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ARROWS IN THE "FINITE PRODUCT THEOREM FOR CERTAIN EPIREFLECTIONS" OF R. FRIČ AND D. C. KENT

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ABSTRACT. The t-envelope, a kind of completion for t-embedded convergence spaces devised by D. C. Kent and G. D. Richardson, has its origins in J. Novák's sequential envelope. Recently, R. Frič and D. C. Kent found that the t-envelope preserves finite Cartesian products, under a rather technical condition. By placing their ideas in a categorial framework, we clarify their work, improve and extend the entire theory, and discover the meaning of their technical condition: it makes their t-embedded spaces form a Cartesian-closed category.

§0. Introduction

In 1979, Frič, K. McKennon and Richardson [FMR] used sequential continuous convergence to construct another sequential envelope, equivalent to Novák's one $[N_1]$, $[N_2]$. In the same year, Kent and Richardson [KR] devised their t-envelopes, starting with a hereditary coreflector (in their terms, an HIU-modifier) ϑ on the category of filter convergence spaces. Then in 1986, Frič derived the product theorem for the sequential envelope, and finally in 1988, Frič and Kent found the product theorem for the t-envelope [FK; Theorem 3.10]: *Assume that products of t-embedded spaces are ϑ-spaces. Then [the t-envelope] E_t is finitely productive.*

Like E. Binz's theory of c-embedded spaces [B], the t-theory of [KR] and [FK] relies on continuous convergence and real-valued functions. However, neither theory needs all these limitations. To show this, we build a theory for any coreflector around almost any space at all, expressing it in the language of [KR] and [FK], and deriving the same results (often, more simply).

In §1, we recall some significant features of continuous convergence, and in §2, we do the same for convergence in general, and for coreflectors. In §3, we

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introduce the main ideas of t-embedded and t-complete spaces. This leads us to t-envelopes and the completion theory for t-embedded spaces described in §4. We review the product theorem in §5 and work on Cartesian-closedness in §6.

Throughout, we compare our theory with earlier ones, applying it to familiar examples. None of our techniques requires deep results from recondite sources: [B; §0] covers the basic ideas of filter convergence, as distinct from topology or sequential convergence; and categorially, we seldom go beyond the universal properties of subspaces and products.

§1. Background – continuous convergence

Let us summarize some key features of continuous convergence, mainly to display our notation – for details and definitions, see [B], [BK] or many other sources. To exploit the common properties of several familiar classes of convergence spaces, we assume only what we need in order to get results. In particular, the categories of filter convergence spaces[1)], of limit spaces (under the axioms given in [B]), of Choquet spaces, and of c-embedded spaces [B] all satisfy our assumptions, marked **[ass]**.

For any spaces W, X and Y in the category **cvk** of convergence spaces (in that symbol, the **k** acts as a reminder of Kent's axioms), continuous convergence transforms the set $C(X,Y)$ of all arrows[2)] from X to Y into the space $C_c(X,Y)$. By definition, $C_c(X,Y)$ carries the coarsest convergence making the evaluation map ω from $C(X,Y) \times X$ to Y continuous – as usual, $\omega\colon f \times x \mapsto f(x)$.

For any full subcategory **cv** of **cvk**, one calls W a *natural exponential object* in **cv** if $C_c(Z,W) \in$ **cv** for all $Z \in$ **cv**. Obviously, **cvk** itself consists entirely of natural exponential objects; so do all the other categories mentioned above. Now let us state some of our basic assumptions:

> **[ass]** **cv** is a full subcategory of **cvk**; it contains the finite discrete spaces; it inherits its subspaces from **cvk**; it is closed under finite Cartesian products and under homeomorphisms; it consists entirely of natural exponential objects – in short, **cv** is Cartesian-closed [AHS].

We emphasize: though most of the results in this section were first established for limit spaces, they remain true for **cv**. So from now on, all our spaces belong to **cv**.

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1) These spaces fulfil Kent's axioms:

(1) point ultrafilters converge to their base point; and
(2) if a filter converges to some point, then all finer filters converge to the same point.

2) We often refer to *arrows* instead of continuous maps.

Each map $g\colon W \to C(X,Y)$ defines the map $\widetilde{g} = \omega \circ (g \times \mathrm{id})\colon W \times X \to Y$. This construction leads to the universal property that characterizes continuous convergence:

1.1. *A map $g\colon W \to C_c(X,Y)$ is continuous if and only if $\widetilde{g}\colon W \times X \to Y$ is continuous.*

Dually, each arrow $f\colon W \times X \to Y$ defines an arrow $\underline{f}\colon W \to C_c(X,Y)$. These procedures (often known as conversion) invert one another.

1.2. THE CONVERSION THEOREM. *For all spaces W, X and Y, the rules of conversion $g \mapsto \widetilde{g}$ and $f \mapsto \underline{f}$ establish a homeomorphism between the spaces $C_c\big(W, C_c(X,Y)\big)$ and $C_c(W \times X, Y)$.*

1.3. *For x in X and $f \in C(X,Y)$, the formula $@(x) = \widehat{x}\colon f \mapsto f(x)$ defines $@$ as a continuous map*[3] *from X to $C_c\big(C_c(X,Y),Y\big)$.*

For x in X and $y \in Y$, the formula $k(y)(x) = y$ defines k as a continuous map[4] from Y to $C_c(X,Y)$. In fact, since $@(x) \circ k = \mathrm{id}_Y$ for all $x \in X$,

1.4. *Y is a retract of $C_c(X,Y)$.*

In Hausdorff topology, retracts behave nicely; they behave just as well in convergence theory too.

1.5. *Let S be a retract of a Hausdorff space T, defined by a section $j\colon S \to T$. Then j embeds S onto a closed subset of T.*

Now, for all arrows $e\colon W \to X$, $f\colon X \to Y$ and $g\colon Y \to Z$ in **cv**, let $f^*(g) = g \circ f$ and $f_*(e) = f \circ e$. This sets up the familiar functorial behaviour:

1.6. *$f^*\colon C_c(Y,Z) \to C_c(X,Z)$ and $f_*\colon C_c(W,X) \to C_c(W,Y)$ are both continuous.*

1.7. THE C_c-EMBEDDING THEOREM. *Let j embed W in Z. Then for all spaces Y,*

(a) *j_* embeds $C_c(Y,W)$ in $C_c(Y,Z)$, and*
(b) *$j_*\big(C(Y,W)\big)$ is closed in $C_c(Y,Z)$ if Z is Hausdorff and $j(W)$ is closed in Z.*

Proof. If I knew a reference to this, I would omit the proof. We use the universal property of subspaces. Let $h\colon H \to C(Y,W)$ be a map such that $j_* \circ h\colon H \to C_c(Y,Z)$ is an arrow in **cv**. On conversion, we get the arrow

[3] To verify its continuity, convert it: $\widetilde{@}\colon X \times C_c(X,Y) \to Y$ is continuous, being the transpose of ω.

[4] To verify its continuity, just convert it – $\widetilde{k}$ is the projection from $X \times Y$ onto Y.

$j \circ \widetilde{h}\colon H \times Y \to Z$. Thus $\widetilde{h}\colon H \times Y \to W$ in **cv**, and on converting again, $h\colon H \to C_c(Y, W)$, as desired.

Part (b) follows directly from an obvious counterpart of (a) for pointwise convergence. For all spaces Y and Z, let $C_s(Y, Z)$ denote the set $C(Y, Z)$, equipped with pointwise convergence: then j_* embeds $C_s(Y, W)$ in $C_s(Y, Z)$ for all spaces Y. Now let $j(W)$ be closed in Z, let $K = j_*\big(C(Y, W)\big)$, and let g belong to the closure of K in $C_s(Y, Z)$. Then for some filter ψ on $C(Y, W)$, the image $\theta = j_*(\psi) \to g \in C_s(Y, Z)$. Now by definition of pointwise convergence, $\widehat{y}(\theta) \to g(y)$ in Z, for all $y \in Y$, while $\widehat{y}(\theta) = j\big(\widehat{y}(\psi)\big)$, a filter on $j(W)$. Thus $g(y) \in j(W)$ for all $y \in Y$, since $j(W)$ is closed. This means that one can regard g as an arrow from Y to W. More precisely, $g = j \circ f$, an equation that not only defines $f\colon Y \to W$, but also establishes its continuity. So, $g = j_*(f) \in K$. In short, K is closed in $C_s(Y, Z)$, and in $C_c(Y, Z)$ too. □

1.8. PROPOSITION. *For all spaces V and W, if W is topologically Hausdorff,*[5] *then so are $C_s(V, W)$ and $C_c(V, W)$.*

P r o o f. Let W be topologically Hausdorff, and suppose $f \neq g \in C(V, W)$. Choose $v \in V$ such that $f(v) \neq g(v)$, and find disjoint open sets F and G in W containing $f(v)$ and $g(v)$ respectively. Now let U and V be the inverse images of F and G under $\widehat{v}$. Then under pointwise convergence, U and V are disjoint open sets with $f \in U$ and $g \in V$. Thus $C_s(V, W)$ is topologically Hausdorff. So is $C_c(V, W)$, because continuous convergence is finer than pointwise convergence. □

§2. Background – coreflection

Now we can state and emphasize another basic assumption:

[ass] ϑ is a *coreflector* or more correctly, a coreflective modifier, on **cv**.

This means: ϑ is a functor from **cv** to **cv**, such that for all spaces $Z \in$ **cv** and all arrows $f\colon X \to Y$ in **cv**,

(M) Z and ϑZ overlie the same set, and $f = \vartheta f$,
(C) $\vartheta Z \geq Z$, and $\vartheta\vartheta Z = \vartheta Z$.

Here and elsewhere, we write $A \geq B$ when $A \subset B$, as sets, and the inclusion $j\colon A \to B$ is continuous. Let **tcv** denote the class of all ϑ-spaces – namely, those such that $Z = \vartheta Z$. Under these assumptions, **tcv** forms a full coreflec-

[5] The open subsets of a space W form a topology, and its *topological modification* τW carries that topology. The *topological closure* of a subset of W is its closure in τW. One calls W *topologically Hausdorff* if it has at least two points and τW is Hausdorff.

tive subcategory of **cv**. The original work in [KR] and [FK] requires an extra assumption, that ϑ is *hereditary* too:[6)]

(H) if $X \hookrightarrow Y$, then $\vartheta X \hookrightarrow \vartheta Y$, both subspaces taken in **cv**.

Some simple coreflectors based on set-theoretic constructions are hereditary. Examples include

- the identity modifier ι,
- the discrete modifier δ, which assigns to each space Z, the space δZ in which $\zeta \to z$ if and only if ζ is the ultrafilter over z,
- the sequential modifier ψ, which assigns to each space Z, the space ψZ in which $\zeta \to z$ if and only if $\zeta \supset \phi$, where $\phi \to z$ in Z and ϕ is the Fréchet filter of some sequence, and
- the antisequential modifier ϖ, which assigns to each space Z, the P-space ϖZ in which $\zeta \to z$ if and only if $\zeta \supset \phi$, where $\phi \to z$ in Z and ϕ is closed under countable intersections.
- However, the local compactifier κ is a coreflector, but not hereditary: it assigns to each Hausdorff space Z, the convergence inductive limit of its compact Hausdorff subobjects.

Most of these have received some attention. Novák, Frič and others deal with ψ and its envelopes, while ι, κ and ϖ appeared in [KR; §4], together with the ι- and ϖ-envelopes.

2.1. *For each coreflector ϑ and all spaces $X, Y \in$ **cv**,*

(a) *$C(X,Y) \subset C(\vartheta X, \vartheta Y) \subset C(\vartheta X, Y)$, and*

(b) *$C(X,Y) = C(X, \vartheta Y)$ if X is a ϑ-space.*

We adopt the notation of [KR] and [FK]: let $C_t(X,Y) = \vartheta C_c(X,Y)$ for all spaces $X, Y \in$ **cv**. This preserves the functorial behaviour noted in 1.6 – for all $W, Z \in$ **cv** and each arrow $f \colon X \to Y$,

2.2. *$f^* \colon C_t(Y,Z) \to C_t(X,Z)$ and $f_* \colon C_t(W,X) \to C_t(W,Y)$ are arrows in* **cv** *and in* **tcv**.

Several difficulties that [FK] strove to overcome lie in a simple fact – subspaces and products in **tcv** need not coincide with those in **cv**. Heredity disposed of their subspace problem, but they often assumed ... *that products of t-embedded spaces are ϑ-spaces* ... After some experiment with hybrid theories, in which certain key features (subspaces, products, and the t-constructs described below) are tied to **cv** or **cvk**, we took a categorially cleaner approach: we work within **tcv**, and then we look for the links to **cv**.

[6)] They used the term *HIU-modifier* instead of hereditary coreflector; our (C) combines their axioms (I) and (U), letters which stand for *idempotent* and *upward*, I believe.

This means: we use universal properties to define embeddings and products in **tcv**. So, an arrow $j\colon S \to T$ in **tcv** embeds S in T if and only if for each $W \in$ **tcv**, a map f from W to S is continuous if the composite $j \circ f\colon W \to T$ is an arrow in **tcv**. This gives us a form of heredity at no cost – embeddings lift from **cv** to **tcv**, and sections work better still.

2.3. *If $S, T \in$* **cv** *and j embeds S in T in* **cv**, *then j embeds ϑS in ϑT in* **tcv**.

2.4. *If $j\colon S \to T$ is a section in* **cv** *and $T \in$* **tcv**, *then $S \in$* **tcv** *too.*

So, to obtain subspaces in **tcv**, simply form the subspace in **cv**, then apply ϑ to it. Because the next two facts rely more on universal properties than on details of convergence, they apply equally to **cv** and to **tcv**.

2.5. The Embedding Lemma. *Take arrows $v\colon S \to V$ and $p\colon V \to U$. Let $u = p \circ v$ and suppose that u embeds S in U. Then v embeds S in V.*

Proof. Take any map q from a space Z to S that makes $v \circ q$ continuous. Then $p \circ v \circ q = u \circ q\colon Z \to U$. So $q\colon Z \to S$, as $u\colon S \hookrightarrow U$. Hence v embeds S in V, as desired. □

2.6. The Extremal Lemma. *Take embeddings u and v, with $u = p \circ v$ as above. Suppose that $u(S)$ is closed in U and that V is Hausdorff. Then $v(S)$ is closed in V.*

Proof. Let $H = u(S)$ and let K be the closure of the set $v(S)$ in V. Consider this diagram.

$$\begin{tikzcd}
S \arrow[r, shift left, "\overline{u}"] \arrow[dr, "\overline{v}"'] & H \arrow[l, dashed, shift left, "h"] \arrow[r, hook, "\varepsilon"] & U \\
& K \arrow[u, dashed, "q"] \arrow[r, hook, "\eta"] & V \arrow[u, "p"']
\end{tikzcd}$$

The inclusions $\varepsilon\colon H \hookrightarrow U$ and $\eta\colon K \hookrightarrow V$ make H and K subspaces of U and V; they also define arrows $\overline{u}\colon S \to H$ and $\overline{v}\colon S \to K$ such that $u = \varepsilon \circ \overline{u}$ and $v = \eta \circ \overline{v}$. Now as a map, $\overline{u}\colon S \to H$ has an inverse, h. Obviously, h is continuous,[7] since u is an embedding and $u \circ h = \varepsilon$.

By continuity, $p(K) \subset H$, as H is closed. This enables us to define a map $q\colon K \to H$ such that $\varepsilon \circ q = p \circ \eta$, and because of this, q is continuous. Let $k = \overline{v} \circ h$. Now $\varepsilon \circ q \circ k = p \circ \eta \circ k = p \circ v \circ h = \varepsilon$. Hence $q \circ k = \mathrm{id}_H$, as ε is monic. In other words, k is a section. Now as V is Hausdorff and $k(H) = v(S)$, the set $v(S)$ is both closed and dense in K, by 1.5. In short, $K = v(S)$, as desired. □

[7] So, every embedding in **tcv** is a product of a homeomorphism and an inclusion.

Thus in categorial language, closed embeddings in Hausdorff spaces or Hausdorff ϑ-spaces are extremal monomorphisms, just as in topology (even though in **tcv**, we no longer use the usual idea of embedding). Compare this with 1.5: sections are extremal monomorphisms.

Deal with products in much the same way: for clarity, let $\times$ and $\sqcap$ stand for the product operators in **cv** and in **tcv** respectively. Let T be the product in **cv** of a non-trivial collection Υ of **cv**-spaces: $T = \times\Upsilon$. Then the family $\{\vartheta Y : Y \in \Upsilon\}$ has a product in **tcv**, namely, ϑT.

2.7. *If $T \in$ **tcv**, then each member of Υ belongs to **tcv** too, and if each member of Υ belongs to **tcv**, then $\vartheta T = \sqcap\Upsilon$.*

P r o o f. Choose $S \in \Upsilon$, and let $p\colon T \to S$ be the projection. For every other $R \in \Upsilon$, choose $q_R \in R$. Use these choices to define the section $q\colon S \to T$ such that $\mathrm{id}_S = p \circ q$. Now suppose $T \in$ **tcv**. Then by 2.4, S is a ϑ-space. We leave the rest as an exercise. □

Let $e_v\colon S_v \to T_v$ be arrows in **tcv**, for all v in an index set Υ, let S and T denote their products in **tcv**, and let $e\colon S \to T$ be the usual product map which commutes with the projections p_v and q_v (in other words, $q_v \circ e = e_v \circ p_v$, as in the diagram below).

2.8. *In this situation,*

(a) *e is an arrow in* **tcv**,
(b) *if each e_v is an embedding, then so is e, and*
(c) *if each image set $e_v(S_v)$ is closed in T_v, then $e(S)$ is closed in T.*

P r o o f. The commutation rules and the universal property of T as a product ensure the continuity of e. Next, let $W \in$ **tcv**, and take a map $w\colon W \to S$ making $e \circ w$ continuous:

$$\begin{tikzcd} W \arrow[r, dashed, "w"] & S \arrow[r, "e"] \arrow[d, "p_v"'] & T \arrow[d, "q_v"] \\ & S_v \arrow[r, "e_v"'] & T_v \end{tikzcd}$$

Now consider the arrows $q_v \circ e \circ w = e_v \circ p_v \circ w$: since each e_v is an embedding, the maps $p_v \circ w$ are all continuous. So by the universal property of S as a product, w is continuous. Finally, as $e(S)$ is the intersection of the inverse images under q_v of the closed sets $e_v(S_v)$, it is closed too. □

§3. Embedded and complete spaces

The real line $\mathbb{R}$, with its usual topology, plays an integral part in N o v á k 's work on the sequential envelope, in B i n z 's work on continuous convergence and c-embeddedness, and in recent work on more general envelopes. Obviously, $\iota\mathbb{R} = \psi\mathbb{R} = \kappa\mathbb{R}$, but $\mathbb{R}$ is neither a P-space nor discrete.

However, $\mathbb{R}$ has more properties than we need. To clarify this, choose any space $B \in$ **tcv** whatsoever, call it the *base*, and let B take the place of $\mathbb{R}$ in the theory. Now we can state and emphasize the last of our basic assumptions:

[**ass**] the base B is a topologically Hausdorff ϑ-space.

Without this axiom, some easier parts of the theory remain, but the envelopes lose their nicer properties.

As in the usual theory based on $\mathbb{R}$, the base stays out of sight – in other words, $C(Z)$ stands for $C(Z,B)$ for all spaces Z. This makes it easy for us to survey the results of [KR] and [FK] on t-embedded and t-complete spaces, and to present them for all bases, not just $\mathbb{R}$.

Having chosen the base B, we call the rule $@ = @_Z : z \mapsto \widehat{z}$, the *carrier* of Z. Let $\widehat{Z}$ denote its image, the set $@(Z)$. When $B = \mathbb{R}$ and $\vartheta = \iota$, the carrier $@_Z$ of each space Z is continuous and $\widehat{Z}$ is closed in $C_c\bigl(C_c(Z)\bigr)$ [B; Theorem 17]. More generally, the point evaluations $@(z) : C_t(Z) \to B$ are always continuous. However, the carrier $@ : Z \to C_t\bigl(C_t(Z)\bigr)$ may be discontinuous and $\widehat{Z}$ need not be closed, even when $B = \mathbb{R}$. Most modifiers exhibit this behaviour to some extent: under the discrete modifier δ, the carrier is continuous only for discrete spaces, but its image is always closed; under the sequential modifier ψ, both possibilities can occur independently $[N_2]$.

We deal with discontinuous carriers by definition: we call a space *t-admissible* if its carrier is continuous.

3.1. THEOREM. *All ϑ-spaces are t-admissible.*

P r o o f . Let us see conversion in action, line-by-line. For any ϑ-space Z, let q be the transpose of the evaluation map, under which $q(z,f) = f(z)$. Then

$q : Z \times C_c(Z) \to B$ in **cv**, because $Z \times C_c(Z) \longleftrightarrow C_c(Z) \times Z$,
$q : Z \times C_t(Z) \to B$, because $C_t(Z) \geq C_c(Z)$,
$\underline{q} : Z \to C_c\bigl(C_t(Z), B\bigr) = C_c\bigl(C_t(Z)\bigr)$, under conversion in **cv**,
$\underline{q} : Z \to C_t\bigl(C_t(Z)\bigr)$, because Z is a ϑ-space.

Finally, simply verify that $\underline{q} = @$. □

Given a ϑ-space Z, we call it *t-embedded* if its carrier $@_Z$ is an embedding in **tcv**; we call it *t-complete* if, in addition, $\widehat{Z}$ is closed in $C_t\bigl(C_t(Z)\bigr)$. These spaces[8] form the categories **tem** and **tcm** respectively, full subcategories of **cv** and of **tcv**. By assumption, the base B is topologically Hausdorff, and by 1.8, many other spaces share this property.

3.2. *Every t-embedded space is topologically Hausdorff.*

Binz's original c-theory forms a special case, in which the coreflector is the identifier, ι. A space is i-embedded in our sense if and only if it is c-embedded [B], and because $\widehat{Z}$ is always closed in $C_c(C_c(Z))$, each i-embedded space is i-complete. In the c-theory, the function spaces $C_c(Z)$ are c-embedded, completely regular topological spaces are c-embedded, and in particular, the base $\mathbb{R}$ is c-embedded. In our t-theory, we have to work harder for less information.

3.3. THEOREM. *Let Z be t-admissible, and let $C = C_t(Z)$. Then*

(a) $@_Z{}^* \circ @_C = \mathrm{id}_C$,
(b) *$C_t(Z)$ is t-complete, and*
(c) *the base B is t-complete.*

Proof. To prove (a), check that $@_Z{}^* \circ @_C(f) = f$, for all $f \in C$. To do this, take $z \in Z$ and calculate:

$$@_Z{}^* \circ @_C(f)(z) = @_C(f)\big(@_Z(z)\big) = @_Z(z)(f) = f(z).$$

By (a), $C_t(Z)$ is a retract of $C_t\big(C_t(C)\big)$, and the latter is Hausdorff, by 3.2. So by 1.5, $@_C$ embeds C as a closed subspace of $C_t\big(C_t(C)\big)$. Thus $C = C_t(Z)$ is t-complete. Now to prove (c), take a singleton space Z. Clearly, B and $C_c(Z)$ are homeomorphic. Thus $B = \vartheta B \longleftrightarrow C_t(Z)$. Being discrete, Z is a ϑ-space. So by (b), $C_t(Z)$ and B are both t-complete. $\square$

3.4. *Let $i\colon Z \to I$ and $j\colon I \to C_t\big(C_t(Z)\big)$ in* **cv**. *Suppose $@ = j \circ i$ and $i^*\colon C(I) \to C(Z)$ is injective. Then $i^*\colon C_t(I) \to C_t(Z)$ is a homeomorphism.*

Proof. Because $@ = j \circ i\colon Z \to C_t\big(C_t(Z)\big)$, the space Z is t-admissible. By contravariance, $(j \circ i)^* = i^* \circ j^*$ and, by 3.3, $@_Z{}^* \circ @_C = \mathrm{id}_C$. Now i^* is injective and $i^* \circ j^* \circ @_C = \mathrm{id}_C$. Thus i^* has an inverse,[9] $j^* \circ @_C$ is that inverse and further, the inverse is continuous. $\square$

Frič and Kent [FK] discussed the permanence properties of **tem** and of **tcm**. We obtain similar results in our more general situation: ours look nicer (and

[8] Guided by [FK], one might call a space t-embedded if its carrier were an embedding in **cv**. The two definitions agree, for hereditary coreflectors. In other cases, t-embedded spaces might not be ϑ-spaces. For example, consider the case of the local compactifier κ: though all k-complete spaces would be locally compact, as desired, most k-embedded ones would not be locally compact.

[9] An injective map $I\colon A \to B$ with a right inverse $J\colon B \to A$ has an inverse, J.

have an easier proof) because we work in **tcv**, but they have a drawback: they do not apply to ordinary subspaces or Cartesian products unless these objects actually belong to **tcv**. (The need to check this appears explicitly in their results about products, but not in those about subspaces – recall their assumption of heredity.)

To derive all this, we need several facts, some from §2. Take an arrow $f\colon X \to Y$ in **cv**, and let f^{**} denote the map from $C_c(C_c(X))$ to $C_c(C_c(Y))$ or from $C_t(C_t(X))$ to $C_t(C_t(Y))$. In both cases, f^{**} is continuous, even though the commutation law below does not imply the continuity of the carriers:

3.5. $@_Y \circ f = f^{**} \circ @_X$.

Next, we assign to each t-admissible space Z its *t-modification* $A_t(Z)$, the set $\widehat{Z}$ as a **tcv**-subspace of $C_t(C_t(Z))$; we call A_t the *t-embedder*. Further, let $a_Z\colon Z \to A_t(Z)$; $z \mapsto \widehat{z}$. Clearly, a_Z is continuous; and by Embedding Lemma 2.5, it is a homeomorphism in **tcv** if and only if Z is t-embedded.

3.6. *For each ϑ-space Z and each arrow $f\colon X \to Y$ in* **tcv**,

(a) *the arrow $a_Z{}^*\colon C_t(A_t(Z)) \to C_t(Z)$ is a homeomorphism, and*
(b) *exactly one arrow $A_t(f)\colon A_t(X) \to A_t(Y)$ makes $A_t(f) \circ a_X = f \circ a_Y$.*

P r o o f. Take a ϑ-space Z. Since a_Z is surjective, $a_Z{}^*$ is injective. So by 3.4, $a_Z{}^*$ is a homeomorphism. The second claim follows directly from 3.5. □

3.7. THEOREM. *The t-embedder A_t reflects* **tcv** *onto* **tem**.

P r o o f. Take a ϑ-space Z, a t-embedded space E, and an arrow $f\colon Z \to E$. By the previous result, the space $A_t(Z)$ is t-embedded. Let $h\colon A_t(E) \to E$ be the inverse of the homeomorphism $a_E\colon E \to A_t(E)$, and let $\alpha(f) = h \circ A_t(f)\colon A_t(Z) \to E$. Since $a_E \circ f = A_t(f) \circ a_Z$, by 3.6, $f = \alpha(f) \circ a_Z$. Because a_Z is surjective, there is exactly one such arrow. In short, A_t has the universal property of a reflector. □

3.8. COROLLARY – THE FRIČ-KENT PERMANENCE THEOREM. *Under our definitions,*

(a) **tem** *is hereditary and productive, in* **tcv**,
(b) **tcm** *is closed–hereditary and productive, in* **tcv**.

P r o o f. Since **tem** is reflective in **tcv**, it is closed under products taken in **tcv**. Next, suppose that $T \in$ **tem** and that j embeds S in T, in **tcv**. By the reflective property, $j = \alpha(j) \circ a_S$. Hence $@_S$ embeds S in $C_t(C_t(S))$, by 2.5.

Now suppose that T is t-complete and S is a closed subspace of T. We use the Extremal Lemma 2.6, with $u = @_T \circ j\colon S \hookrightarrow T \hookrightarrow C_t(C_t(T))$. Then

u embeds S in $C_t\big(C_t(T)\big)$ as a closed subspace and $j^{**} \circ @_S = u$. So by 2.6, $\widehat{S} = @_S(S)$ is closed in $C_t\big(C_t(S)\big)$. In short, S is t-complete.

To obtain the rest of (b), we could rely on 4.4, that **tcm** is reflective in **tcv**, and hence, productive. Alternatively, we use 2.8: for each v in an index set Υ, let S_v be t-complete, let $T_v = C_t\big(C_t(S_v)\big)$, and let $@_v \colon S_v \hookrightarrow T_v$ be the carrier. Next, let $p_v \colon S \to S_v$ and $q_v \colon T \to T_v$ denote the projections. As usual, the commutation laws $@_v \circ p_v = q_v \circ e = {p_v}^{**} \circ @_S$ hold, for all v. Now as the diagram

$$\begin{tikzcd}
& S_v \arrow[dr, "@_v"] & \\
S \arrow[ur, "p_v"] \arrow[r, "e"] \arrow[dr, "@_S"'] & T \arrow[r, "q_v"] & T_v \\
& C_t\big(C_t(S)\big) \arrow[u, dashed, "l"] \arrow[ur, "{p_v}^{**}"'] &
\end{tikzcd}$$

commutes, the universal property of the product T provides the arrow l such that $q_v \circ l = {p_v}^{**}$, for all v. So $q_v \circ l \circ @_S = {p_v}^{**} \circ @_S = q_v \circ e$, for all v. Hence[10] $e = l \circ @_S$. By part (a) – and by the Embedding Lemma 2.5 as well – $@_S$ embeds S into $C_t\big(C_t(S)\big)$. So by 2.8, 3.2, and the Extremal Lemma 2.6, as the image of S is closed in T, it is closed in $C_t\big(C_t(S)\big)$ as well. In short, S is t-complete. □

The Cartesian version of the Frič-Kent Permanence Theorem in [FK] follows directly from this. In our context, as ϑ need not be hereditary, we must take care with subspaces:

- a subspace or product in **cv** of t-embedded spaces is t-embedded if and only if it is a ϑ-space, and
- a closed subspace or product in **cv** of t-complete spaces is t-complete if and only if it is a ϑ-space.

Note: two of the results in this section (3.1 and 3.5) are always true; 3.3 (a), 3.6, 3.7 and 3.8 (a) are true even if the base B is just a ϑ-space; and the other parts of 3.3 and 3.8 are true if the base is a Hausdorff ϑ-space.

§4. Extensions and envelopes

For any ϑ-space Z, we define its *t-envelope* $E_t(Z)$ to be the topological closure[11] of $\widehat{Z}$ in $C_t\big(C_t(Z)\big)$. Working in **tcv**, we regard $E_t(Z)$ as a subspace of $C_t\big(C_t(Z)\big)$. As before, we obtain the arrow $e_Z \colon Z \to E_t(Z)$ by factoring the

[10] One can "cancel the projections" q_v collectively, as they form a mono-source [AHS].

[11] Though the topological modifier τ reflects **cvk** onto the category of topological spaces,

carrier $@\colon Z \to C_t(C_t(Z))$ through $E_t(Z)$. Clearly, Z is t-complete if and only if e_Z makes Z and $E_t(Z)$ homeomorphic, while $E_t(Z)$ is t-complete for all $Z \in \mathbf{tcv}$, as a closed subspace of the t-complete space $C_t(C_t(Z))$.

For the base $\mathbb{R}$, the results in [FK] may establish the t-enveloper E_t as a completion functor. We confirm this below, for all our bases. Our discussion includes extensions of maps as well as spaces. Take a subcategory **sb** of **cv**. We call an arrow $i\colon Z \to I$ in **cv** *epic over* **sb** if for any space $W \in \mathbf{sb}$ and any arrows $p, q\colon I \to W$, if $p \circ i = q \circ i$, then $p = q$.

4.1. *Let $i\colon Z \to I$ in* **tcv**. *Then*

(a) *i is epic over* **sb** *if and only if $i^*\colon C(I,W) \to C(Z,W)$ is injective, for all $W \in \mathbf{sb}$, and*

(b) *i is epic over* **tem** *if and only if it is epic over* **tcm**.

Proof.

(a) Let $W \in \mathbf{sb}$, and let $p, q \in C(I,W)$. By definition, $i^*(p) = p \circ i$ and $i^*(q) = q \circ i$. Suppose i^* is injective. Then $p \circ i = q \circ i$ implies $i^*(p) = i^*(q)$, and in turn, this implies $p = q$. So i is epic over **sb**. The converse is just as easy.

(b) One direction is trivial. So, suppose i is epic over **tcm**. Take a t-embedded space W and arrows $p, q\colon I \to W$ such that $p \circ i = q \circ i$. Then $e_W \circ p$ and $e_W \circ q$ are both arrows from I to the t-complete space $E_t(W)$, and $e_W \circ p \circ i = e_W \circ q \circ i$. So $e_W \circ p = e_W \circ q$. So $p = q$, as e_W is injective – after all, $@_W$ embeds W in $C_t(C_t(W))$. In short, i is epic over **tem**. □

So, we call an arrow *t-epic* if it is epic over **tem** (or equivalently, over **tcm**). Further, we call it *weakly epic* if its image is topologically dense.

4.2. *Each weakly epic arrow is t-epic. For each t-epic arrow $i\colon Z \to I$, the map $i^*\colon C(I) \to C(Z)$ is injective. For each ϑ-space Z, the arrow e_Z is weakly epic and $e_Z^*\colon C_t(E_t(Z)) \to C_t(Z)$ is a homeomorphism.*

Proof. First, let $i(Z)$ be dense in τI, let W be t-embedded, and let $p \circ i = q \circ i\colon Z \to I \to W$. In topology, each arrow with dense image is epic over the category of Hausdorff spaces. So, apply the topological modifier, note that τW is Hausdorff, by 3.2, and conclude: $p = q$.

The second claim follows from 4.1. Finally, take a ϑ-space Z. Since $\widehat{Z}$ is topologically dense in $E_t(Z)$, the arrow e_Z is t-epic and $e_Z{}^*$ is injective, by 4.1. So by 3.4, $e_Z{}^*$ is a homeomorphism. □

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it is not a hereditary reflector. However, if X is open or closed in Y, then τX is a subspace of τY. In particular, $\widehat{Z}$ is topologically dense in $E_t(Z)$: to prove this, take $X = E_t(Z)$ and $Y = C_t(C_t(Z))$.

4.3. The Frič-Kent Extension Theorem. *Each arrow $f\colon X \to Y$ in* **cv** *induces exactly one arrow $E_t(f)\colon E_t(X) \to E_t(Y)$ such that $e_Y \circ f = E_t(f) \circ e_X$.*

P r o o f . We follow the proof in [FK]. By 3.5, the outer trapezium commutes.

$$\begin{tikzcd}
 & X \arrow[r, "f"] \arrow[d, "e_X"] \arrow[ddl, "@_X"'] & Y \arrow[d, "e_Y"'] \arrow[ddr, "@_Y"] & \\
 & E_t(X) \arrow[r, "E_t(f)"] \arrow[dl] & E_t(Y) \arrow[dr] & \\
C_t(C_t(X)) \arrow[rrr, "f^{**}"] & & & C_t(C_t(Y))
\end{tikzcd}$$

Because $f^{**}\colon \tau C_t(C_t(X)) \to \tau C_t(C_t(Y))$ is continuous, it maps $E_t(X)$ into $E_t(Y)$. Thus the restriction of f^{**} to $E_t(X)$ is an arrow $E_t(f)$ making the inner square commute. It is unique, because e_X is weakly epic. □

4.4. Theorem. *The t-enveloper E_t reflects* **tcv** *onto* **tcm**.

P r o o f . Take a ϑ-space Z, a t-complete space C, and an arrow $f\colon Z \to C$. As we noted above, $E_t(Z)$ is t-complete. By assumption, $e_C\colon C \to E_t(C)$ is a homeomorphism. Let h be its inverse, and let $\varepsilon(f) = h \circ E_t(f)$. Then $f = \varepsilon(f) \circ e_Z$, because by 4.3, $e_C \circ f = E_t(f) \circ e_Z$. Because e_Z is weakly epic, there is exactly one such arrow. In short, E_t has the universal property of a reflector. □

Given arrows $i\colon Z \to I$ and $j\colon Z \to J$, we call i and j *equivalent* if $j = h \circ i$ for some homeomorphism $h\colon I \to J$, and we write $i \prec j$ if $j = h \circ i$ for some arrow $h\colon I \to J$. Obviously,

(a) if $i \prec j$ and $j \prec k$, then $i \prec k$, and
(b) if i and j are equivalent, then $i \prec j$ and $i \succ j$.

Sometimes, we can do better.

4.5. *Let I and J be t-embedded, and let $i\colon Z \to I$ and $j\colon Z \to J$ be t-epic. Then i and j are equivalent if and only if $i \prec j$ and $i \succ j$.*

P r o o f . Suppose $i \prec j$ and $i \succ j$. Choose $\bar{\jmath}\colon I \to J$ and $\bar{\imath}\colon J \to I$ such that $j = \bar{\jmath} \circ i$ and $i = \bar{\imath} \circ j$. Let $p = \bar{\imath} \circ \bar{\jmath}$ and $q = \mathrm{id}_I$. Now $p \circ i = \bar{\imath} \circ j = i = q \circ i$. Hence $\bar{\imath} \circ \bar{\jmath} = \mathrm{id}_I$, because i is t-epic. Similarly, $\bar{\jmath} \circ \bar{\imath} = \mathrm{id}_J$. In short, $\bar{\imath}$ makes i and j equivalent. □

Next, given a ϑ-space Z and an arrow $i\colon z \mapsto @(z)$ from Z to a subspace I of $E_t(Z)$, we call i a *standard extension* of Z if it is t-epic. Clearly, a_Z and e_Z

are standard extensions of Z, but there may be others. More generally, we call an arrow i from Z to a t-embedded space I a *t-extension* of Z if it is t-epic and $i \prec e_Z$. In fact, this extra generality is more apparent than real, as each t-extension i has a standard counterpart $\bar{\imath}$ – see 4.7 (iv) below. Meanwhile, we record the universal description.

4.6. *An arrow $i\colon Z \to I$ in* **tcv** *is a t-extension if and only if for each t-complete space C and each arrow $g\colon Z \to C$, exactly one arrow $\overline{g}\colon I \to C$ makes $g = \overline{g} \circ i$.*

Proof. Suppose $i\colon Z \to I$ is a t-extension, with $e_Z = h \circ i$. Take a t-complete space C, an arrow $g\colon Z \to C$ in **cv**, and its universal extension $\varepsilon(g)\colon E_t(Z) \to C$.

$$\begin{tikzcd}
I \arrow[rr, bend left, "h"] \arrow[dr, "\overline{g}"'] & Z \arrow[l, "i"'] \arrow[r, "e_Z"] \arrow[d, "g"] & E_t(Z) \arrow[dl, "\varepsilon(g)"] \\
& C &
\end{tikzcd}$$

Put $\overline{g} = \varepsilon(g) \circ h$, and calculate: $\overline{g} \circ i = \varepsilon(g) \circ e_Z = g$, as desired. Uniqueness follows, because i is t-epic.

Conversely, let i have the given property. Then $e_Z\colon Z \to E_t(Z)$ has an extension $h = \overline{e_Z}\colon I \to E_t(Z)$ such that $e_Z = h \circ i$. Thus $i \prec e_Z$. Next, take a t-embedded space W and arrows $p, q\colon I \to W$, such that $p \circ i = q \circ i = f$, say. Let $g = e_W \circ f\colon Z \to E_t(W)$, a t-complete space. Then there is exactly one arrow $\overline{g}\colon I \to E_t(W)$ making $g = \overline{g} \circ i$. Now $e_W \circ p$ also has this property: $g = e_W \circ p \circ i$. So $\overline{g} = e_W \circ p$, and similarly, $\overline{g} = e_W \circ q$. So $p = q$, as desired, because e_W is injective. □

4.7. Facts. *For any ϑ-space Z and any t-extension $i\colon Z \to I$,*

(i) *$a_Z \prec i$ (and by definition, $i \prec e_Z$),*
(ii) *$i^*\colon C_t(I) \to C_t(Z)$ is a homeomorphism,*
(iii) *$E_t(Z)$ and $E_t(I)$ are homeomorphic,*
(iv) *i is equivalent to a standard extension $\bar{\imath}\colon Z \to E_i$,*
(v) *$E_u \hookrightarrow E_v$ if and only if $u \prec v$, for all t-extensions u and v of Z,*
(vi) *i is minimal under $\prec$ if and only if $E_i = A_t(Z)$, and*
(vii) *i is maximal under $\prec$ if and only if I is t-complete, if and only if $E_i = E_t(Z)$.*

Proof.

(i) By the universal property 3.7, $i = \alpha(i) \circ a_Z$, and so $a_Z \prec i$.

(ii) By 4.1, $i^*\colon C_t(I) \to C_t(Z)$ is injective. As $i \prec e_Z$, there is an arrow $h\colon I \to E_t(Z)$ such that (a) $e_Z = h \circ i$. As i is t-epic, there is exactly one such arrow – call it h_i. Combine h_i with the inclusion $E_t(Z) \to C_t\big(C_t(Z)\big)$ to form the arrow $j\colon I \to C_t\big(C_t(Z)\big)$. Clearly, $@ = j \circ i$. So by 3.4, $i^*\colon C_t(I) \to C_t(Z)$ is a homeomorphism.

(iii) By the universal property 4.4, the arrow $h = h_i$ of (ii) induces the arrow $\varepsilon(h)$ from $E_t(I)$ to the t-complete space $E_t(Z)$ such that (b) $h = \varepsilon(h) \circ e_I$. On the other hand, consider the commuting diagram

$$\begin{tikzcd}
Z \arrow[r, "i"] \arrow[d, "e_Z"'] & I \arrow[dl, "h"'] \arrow[d, "e_I"] \\
E_t(Z) & E_t(I) \arrow[l, "\varepsilon(h)"]
\end{tikzcd}$$

and note that $E_t(i) \circ h \circ i = E_t(i) \circ e_Z = e_I \circ i$. Thus (c) $E_t(i) \circ h = e_I$, as i is t-epic. Put (b) and (c) together: $e_I = E_t(i) \circ \varepsilon(h) \circ e_I$. So as e_I is weakly epic, $E_t(i) \circ \varepsilon(h)$ is the identity on $E_t(I)$. Similarly, $\varepsilon(h) \circ E_t(i)$ is the identity on $E_t(Z)$, because by (a) in part (ii) above,

$$\varepsilon(h) \circ E_t(i) \circ e_Z = \varepsilon(h) \circ e_I \circ i = h \circ i = e_Z$$

and e_Z is weakly epic. In short, as $\varepsilon(h)$ and $E_t(i)$ invert one another, they make $E_t(Z)$ and $E_t(I)$ homeomorphic.

(iv) With the same data, $h_i = h = \varepsilon(h) \circ e_I = \big(E_t(i)\big)^{-1} \circ e_I$, say. This formula does three things: it emphasizes the unique dependence of h on i; it displays $h\colon I \to E_t(Z)$ as an embedding; and it shows the existence and uniqueness of the standard extension $\bar{\imath}$ equivalent to i (namely, $\bar{\imath}\colon Z \to E_i = h_i(I) \hookrightarrow E_t(Z)$).

(v) Clearly, if $E_u \subset E_v$, then $u \prec v$. Conversely, let $v = k \circ u\colon Z \to V$. Form the embeddings h_u and h_v as above. Then $h_v \circ k = h_u$, because $h_v \circ k \circ u = h_v \circ v = e_Z = h_u \circ u$ and u is t-epic. So $E_u = h_u(U) = h_v\big(k(U)\big) \subset h_v(V) = E_v$.

(vi) Clearly, $A_t(Z)$ is the smallest standard extension. Suppose i is minimal among t-extensions. Then so is its standard counterpart E_i, by (v). On the other hand, as $A_t(Z) \subset E_i$ in all cases, $A_t(Z) = E_i$.

(vii) Trivially, $i \prec e_I \circ i$. So, if i is maximal, then by 4.5, there is a homeomorphism $h\colon I \to E_t(I)$ such that $e_I \circ i = h \circ i$. Now cancel the arrow i, as it is t-epic. So as e_I is a homeomorphism, I is t-complete.

Next, suppose I is t-complete. Consider the arrow discussed in (ii) and (iii), $h = \varepsilon(h) \circ e_I$. Because $\varepsilon(h)$ and e_I are both homeomorphisms, $E_i = h(I) = \varepsilon(h)\big(E_t(I)\big) = E_t(Z)$. Finally, if $E_i = E_t(Z)$, then i is maximal, by (v). □

One can extend 4.7 (ii): first, $i^*\colon C(I, W) \to C(Z, W)$ is bijective, for each t-extension i and each t-complete space W, by 4.1 and 4.6. Later, under mild conditions on ϑ, we find that $i^*\colon C_t(I, W) \to C_t(Z, W)$ is a homeomorphism. Next, we show that 4.7 (ii) characterizes t-extensions among weakly epic arrows.

4.8. *Take a t-space Z, a t-embedded space I, and a weakly epic arrow $i\colon Z \to I$ such that i^* is a homeomorphism. Then i is a t-extension.*

Proof. By 4.2, i is t-epic. Clearly, $i^{**}\colon C_t\big(C_t(Z)\big) \to C_t\big(C_t(I)\big)$ is a homeomorphism. Let g be the inverse of i^{**}, and let $J = i(Z)$. By 3.5, $@_Z = g \circ @_I \circ i$. Since $g \circ @_I$ is continuous and $I = \overline{J}$, the closure of J in τI,

$$g \circ @_I(I) = g \circ @_I\big(\overline{J}\big) \subset \overline{g \circ @_I\big(i(Z)\big)} = \overline{@_Z(Z)} = E(Z),$$

the closure of $@_Z(Z)$ in $\tau C_t\big(C_t(Z)\big)$. Similarly,

$$g\big(E(I)\big) = g\big(\overline{@_I(I)}\big) \subset \overline{g \circ @_I(I)} \subset \overline{E(Z)} = E(Z),$$

because $E(Z)$ is closed. So, $E_t(i)\colon E_t(Z) \to E_t(I)$ is a homeomorphism. Let g_0 stand for its inverse. Now, $e_Z = g_0 \circ e_I \circ i$, since $E_t(i) \circ e_Z = e_I \circ i$ by 4.3. In other words, $i \prec e_Z$. □

§5. The Frič-Kent Product Theorem

Here we prove this theorem without relying on the minute calculations in [FK]. In doing so, we use only the usual Cartesian product $\times$: the product $\sqcap$ in **tcv** does not appear again until the next section. As before, we write $S \geq T$ when the inclusion from S to T is continuous.

5.1. *Let $C_t(Z, W) \times Z$ be a ϑ-space. Then $C_t(Z, \vartheta W) = C_t(Z, W)$.*

Proof. As Z is a ϑ-space, by 2.7, $C(Z, \vartheta W) = C(Z, W)$, by 2.1. Clearly,

(a) $C_t(Z, \vartheta W) \geq C_t(Z, W)$.

Conversely, ϑ lifts the evaluation arrow $\omega\colon C_c(Z, W) \times Z \to W$ from **cv** to **tcv**; in other words, $\omega\colon \vartheta\big(C_c(Z, W) \times Z\big) \to \vartheta W$ in **tcv**. Now $\vartheta\big(C_c(Z, W) \times Z\big) = C_t(Z, W) \times Z$, as the latter is a ϑ-space. In short, $\omega\colon C_t(Z, W) \times Z \to \vartheta W$ is continuous. So by definition of continuous convergence, $\mathrm{id}\colon C_t(Z, W) \to C_c(Z, \vartheta W)$ in **cv**. Hence

(b) $C_t(Z, W) \geq C_t(Z, \vartheta W)$.

Together, (a) and (b) complete the proof. □

5.2. *Suppose X is a ϑ-space, $i\colon X \to I$ is a t-extension, and $C_t\big(Y, C_c(X)\big) \times Y$ is a ϑ-space. Then $(i \times \mathrm{id}_Y)^*\colon C_t(I \times Y) \to C_t(X \times Y)$ is a homeomorphism.*

Proof. Consider the diagram sketched below:

$$\begin{array}{ccc} C_t(I\times Y) & \xrightarrow{j_1} & C_t(X\times Y) \\ \uparrow & & \uparrow \\ C_t\big(Y, C_c(I)\big) & \xrightarrow{j_2} & C_t\big(Y, C_c(X)\big) \\ {\scriptstyle \mathrm{id}_1}\uparrow & & \uparrow{\scriptstyle \mathrm{id}_2} \\ C_t\big(Y, C_t(I)\big) & \xrightarrow{j_3} & C_t\big(Y, C_t(X)\big) \end{array}$$

In the top half, the vertical arrows are conversions and as such, they are homeomorphisms (both before and after coreflection), while $j_1 = (i \times \mathrm{id}_Y)^*$ and $j_2 = (i^*)_*$ are both continuous. Moreover, the top half commutes. The bottom half commutes too, because the vertical arrows are identity maps and because $j_2 = (i^*)_* = j_3$. Now

id_2 is a homeomorphism, by 5.1,
$i^*\colon C_t(I) \to C_t(X)$ is a homeomorphism, by 4.7, and hence
j_3 is a homeomorphism.

As a result, id_1 and j_2 are both homeomorphisms. Thus, back in the top half, j_1 is a homeomorphism, as desired. □

Our version of the Frič-Kent Product Theorem improves the original one in three ways: it applies to weakly epic t-extensions as well as t-envelopes (a minor matter, as the proof shows), the base may be any topologically Hausdorff ϑ-space, and heredity is not needed at all. However, we still need some kind of productivity: we call ϑ *finitely productive* on a subcategory **sb** of **cv** if the Cartesian product $\vartheta X \times \vartheta Y$ is a ϑ-space, for all spaces X and Y in **sb**. By 2.7, ϑ is finitely productive on **tcv** if and only if **tcv** is closed under finite Cartesian products; by the Frič-Kent Permanence Theorem, ϑ is finitely productive on **tem** (or on **tcm**) if **tem** (or **tcm**) is closed under finite Cartesian products.

The identifier ι is productive on **cvk**; the discrete modifier δ, the sequential modifier ψ, the antisequential modifier ϖ, and the local compactifier κ are finitely productive too.

5.3. THEOREM. *Let ϑ be finitely productive on* **tem**, *and let $X, Y \in$* **tem**. *Let $i\colon X \to I$ and $j\colon Y \to J$ be weakly epic t-extensions. Then $i\times j\colon X\times Y \to I\times J$ is a weakly epic t-extension.*

Proof. Clearly, $i \times j \prec e_X \times e_Y$, since $i \prec e_X$ and $j \prec e_Y$. Next, $i \times j$ is weakly epic, as the image of $X \times Y$ is topologically dense[12] in $I \times J$. To

[12] This innocent fact deserves some comment. The topological modifier τ is not finitely productive – in other words, $\tau(U\times V)$ can have a strictly finer topology than $\tau U \times \tau V$. Despite this, the closure of any box $A \times B$ in $\tau(U \times V)$ coincides with its closure in $\tau U \times \tau V$.

conclude, we must show that $i \times j \prec e_Z$, where $Z = X \times Y$. We do better, we show that $e_X \times e_Y \prec e_Z$.

By 4.7 (iv), we can work with standard extensions. So, we assume $I = E_t(X)$ and $i = e_X$, and $J = E_t(Y)$ and $j = e_Y$. Let $K = I \times J$, and $k = i \times j$. Together, finite productivity and the Permanence Theorem 3.8 ensure that Z is t-embedded and that K is t-complete. We aim to show $k^*\colon C_t(K) \longleftrightarrow C_t(Z)$ and then, to compare k with e_Z.

Under conversion, $C_c(Z) = C_c(X \times Y) \longleftrightarrow C_c\big(X, C_c(Y)\big)$. By 3.3, $C_t(Z)$ is t-complete, because Z is a ϑ-space. Hence $C_t\big(X, C_c(Y)\big)$ is also t-complete. This makes $C_t\big(X, C_c(Y)\big) \times X$ a t-embedded space. So by a mirror version of 5.2, $(\mathrm{id}_X \times j)^*\colon C_t(X \times J) \longleftrightarrow C_t(X \times Y)$, and $(i \times \mathrm{id}_J)^*\colon C_t(I \times J) \longleftrightarrow C_t(X \times J)$, by 5.2 itself. As the composite of these homeomorphisms, k^* makes $C_t(I \times J)$ and $C_t(X \times Y)$ homeomorphic.

So by 4.8, $k\colon Z \to K$ is a t-extension, and in particular, $k \prec e_Z$. Now as I and J are t-complete, so is their Cartesian product $K = I \times J$. By 4.4, $k\colon Z \to K$ has an extension $\varepsilon(k)\colon E_t(Z) \to K$ such that $k = \varepsilon(k) \circ e_Z$, and this equation ensures $e_Z \prec k$. So by 4.5, e_Z and k are equivalent. (We really do not need 4.5, because we can give a more explicit formula: the homeomorphism $\varepsilon(k) = (e_K)^{-1} \circ E_t(k)$ makes e_Z and k equivalent.) $\square$

5.4. Corollary – the Frič-Kent Product Theorem. *Let ϑ be finitely productive on* **tem**. *Then the t-enveloper E_t is finitely productive on* **tem** *too – more precisely, the universal extension of the embedding $e_X \times e_Y$ from $X \times Y$ into $E_t(X) \times E_t(Y)$ makes $E_t(X \times Y)$ and $E_t(X) \times E_t(Y)$ homeomorphic.*

§6. Closure

The assumptions in the previous section seem quite artificial – and one might suspect that they only betray mathematical incompetence. However, they do have some significance: they help make **tcv**, **tem** and **tcm** Cartesian closed.

6.1. Theorem. *For any coreflector ϑ, the following are equivalent:*

(i) *ϑ is finitely productive on* **tcv**.
(ii) **tcv** *is closed under finite Cartesian products.*
(iii) *$C_t(X, Y) \times X \in$* **tcv**, *for all $X, Y \in$* **tcv**.
(iv) *$C_t(-,-)$ makes* **tcv** *Cartesian-closed.*

Proof. We noted some of this earlier: by 2.7, (ii) $\Longrightarrow$ (i). Trivially, (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii). To close the cycle, (iii) $\Longrightarrow$ (ii), since $Y \times X$ is a retract of $C_t(X, Y) \times X$.

Next, recall the two distinct products, $\sqcap$ in **tcv** and $\times$ in **cv**. Conversion offers a clean proof of (ii) $\Longrightarrow$ (iv). For all W, X, Y in **tcv**,

$$g\colon W \to C_t(X, Y) \text{ in } \mathbf{tcv},$$

$g\colon W \to C_c(X,Y)$ in **cv**,
$\widetilde{g}\colon W \times X \to Y$ in **cv**, after conversion,
$\widetilde{g}\colon W \sqcap X \to Y$ in **tcv**.

This half of the conversion law is free: for the other half,

$f\colon W \sqcap X \to Y$ in **tcv**,
$f\colon W \times X \to Y$ in **cv** because, by (ii), $W \sqcap X = W \times X$,
$\underline{f}\colon W \to C_c(X,Y)$ in **cv**, after conversion,
$\underline{f}\colon W \to C_t(X,Y)$ in **tcv**, because $W \in$ **tcv**.

In short, **tcv** upholds conversion. Finally, consider (iv) $\implies$ (ii). For all X, Y, Z in **tcv**,

$h\colon X \sqcap Y \to Z$ in **tcv**,
$\underline{h}\colon X \to C_t(Y,Z)$ in **tcv**, by (iv),
$\underline{h}\colon X \to C_c(Y,Z)$ in **cv**,
$h\colon X \times Y \to Z$ in **cv**.

In particular, the choice $Z = X \sqcap Y$ and $h = \mathrm{id}_Z$ implies $\mathrm{id}_Z\colon X \times Y \to X \sqcap Y$ in **cv**. Hence (ii) holds: $X \sqcap Y = X \times Y$. □

The function space $C_c(X,Y)$ inherits c-embeddedness from Y: let us investigate this more generally, for t-embedded and t-complete spaces. Suppose ϑ is finitely productive on **tcv**. In this situation, we could use [AHS; Theorem 27.9]: the reflective subcategories **tem** and **tcm** of **tcv** would inherit its Cartesian-closedness if the reflectors A_t and E_t were finitely productive too. However, I can see no way of using the finite productivity of ϑ to establish finite productivity for A_t or E_t (indeed, I have my doubts, except perhaps when the base is the real line $\mathbb{R}$). This leaves us little choice: we must find out when **tem** and **tcm** are Cartesian-closed directly. The key to this is a t-counterpart of 1.7.

6.2. THE C_t-EMBEDDING THEOREM. *Let j embed W in Z in* **tcv**.

(a) *Suppose* **tcv** *is closed under finite Cartesian products. Then j_* embeds $C_t(Y,W)$ in $C_t(Y,Z)$, for all ϑ-spaces Y.*
(b) *Suppose Z is t-embedded, and* **tem** *is closed under finite Cartesian products. Then j_* embeds $C_t(Y,W)$ in $C_t(Y,Z)$, for all $Y \in$* **tem**.

P r o o f. We deal with (b), the case of interest here, but one can derive (a) in much the same way. As before, we use the universal property of subspaces. Take t-embedded spaces H and Y, and a map $h\colon H \to C(Y,W)$ such that $j_* \circ h = k\colon H \to C_t(Y,Z)$ is continuous. Then just as in 1.7, $\widetilde{h}\colon H \times Y \to W$ is continuous, since $\widetilde{k} = j \circ \widetilde{h}$. Now $H \times Y = H \sqcap Y$, by assumption. Hence as in the proof of 6.1, $h\colon H \to C_t(Y,W)$ is continuous. In short, j_* is an embedding in **tem** (and in **tcv** too), as claimed. □

6.3. THEOREM. *The coreflector ϑ is finitely productive on* **tem** *if and only if* **tem** *is closed under finite Cartesian products, if and only if $C_t(-,-)$ makes* **tem** *Cartesian-closed.*

Proof. As the proof of 6.1 shows, if **tem** is Cartesian-closed, then the products in **tem** and in **cv** coincide: in other words, **tem** is closed under finite Cartesian products. Conversely, suppose **tem** is closed under finite Cartesian products. Then one deduces the conversion laws as in 6.1, but they do not imply that the function space is t-embedded.

To prove this, take t-embedded spaces X and Y, and let T be the t-complete space $C_t(Y)$. Then by 3.3, $C_t(X \times T)$ is t-complete, as $X \times T \in \mathbf{tem}$. Similarly, $C_t(X \times T) \times X$ is t-embedded. So on conversion, $C_t\big(X, C_c(T)\big) \times X \in \mathbf{tcv}$. Thus by 5.1, $C_t\big(X, C_t(T)\big) \longleftrightarrow C_t\big(X, C_c(T)\big) \longleftrightarrow C_t(X \times T)$. Now let $j = @_Y : Y \hookrightarrow C_t(T)$. By 6.2, j_* embeds $C_t(X,Y)$ in $C_t\big(X, C_t(T)\big)$ in **tcv**. In short, $C_t(X,Y)$ is t-embedded. □

This theorem almost applies to t-completeness. Suppose the coreflector ϑ is finitely productive on **tem**. Then by the Frič-Kent Product Theorem, the reflector E_t is finitely productive on **tem**, up to homeomorphism. This differs from the finite productivity required in [AHS; Theorem 27.9], but one could probably extend that theorem to cover the situation. If so, then we could conclude that $C_t(-,-)$ makes **tcm** Cartesian-closed. However, we can get a better result by repeating the proof of 6.2, using both parts of 1.7.

6.4. THEOREM. *The coreflector ϑ is finitely productive on* **tcm** *if and only if* **tcm** *is closed under finite Cartesian products, if and only if $C_t(-,-)$ makes* **tcm** *Cartesian-closed.*

In short, the conditions for the product theorem in [FK] are exactly those that determine when **tem** and **tcm** are Cartesian-closed. Several questions remain:

- are our results about weakly epic t-extensions true, without this side condition; can one find a t-epic arrow that is not weakly epic;
- can one find a (hereditary) coreflector that is not finitely productive; if so, are **tcv** and/or **tem** Cartesian-closed; and
- do the main results of [FK] follow from Cartesian-closure?

The point is simple: if **tcv** is Cartesian-closed (with its own products and subspaces), then its function spaces might carry a coarser convergence than the one handled here. This would make it harder to link the theory with the familiar c-theory.

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