{u}$$

if for $k \le n-1$ we use (5.1.1) from Lemma 5.1 in which we let $u \to 0^+$ and for $k = n$ we use (3.0.10), then we can continue in the last inequality:

$$> \frac{\partial F_k}{\partial u_k^+}\big(x,u_1(x),\dots,u_{k-1}(x),0,\overbrace{0,\dots,0}^{n-k \text{ times}}\big) \cdot u$$

if besides (5.3.2) in induction assumptions we use (5.2.1) for $k \le n-1$ and (3.0.12) for $k = n$, then we can continue in inequality:

$$\geq \frac{\partial F_k}{\partial u_k^+}\big(x,v_1(x),\dots,v_{k-1}(x),0,\overbrace{0,\dots,0}^{n-k \text{ times}}\big) \cdot \frac{u+|u|}{2}\,.$$

Hence, for $u > 0$ we have:

$$g(x,u) > f(x,u)\,. \tag{5.3.5}$$

Assumption (2.3.4) is verified and we can use Lemma 2.3. Because $u_k(0) = 0 = w(0)$, $u_k'(0) = w'(0)$ holds, possibility (2.3.8) cannot be true. So statement (2.3.9) must be true and using $w'(0) = u_k'(0) > 0$ we obtain $\alpha = 1$. Then $u_k(\cdot) = w(\cdot)$ in the interval $\langle 0,T_k\rangle$, what by already proved condition (5.3.5) and equations (2.3.5) gives us a contradiction. So, we have proved that $w(\cdot)$ (solution of (5.3.4)) has the zero point in the interval $(0,T_k)$, what together with (5.3.1) in the induction assumption and with [8; Definition 5.6] for $k \ge 2$ or with [8; Definition 5.9] for $k = 1$ imply $T_k > a_k$. Hence, the second step of induction for condition (5.3.3) is proved. If we now use analogously [8; Definition 5.1] of mapping $\vec{R}_k$ we obtain:

$$\vec{R}_k(\alpha_1,\dots,\alpha_{k-1},\hat{\alpha}_k) = \big(T_1,\dots,T_{k-1},a_k(T_1,\dots,T_{k-1})\big)$$

where

$$\alpha_1 := v_1'(0) > 0\,,\ \ \alpha_2 := v_2'(0) > 0\,,\ \dots\,,\ \alpha_{k-1} := v_{k-1}'(0) > 0\,,\ \ \hat{\alpha}_k := 0\,.$$

From [8; Lemma 5.5] continuity of $\vec{R}_k(\cdot)$ follows, so if we let $\hat{\alpha}_k \to \infty$, then the right-hand side must converge to infinity and because according to the definition the first components of right-hand side are always $T_1, \dots, T_{k-1}$, the last component must grow up to infinity from what we obtain, if we consider

$$a_k(T_1, \dots, T_{k-1}) < T_k,$$

that for suitable $\alpha_k > 0$ the following statement will be true:

$$\vec{R}_k(\alpha_1, \dots, \alpha_{k-1}, \alpha_k) = (T_1, \dots, T_{k-1}, T_k).$$

From the definition it follows that we can complete $k-1$ functions $v_1, \dots, v_{k-1}$ to k functions so that (5.3.1) will hold for new k. Hence, induction step for condition (5.3.1) is proved. If $k \le n-1$, we need to do induction step also in condition (5.3.2): Functions $u_k(\cdot)$, $v_k(\cdot)$ are solutions of the following two problems:

$$\begin{gathered} v_k''(x) \overset{\text{a.e.}}{=} F_k\big(x, v_1(x), \dots, v_{k-1}(x), v_k(x), \overbrace{0, \dots, 0}^{n-k \text{ times}}\big), \\ v_k(0) = 0, \qquad v_k(T_k) = 0, \qquad v_k(x) > 0, \quad \text{for all } x \in (0, T_k) \end{gathered}$$

$$\begin{gathered} u_k''(x) \overset{\text{a.e.}}{=} F_k\big(x, u_1(x), \dots, u_{k-1}(x), u_k(x), u_{k+1}(x), \dots, u_n(x)\big), \\ u_k(0) = 0, \qquad u_k(T_k) = 0, \qquad u_k(x) > 0 \quad \text{for all } x \in (0, T_k). \end{gathered}$$

Now we can use Lemma 2.3, in which we put $\langle a, b \rangle = \langle 0, T_k \rangle$, $u(\cdot) = v_k(\cdot)$, $v(\cdot) = u_k(\cdot)$, $f(x, \hat{u}_1, \hat{u}_2) = F_k\big(x, v_1(x), \dots, v_{k-1}(x), \hat{u}_1, \overbrace{0, \dots, 0}^{n-k \text{ times}}\big)$, $g(x, \hat{v}_1, \hat{v}_2) = F_k\big(x, u_1(x), \dots, u_{k-1}(x), \hat{v}_1, u_{k+1}(x), \dots, u_n(x)\big)$. Let us verify assumptions of Lemma 2.3: (2.3.1) follows from (3.0.2). Condition (2.3.3) follows from (3.0.6) and from (3.0.7). Conditions (2.3.5), (2.3.6) and (2.3.7) hold evidently. Let us verify assumption (2.3.2): So, let $u > 0$, $\alpha > 1$. Using Lemma 5.1 we easily obtain

$$\begin{aligned} &F_k\big(x, v_1(x), \dots, v_{k-1}(x), \alpha \cdot u, \overbrace{0, \dots, 0}^{n-k \text{ times}}\big) \\ &> \alpha \cdot F_k\big(x, v_1(x), \dots, v_{k-1}(x), u, \overbrace{0, \dots, 0}^{n-k \text{ times}}\big) \end{aligned} \tag{5.3.6}$$

from what we have (2.3.2). Now it remains to verify (2.3.4):

$$\begin{aligned} &F_k\big(x, u_1(x), \dots, u_{k-1}(x), u, u_{k+1}(x), \dots, u_n(x)\big) \\ \overset{(3.0.13),(3.0.3)}{\ge}\; &F_k\big(x, u_1(x), \dots, u_{k-1}(x), u, \overbrace{0, \dots, 0}^{n-k \text{ times}}\big) \end{aligned}$$

by using (5.3.2) and induction assumption and assumption (3.0.11)

$$\geq F_k\big(x, v_1(x), \dots, v_{k-1}(x), u, \overbrace{0, \dots, 0}^{n-k \text{ times}}\big)$$

and (2.3.4) is verified. So Lemma 2.3 can be used and it gives us that at least one of the following cases must be true:

1. Assertion (2.3.8) is true. In this case $u_k(x) < v_k(x)$ holds for all $x \in (0, T_k)$, and the induction step in condition (5.3.2) is complete.

2. Assertion (2.3.9) is true. In this case we obtain (using notation from Lemma 2.3) there exists $\alpha \geq 1$: $v(\cdot) = \alpha \cdot u(\cdot)$. We shall eliminate the case $\alpha > 1$. By contradiction: Let $x \in (0, T_k)$.

$$v''(x) \overset{\text{a.e.}}{=} g\big(x, v(x)\big) = g\big(x, \alpha \cdot u(x)\big) \overset{(2.3.4)}{\geq} f\big(x, \alpha \cdot u(x)\big)$$

by use (2.3.6), $u(x) > 0$ and $\alpha > 1$ in (5.3.6) we can continue

$$> \alpha \cdot f\big(x, u(x)\big) \overset{\text{a.e.}}{=} \alpha \cdot u''(x) = v''(x)\,.$$

Because this holds for almost every $x \in (0, T_k)$, so we got contradiction which we needed. Hence $\alpha = 1 \implies u_k(\cdot) = v_k(\cdot)$, and induction step in condition (5.3.2) is complete.

So, we have shown validity of all three induction assumptions also for this new k. From induction principle it follows, that the proved assertion holds also for $k = n$ and then from condition (5.3.3) and from [8; Definition 5.12] we obtain $(T_1, \dots, T_n) \in \Omega_n^o$, what we needed to prove. □

The following lemma can be applied in order to simplify assumption (3.0.8) by easier assumption (5.4.1), which can be verified separately for every component F_i.

Lemma 5.4. *Let $F_1, \dots, F_n$ fulfil* (3.0.13) *and also the following condition:*

$$\begin{aligned} &\forall\, k \in \{1, \dots, n\} \quad \forall\, T > 0 \quad \forall\, R > 0 \quad \exists\, C > 0 \\ &\quad \Big(\forall\, x \in \langle 0, T\rangle \quad \forall\, u_i \in \langle 0, R\rangle,\ i = 1, \dots, k-1 \quad \forall\, u_k \geq C \\ &\qquad\qquad F_k(x, u_1, \dots, u_{k-1}, u_k, \underbrace{0, \dots, 0}_{n-k \text{ times}}) \geq 0\Big)\,. \end{aligned} \tag{5.4.1}$$

Then $F_1, \dots, F_n$ fulfil also (3.0.8).

P r o o f. Let $T > 0$ be chosen arbitrarily, but fixed. We shall construct some functions $c_1(\lambda), \dots, c_n(\lambda)$ which will fulfil all conditions in form (3.0.8). To do it, we prove the following statement for $k = 1, \dots, n$.

Statement:
There exist k functions $c_1(\cdot),\dots,c_k(\cdot)\colon \langle 0,\infty) \mapsto (0,\infty)$ such that

$$\forall j \in \{1,\dots,k\} \qquad \lim_{\lambda\to\infty} c_j(\lambda)=\infty$$

and

$$\forall x \in \langle 0,T\rangle \quad \forall \lambda \geq 0 \quad \forall u_i \in \langle 0, c_i(\lambda)\rangle,\ i=1,\dots,j-1$$
$$F_j\big(x,u_1,\dots,u_{j-1},c_j(\lambda),\ \underbrace{0,\dots,0}_{n-j \text{ times}}\big) \geq 0\,.$$

We shall prove this by induction.

The case ($k=1$).
Let us choose $C>0$ according to assumption (5.4.1), where we put $k=1$ and R arbitrary (for $k=1$ it has no meaning). If we choose continuous function $c_1(\cdot)$ such that

$$\forall \lambda \geq 0 \quad c_1(\lambda) \geq C \qquad \text{and} \qquad \lim_{\lambda\to\infty} c_1(\lambda)=\infty\,,$$

then the statement will be true for $k=1$.

Induction step ($2 \leq k \leq n$).
We assume that the statement is true for $k-1$, so we have already constructed functions $c_1(\cdot),\dots,c_{k-1}(\cdot)$. Let us put for every $l \in \mathbb{N}$

$$R_l := \max\big\{c_i(\lambda)\,;\ 1 \leq i \leq k-1\,,\ l-1 \leq \lambda \leq l\big\}\,. \tag{5.4.2}$$

We shall choose C_l for every $l \in \mathbb{N}$ according to assumption (5.4.1), in which instead of R we shall put R_l. Then it will be true:

$$\forall l \in \mathbb{N} \quad \forall x \in \langle 0,T\rangle \quad \forall u_i \in \langle 0,R_l\rangle,\ i=1,\dots,k-1 \quad \forall u_k \geq C_l$$
$$F_k(x,u_1,\dots,u_{k-1},u_k,\ \underbrace{0,\dots,0}_{n-k \text{ times}}) \geq 0\,. \tag{5.4.3}$$

If we now construct continuous function $c_k(\cdot)$ such that

$$\lim_{\lambda\to\infty} c_k(\lambda)=\infty \qquad \text{and} \qquad \forall l \in \mathbb{N} \quad \forall \lambda \in \langle l-1,l\rangle \quad c_k(\lambda) \geq C_l\,,$$

then by (5.4.2), (5.4.3) we easily verify that functions $c_1(\cdot),\dots,c_k(\cdot)$ fulfil the statement for k. Hence, induction step is complete and the statement holds also for $k=n$. By it and by assumption (3.0.13) we can easily verify that functions $c_1(\cdot),\dots,c_n(\cdot)$ fulfil all conditions from (3.0.8) for the above chosen T. Hence, the lemma is proved. □

6. Example of weak assumptions on F_i

In this section we present a special form of functions F_i, which we can apply in already proved [8; Theorem 6.1] and in Theorem 5.3 in present paper. We summarize it in the following theorem:

THEOREM 6.1[2]**.** *Let the functions F_k have the following form for all $(x, u_1, \dots, u_n) \in \mathbb{R}_0^+ \times E_n$:*

$$F_k(x, u_1, \dots, u_n) \stackrel{\text{def}}{=} \begin{cases} u_k \cdot [f_k(x, u_1, \dots, u_k) + g_k(x, u_1, \dots, u_n)] & \text{for } k \le n-1\,, \\ u_n \cdot f_n(x, u_1, \dots, u_n) & \text{for } k = n\,, \end{cases}$$

where functions $f_k \in C^1\big(\mathbb{R}_0^+ \times (\mathbb{R}_0^+)^k, \mathbb{R}\big)$ fulfil the following conditions:

$$\forall\, k \in \{1, \dots, n\} \quad \forall\, (x, u_1, \dots, u_k) \in \mathbb{R}_0^+ \times E_k,\ u_k > 0 \qquad \frac{\partial f_k}{\partial u_k^+}(x, u_1, \dots, u_k) > 0\,,$$

$$\forall\, k \in \{2, \dots, n\} \quad \forall\, i \in \{1, \dots, k-1\} \quad \forall\, (x, u_1, \dots, u_k) \in \mathbb{R}_0^+ \times E_k \qquad \frac{\partial f_k}{\partial u_i^+}(x, u_1, \dots, u_k) \le 0\,,$$

$$\forall\, k \in \{1, \dots, n\} \quad \forall\, x \in \mathbb{R}_0^+ \quad \forall\, u_i \in \mathbb{R}_0^+,\ i = 1, \dots, k-1 \qquad \lim_{u \to \infty} f_k(x, u_1, \dots, u_{k-1}, u) > 0\,,$$

and where functions $g_k \in C^1\big(\mathbb{R}_0^+ \times (\mathbb{R}_0^+)^n, \mathbb{R}\big)$ fulfil:

$$\forall\, k \in \{1, \dots, n-1\} \quad \forall\, (x, u_1, \dots, u_n) \in \mathbb{R}_0^+ \times E_n \qquad g_k(x, u_1, \dots, u_n) \ge 0\,,$$

$$\forall\, k \in \{1, \dots, n-1\} \quad \forall\, (x, u_1, \dots, u_n) \in \mathbb{R}_0^+ \times E_n,\ u_k \cdot u_{k+1} \cdot \dots \cdot u_n = 0 \qquad g_k(x, u_1, \dots, u_n) = 0\,.$$

We shall define functions F_k for all $k \in \{1, \dots, n\}$ and for all $(x, u_1, \dots, u_n) \in \mathbb{R}_0^+ \times \mathbb{R}^n$ in the following way:

$$F_k(x, u_1, \dots, u_n) \stackrel{\text{def}}{=} F_k\left(x, \tfrac{u_1 + |u_1|}{2}, \dots, \tfrac{u_n + |u_n|}{2}\right).$$

Then problem (3.0.1) *has a positive solution if and only if $(T_1, \dots, T_n) \in \Omega_n^o$, where the set Ω_n^o is defined so as in* [8; Definition 5.12] *and practical "algorithm" for verifying whether $(T_1, \dots, T_n)$ belong to this set is shown at the end of Section* 5 *in the first part of this paper* [8].

[2] In [8] mentioned as Theorem 8.1.

P r o o f . Let us verify assumptions of [8; Theorem 6.1] and Theorem 5.3. Conditions (3.0.2), (3.0.3) – (3.0.7), (3.0.9) – (3.0.13) can be easily proved with using assumptions which are stated on f_k, g_k. From the first and the third assumption on f_k we can easily verify condition (5.4.1) when we use compactness of the set $\langle 0,T\rangle \times \langle 0,R\rangle^{k-1}$ in it. Hence, according to Lemma 5.4 also condition (3.0.8) is true. All assumptions of [8; Theorem 6.1] and Theorem 5.3 are verified. □

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