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Mathematica Slovaca, Vol. 41 (1991), No. 1, 29--33

Persistent URL: http://dml.cz/dmlcz/130631

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VARIETIES WITH RECTANGULAR IDEALS

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ABSTRACT. A variety $\boldsymbol{V}$ has rectangular ideals whenever every ideal I in $A \times B$, A, $B \in \boldsymbol{V}$, is product $I = I_A \times I_B$ of ideals I_A, I_B in A, B, respectively. The paper gives a Mal'cev characterization of varieties having rectangular ideals.

Ideals in arbitrary universal algebras were studied in [1], [2], and [4]. In particular, it was shown in [1], [2] that the rectangularity of ideals (see the definition below) in bidual varieties can be expressed by a suitable Mal'cev condition. The aim of this paper is to prove that the rectangularity of ideals is Mal'cev definable in an arbitrary variety having a nullary operation 0. Moreover we prove that the ideals in a variety $\boldsymbol{V}$ (with nullary operation 0) are rectangular iff the square $F_{\boldsymbol{V}}(x) \times F_{\boldsymbol{V}}(x)$ of the $\boldsymbol{V}$-free algebra $F_{\boldsymbol{V}}(x)$ with one free generator x has this property. In addition we find identities characterizing rectangular ideals in permutable varieties. To make this paper selfcontained we begin with some definitions:

Let $\boldsymbol{C}$ be a class of similar algebras having a nullary operation 0. A term $\boldsymbol{p}(\vec{x}, \vec{y})$ ($\vec{x}$ is an abbreviation of a finite sequence $x_1, \ldots, x_n$) is called an *ideal term* in $\vec{x}$ if $0 = \boldsymbol{p}(\vec{0}, \vec{y})$ holds identically in $\boldsymbol{C}$.

A nonempty subset I of an algebra $A \in \boldsymbol{C}$ is and *ideal* in A if for every ideal term $\boldsymbol{p}(\vec{x}, \vec{y})$ in $\vec{x}$, $\vec{i} \in I \times \ldots \times I$, $\vec{a} \in A \times \ldots \times A$ the relation $\boldsymbol{p}(\vec{i}, \vec{a}) \in I$ holds.

An ideal I in the product $A \times B$, A, $B \in \boldsymbol{C}$, is named *rectangular* whenever $I = I_A \times I_B$ for suitable ideals I_A, I_B in A, B, respectively. A class $\boldsymbol{C}$ is said to have *rectangular ideals* if whenever A, $B \in \boldsymbol{C}$, then every ideal of $A \times B$ is rectangular.

Lemma 1. *Let A an algebra with nullary operation* 0. *The ideal $I(S)$ generated by a subset $S \subseteq A$ consists exactly of the elements $\boldsymbol{p}(\vec{s}, \vec{a})$ where $\boldsymbol{p}(\vec{x}, \vec{y})$ is an ideal term in $\vec{x}$ and $\vec{s} \in S \times \ldots \times S$, $\vec{a} \in A \times \ldots \times A$.*

Proof. [4; Lemma 1.2, p. 46].

Lemma 2. *Let A, B be similar algebras having a nullary operation* 0. *Let I be an ideal in the product $A \times B$. The following conditions are equivalent*:

AMS Subject Classification (1985): Primary 08B05
Key words: Variety, Mal'cev condition, Rectangular ideal

(1) *I is rectangular*;

(2) (i) $\langle a, b\rangle \in I$ *implies* $\langle a, 0\rangle, \langle 0, b\rangle \in I$, *and*

(ii) $\langle a, 0\rangle, \langle 0, b\rangle \in I$ *imply* $\langle a, b\rangle \in I$.

Proof. (1) $\Rightarrow$ (2) is evident.

(2) $\Rightarrow$ (1): We have to prove that $\langle a, b\rangle, \langle a', b'\rangle \in I$ imply $\langle a, b'\rangle \in I$ in the product $A \times B$. By (2) (i) we have $\langle a, 0\rangle, \langle 0, b\rangle, \langle a', 0\rangle, \langle 0, b'\rangle \in I$. Further, applying (2) (ii) to $\langle a, 0\rangle, \langle 0, b'\rangle \in I$ we conclude $\langle a, b'\rangle \in I$, as required.

Theorem 1. *Let* $\boldsymbol{V}$ *be a variety with nullary operation* 0. *The following conditions are equivalent*:

(1) $\boldsymbol{V}$ *has rectangular ideals*;

(2) *there exist binary terms* $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_n, \boldsymbol{s}_1, \ldots, \boldsymbol{s}_n$ *and* a $(2+n)$*-ary term* $\boldsymbol{p}$ *such that the identities*

(α) $0 = \boldsymbol{p}(0, 0, \vec{z})$

(β) $x = \boldsymbol{p}(x, y, \vec{\boldsymbol{r}}(x, y))$

(γ) $y = \boldsymbol{p}(x, y, \vec{\boldsymbol{s}}(x, y))$

hold in $\boldsymbol{V}$;

(3) *there exist unary terms* $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n, \boldsymbol{w}_1, \ldots, \boldsymbol{w}_n$ *and* $(2+n)$*-ary term* $\boldsymbol{p}$ *such that the identities*

(α) $0 = \boldsymbol{p}(0, 0, \vec{z})$

(δ) $x = \boldsymbol{p}(x, 0, \vec{\boldsymbol{u}}(x))$

(ε) $x = \boldsymbol{p}(0, x, \vec{\boldsymbol{v}}(x))$

(ζ) $0 = \boldsymbol{p}(0, x, \boldsymbol{w}(x))$

hold in $\boldsymbol{V}$.

Proof. (1) $\Rightarrow$ (2): Let $F_{\boldsymbol{V}}(x, y)$ be the $\boldsymbol{V}$-free algebra with free generators x and y. Consider the ideal $I(\langle x, x\rangle, \langle y, y\rangle)$ generated by the elements $\langle x, x\rangle$ and $\langle y, y\rangle$ in the product $F_{\boldsymbol{V}}(x, y) \times F_{\boldsymbol{V}}(x, y)$. Then $\langle x, y\rangle \in I(\langle x, x\rangle, \langle y, y\rangle)$ follows from the assumption of rectangularity. Applying Lemma 1 we get a $(2+n)$-ary ideal term $\boldsymbol{p}$ (whence the identity (2) (α) follows) such that

$$\langle x, y\rangle = \langle \boldsymbol{p}, \boldsymbol{p}\rangle(\langle x, x\rangle, \langle y, y\rangle, \langle \boldsymbol{r}_1(x, y), \boldsymbol{s}_1(x, y)\rangle, \ldots, \langle \boldsymbol{r}_n(x, y), \boldsymbol{s}_n(x, y)\rangle)$$

for some binary terms $\boldsymbol{r}_1, \ldots, \mathrm{r}_n, \boldsymbol{s}_1, \ldots, \boldsymbol{s}_n$. Writing this separately in each variable we find

(β) $x = \boldsymbol{p}(x, y, \vec{\boldsymbol{r}}(x, y))$

(γ) $y = \boldsymbol{p}(x, y, \vec{\boldsymbol{s}}(x, y))$,

as claimed.

(2) ⇒ (3): The identity (3) (δ) follows from (2) (β) by setting $y = 0$ and $\vec{\boldsymbol{u}}(x) = \vec{\boldsymbol{r}}(x, 0)$.

The identity (3) (ε) follows from (2) (γ) by setting $x = 0$, $y = x$, and $\boldsymbol{v}(x) = \vec{\boldsymbol{s}}(0, x)$.

The identity (3) (ζ) follows from (2) (β) by setting $x = 0$, $y = x$, and $\vec{\boldsymbol{w}}(x) = \vec{\boldsymbol{r}}(0, x)$.

(3) ⇒ (1): Let I be an arbitrary ideal in the product $A \times B \in \boldsymbol{V}$. Following Lemma 2 we have to prove that

(i) $\langle a, b \rangle \in I$ implies $\langle a, 0 \rangle \in I$ and $\langle 0, b \rangle \in I$: By (3) (ε) (ζ) we obtain

(ε) $a = \boldsymbol{p}(0, a, \vec{\boldsymbol{v}}(a))$

(ζ) $0 = \boldsymbol{p}(0, b, \vec{\boldsymbol{w}}(b))$,

which means that

$$\langle a, 0 \rangle = \langle \boldsymbol{p}, \boldsymbol{p} \rangle (\langle 0, 0 \rangle, \langle a, b \rangle, \langle \boldsymbol{v}_1(a), \boldsymbol{w}_1(b) \rangle, \dots, \langle \boldsymbol{v}_n(a), \boldsymbol{w}_n(b) \rangle).$$

Since $\boldsymbol{p}$ is an ideal term and $\langle 0, 0 \rangle, \langle a, b \rangle \in I$ we conclude that also $\langle a, 0 \rangle \in I$.

Similarly $\langle 0, b \rangle \in I$ follows from the identities (2) (ζ) (ε).

(ii) Now suppose that $\langle a, 0 \rangle \in I$ and $\langle 0, b \rangle \in I$. By applying (3) (δ) (ε) we find

(δ) $a = \boldsymbol{p}(a, 0, \vec{\boldsymbol{u}}(a))$

(ε) $b = \boldsymbol{p}(0, b, \vec{\boldsymbol{v}}(b))$,

which means that $\langle a, b \rangle \in I$.

Altogether, I is rectangular, as required.

Theorem 2. *Let* $\boldsymbol{V}$ *be a variety with nullary operation* 0. *The following conditions are equivalent*:

(1) $\boldsymbol{V}$ *has rectangular ideals*;

(2) $F_{\boldsymbol{V}}(x) \times F_{\boldsymbol{V}}(x)$ *has rectangular ideals*;

(3) *the ideal condition*

$\langle x, x, 0 \rangle \in I(\langle x, 0, 0 \rangle, \langle 0, x, x \rangle)$ *holds in the product* $F_{\boldsymbol{V}}(x) \times F_{\boldsymbol{V}}(x) \times F_{\boldsymbol{V}}(x)$.

Proof. (1) ⇔ (2): See Lemma 2.

(1) ⇔ (3): See Theorem 1.

Theorem 3. *Let* $\boldsymbol{V}$ *be a permutable variety such that* $F_{\boldsymbol{V}}(\emptyset) = \{0\}$. *The following conditions are equivalent*:

(1) $\boldsymbol{V}$ *has rectangular ideals*;

(2) *there exists a quaternary term* $\boldsymbol{q}$ *such that the identities*

(α) $0 = \boldsymbol{q}(0, 0, z_1, z_2)$

(β) $x = \boldsymbol{q}(x, y, x, 0)$

(γ) $y = \boldsymbol{q}(x, y, 0, y)$
hold in $\boldsymbol{V}$;

(3) *there exists a quaternary term* $\boldsymbol{q}$ *such that the identities*

(α) $0 = \boldsymbol{q}(0, 0, z_1, z_2)$
(δ) $x = \boldsymbol{q}(x, 0, x, 0)$
(ε) $x = \boldsymbol{q}(0, x, 0, x)$
(ζ) $0 = \boldsymbol{q}(0, x, 0, 0)$
hold in $\boldsymbol{V}$.

Proof. (1) $\Rightarrow$ (2): Let $F_V(e_1, e_2, e_3, e_4)$ ($F_V(x, y)$) be the $\boldsymbol{V}$-free algebra with free generators e_1, e_2, e_3, e_4 (x, y, respectively). As it was already proved in [6; p. 102] the correspondence given by

$$e_1 \mapsto \langle x, x\rangle, e_2 \mapsto \langle y, y\rangle, e_3 \mapsto \langle x, 0\rangle, e_4 \mapsto \langle 0, y\rangle$$

determines the homomorphism φ from $F_V(e_1, e_2, e_3, e_4)$ onto $F_V(x, y) \times \times F_V(x, y)$.

Consider the ideal $I(\langle x, x\rangle, \langle y, y\rangle)$ in the square $F_V(x, y) \times F_V(x, y)$. Then $\langle x, y\rangle \in I(\langle x, x\rangle, \langle y, y\rangle)$ by rectangularity. Further, $I(\langle x, x\rangle, \langle y, y\rangle) = [\langle 0, 0\rangle]\Theta(\langle\langle 0,0\rangle, \langle x, x\rangle\rangle, \langle 0, 0\rangle, \langle y, y\rangle\rangle)$ since any ideal is a congruence block in a permutable variety, see [4; p. 49]. Altogether we have $\langle\langle x, y\rangle, \langle 0, 0\rangle\rangle \in \Theta(\langle\langle 0, 0\rangle, \langle x, x\rangle\rangle, \langle\langle 0, 0\rangle, \langle y, y\rangle\rangle)$. Then [5; p. 113] and [7] guarantee the existence of an element $\boldsymbol{q}(e_1, e_2, e_3, e_4) \in F_V(e_1, e_2, e_3, e_4)$ such that

($*$) $\langle \boldsymbol{q}(e_1, e_2, e_3, e_4),\ 0\rangle \in \Theta(\langle 0, e_1\rangle, \langle 0, e_2\rangle)$

and

($**$) $\varphi(\boldsymbol{q}(e_1, e_2, e_3, e_4)) = \langle x, y\rangle$.

Then the identity (2) (α)

(α) $0 = \boldsymbol{q}(0, 0, e_3, e_4)$

follows from ($*$) and the remaining identities (2) (β) (γ)

(β) $x = \boldsymbol{q}(x, y, x, 0)$
(γ) $y = \boldsymbol{q}(x, y, 0, y)$

are consequences of ($**$).

(2) $\Rightarrow$ (3): The identity (3) (δ) follows from (2) (β) by setting $y = 0$.

The identity (3) (ε) follows from (2) (γ) by setting $x = 0$ and $x = y$.

Finally the identity (3) (ζ) follows from (2) (γ) by setting $x = 0$ and $y = x$.

(3) $\Rightarrow$ (1): Let I be an arbitrary ideal in the product $A \times B \in \boldsymbol{V}$.

(i) Assuming $\langle a, b\rangle \in I$ the identities (3) (ε) (ζ)

(ε) $a = \boldsymbol{q}(0, a, 0, a)$
(ζ) $0 = \boldsymbol{q}(0, b, 0, 0)$

yield $\langle a, 0\rangle = \langle \boldsymbol{q}, \boldsymbol{q}\rangle (\langle 0, 0\rangle, \langle a, b\rangle, \langle 0, 0\rangle, \langle a, 0\rangle)$. Applying (3) (α), the conclusion $\langle a, 0\rangle \in I$ follows.

Analogously $\langle 0, b\rangle \in I$ can be derived by means of (3) (ζ) (ε) and (3) (α).

(ii) Now suppose $\langle a, 0\rangle, \langle 0, b\rangle \in I$. Then the identities (3) (δ) (ε)

(δ) $a = \boldsymbol{q}(a, 0, a, 0)$

(ε) $b = \boldsymbol{q}(0, b, 0, b)$,

i.e. $\langle a, b\rangle = \langle \boldsymbol{q}, \boldsymbol{q}\rangle (\langle a, 0\rangle, \langle 0, b\rangle, \langle a, 0\rangle, \langle 0, b\rangle)$, together with (2) (α) imply $\langle a, b\rangle \in I$. Lemma 2 completes the proof.

Example. Any variety of rings with 1 has rectangular ideals: Evidently the classical ring ideals coincide with ideals mentioned in our paper. Further, for the quaternary term $\boldsymbol{p}(x, y, z_1, z_2) = x \cdot z_1 + y \cdot z_2$ and the unary terms $\boldsymbol{u}_1(x) = \boldsymbol{u}_2(x) = 1$, $\boldsymbol{v}_1(x) = \boldsymbol{v}_2(x) = 1$, $\boldsymbol{w}_1(x) = \boldsymbol{w}_2(x) = 0$ the identities

(α) $\boldsymbol{p}(0, 0, z_1, z_2) = 0 \cdot z_1 + 0 \cdot z_2 = 0$

(δ) $\boldsymbol{p}(x, 0, \boldsymbol{u}_1(x), \boldsymbol{u}_2(x)) = x \cdot 1 + 0 \cdot 1 = x$

(ε) $\boldsymbol{p}(0, x, \boldsymbol{v}_1(x), \boldsymbol{v}_2(x)) = 0 \cdot 1 + x \cdot 1 = x$

(ζ) $\boldsymbol{p}(0, x, \boldsymbol{w}_1(x), \boldsymbol{w}_2(x)) = 0 \cdot 0 + x \cdot 0 = 0$

from Theorem 1 (3) are satisfied.

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Received September 2, 1988

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