Sławomir Bugajski

Statistical maps. II: Operational random variables and the Bell phenomenon

Mathematica Slovaca, Vol. 51 (2001), No. 3, 343--361

Persistent URL: http://dml.cz/dmlcz/130630

Mathematica Slovaca

Math. Slovaca, **51** (2001), No. 3, 343–361

STATISTICAL MAPS II. OPERATIONAL RANDOM VARIABLES AND THE BELL PHENOMENON

SŁAWOMIR BUGAJSKI

(*Communicated by Anatolij Dvurečenskij*)

ABSTRACT. The concept of operational random variable generalizing that of random variable is discussed and shown to be implied by the operational description of measurement. It is proved that any family of operational random variables having independent outcomes can be well represented by a single standard random variable. Nevertheless, it is demonstrated that some families of operational random variables show the Bell phenomenon, what is impossible in the framework of traditional probability theory. That indicates that the proposed generalization of the concept of random variable provides a nontrivial extension of the traditional one.

1. Introduction

1.1. It is clear that the classical (Fréchet-Kolmogorov) probability theory is basic for the classical statistical mechanics. It is equally clear that the classical probability theory does not play the same role with respect to the quantum mechanics (nor to the quantum statistical mechanics). Indeed, it is widely recognized that Bell's analysis [1], see also [2], [3], shows that some objects of microworld generate families of probability measures which cannot be in any way obtained in the framework of the traditional probability theory. The subsequent confirmation of that phenomenon in a series of precise experiments proves that Nature refuses to obey the rules of Kolmogorovian probability theory. We have to agree that the standard probability theory (in the sequel we will use the abbreviation SPT) is of a restricted validity, hence the need for its appropriate generalization becomes evident. Actually there is an almost common agreement

2000 Mathematics Subject Classification: Primary 60A99, 81P99.
Key words: statistical map, operational random variable, Bell phenomenon, operational probability theory.

among physicists and philosophers of physics that the probabilistic structure underlying quantal theories is to some extent, and in a some sense, "nonclassical".

Among various proposals of a generalized "non-Kolmogorovian" probability theory the best developed one is the noncommutative probability theory founded on the condition that generalized random variables form a (noncommutative in general) algebra which is assumed to be an abstract C*- or W*-algebra (among a great variety of relevant publications let us notice [4], [5], [6]). That mathematical model, although powerful and formally elegant, is not free of disadvantages; it does not provide a natural description of the special class of random variables appearing in the operational quantum mechanics and called there unsharp or generalized observables or POV measures (for POV measures see [7] and references quoted therein).

As the generalized observables of operational quantum mechanics become an indispensable tool of quantum physics (it is well demonstrated by the mentioned monograph of Busch, Grabowski and Lahti [7]), it seems natural to apply to SPT the same method of extension which proved successful in transforming standard quantum mechanics into the operational one. The generalization of SPT obtained in that way is called *operational probability theory* (also fuzzy probability theory, OPT for brevity), see [8], [9], [10], [11]. The transformation of SPT into OPT is done essentially by extending the set of traditional random variables; the random variables of OPT are called *operational random variables* (o.r.variables for brevity), they are formally described by statistical maps (see Subsection 1.3 below).

In Section 2 we show how the concept of operational random variable is implied by elementary considerations of statistical experiments, we outline also some basic notions of OPT. Section 3 is devoted to a question of representing o.r.variables by standard random variables; it is proved that an arbitrary family of o.r.variables such that their outcome spaces are mutually independent admits a satisfactory representation in terms of standard random variables. That result could explain why o.r.variables (and OPT as well) have been ignored up to now.

Having in mind that during the last century the well established standard concept of random variable has not been seriously challenged (in spite of warnings issued by quantum mechanical theorists), one can wonder why we actually need to extend it at all, and why in the particular manner mentioned above. In Section 4 we formulate precisely an argument which demonstrates the need for generalizing SPT into OPT. We show namely that OPT, contrary to SPT, is able to describe the so called Bell phenomenon [3], which manifests itself in quantum mechanics in the form of well known Bell inequalities (see Section 5).

1.2. Let $(\Omega, \mathcal{B}(\Omega))$ be a measurable space, we assume that all singletons $\{\omega\}$, $\omega \in \Omega$, are measurable. $M_1^+(\Omega)$ denotes then the convex set of all σ-additive probability measures on $(\Omega, \mathcal{B}(\Omega))$, it will be seen as the base of the base normed Banach space $M(\Omega)$ of all bounded σ-additive signed measures on $(\Omega, \mathcal{B}(\Omega))$. The set of all measurable functions on Ω taking values in the real interval $[0,1]$ will be denoted $\mathcal{E}(\Omega)$; measurable subsets of Ω will be identified with their characteristic functions, so $\mathcal{B}(\Omega) \subset \mathcal{E}(\Omega)$. The set $\mathcal{E}(\Omega)$, called the *set of effects* on Ω, will be considered as an order interval of the vector lattice $\mathcal{F}(\Omega)$ of all real measurable bounded functions on Ω, elements of $\mathcal{B}(\Omega)$ form the set of extreme elements of the convex set $\mathcal{E}(\Omega)$. The integral defines the natural duality between Banach spaces $M(\Omega)$ and $\mathcal{F}(\Omega)$. The dual pair $\langle M(\Omega), \mathcal{F}(\Omega)\rangle$ of Banach spaces provides the basic framework for OPT. For more details see [12].

Standard random variables are represented by measurable functions on measurable spaces. A measurable function, say $F\colon \Omega \to \Xi$, associates with any $\mu \in M_1^+(\Omega)$ a probability measure μ_F on Ξ, called the *distribution* of F at μ. Clearly,

$$\mu_F(X) = \mu(F^{-1}(X)) = \int\limits_{\Omega} \chi_{F^{-1}(X)}(\omega)\, \mu(\mathrm{d}\omega)$$

for any measurable subset $X \in \mathcal{B}(\Xi)$. Thus the measurable function $F\colon \Omega \to \Xi$ extends in a natural way to the affine map

$$D_F\colon M_1^+(\Omega) \to M_1^+(\Xi), \qquad D_F\mu := \mu_F\,,$$

called the *distribution functional* of F. The affine map $D_F\colon M_1^+(\Omega) \to M_1^+(\Xi)$ is an example of a *statistical map* (see [12] for definitions).

1.3. In order to make clear the latter notion, let us consider an affine map $A\colon M_1^+(\Omega) \to M_1^+(\Xi)$. As $M_1^+(\Omega)$ generates linearly the Banach space $M(\Omega)$ of all bounded signed measures on $(\Omega, \mathcal{B}(\Omega))$, the map A extends to the linear map $\overline{A}\colon M(\Omega) \to M(\Xi)$. The map $\overline{A}$ has to be bounded, what implies that there exists the Banach dual $\overline{A}^*\colon M(\Xi)^* \to M(\Omega)^*$ which is necessarily weak*-to-weak* continuous. The dual map $\overline{A}^*$ restricted to $\mathcal{E}(\Xi) \subseteq \mathcal{F}(\Xi)$ will be denoted A^*.

It is clear that for any $f \in \mathcal{E}(\Xi)$

$$(A^* a_f)(\mu) := \int\limits_{\Xi} f(\xi)\, (A\mu)(\mathrm{d}\xi) = a_f(A\mu)\,,$$

where $\mu \in M_1^+(\Omega)$ and a_f is the affine functional on $M_1^+(\Xi)$ defined by

$$a_f(\nu) := \int\limits_{\Xi} f(\xi)\, \nu(\mathrm{d}\xi)\,;$$

for details we refer to [12] again. A^*a_f does not have to belong to $\mathcal{E}(\Omega)$, an affine map $A\colon M_1^+(\Omega) \to M_1^+(\Xi)$ is called *a statistical map* if it satisfies the condition:

$$A^*(\mathcal{E}(\Xi)) \subseteq \mathcal{E}(\Omega). \tag{1}$$

An equivalent characterization of a statistical map (see [12; Proposition of 2.1]) is:

$$(A\mu)(X) = \int_\Omega (A\delta_\omega)(X)\,\mu(\mathrm{d}\omega), \tag{2}$$

what implies that any statistical map is uniquely determined by its restriction to the set $\delta\Omega$ of all Dirac measures on Ω. A statistical map $A\colon M_1^+(\Omega) \to M_1^+(\Xi)$ is called *sharp* if $A(\delta\Omega) \subseteq \partial M_1^+(\Xi)$, where ∂S denotes the set of all extreme points of the convex set S, and *strict* if $A(\delta\Omega) \subseteq \delta\Xi$.

1.4. It is evident that in a case of a finite Ω, there is no difference between general affine maps and statistical maps. Indeed, assume that Ω is finite. According to the assumed convention, all singletons $\{\omega\}$ are measurable, what implies that $\mathcal{B}(\Omega) = 2^\Omega$. Hence $\mathcal{E}(\Omega)$ contains all functions $\Omega \to [0,1]$, and consequently all affine maps $M_1^+(\Omega) \to M_1^+(\Xi)$ with arbitrary measurable spaces Ξ satisfy the condition of formula (1).

1.5. The distribution functional $D_F\colon M_1^+(\Omega) \to M_1^+(\Xi)$ uniquely characterizes the measurable function $F\colon \Omega \to \Xi$, so one could choose statistical maps instead of traditional measurable functions to represent standard random variables. It is easy to realize that distribution functionals of measurable functions form a very special class of statistical maps: a statistical map $A\colon M_1^+(\Omega) \to M_1^+(\Xi)$ equals D_F for some $F\colon \Omega \to \Xi$ if and only if A is strict (see [12]).

Here we find a natural area to extend the standard concept of random variable: all statistical maps $M_1^+(\Omega) \to M_1^+(\Xi)$ are considered as formal representatives of *operational random variables* (also fuzzy random variables, *o.r.variables* for brevity). By *operational* (or *fuzzy*) *probability theory* (OPT for brevity) we will understand a theory based on $M_1^+(\Omega)$ and assuming all statistical maps $M_1^+(\Omega) \to M_1^+(\Xi)$ to represent (operational) random variables. It should be stressed that o.r.variables appear in a natural way in circumstances typical for science, examples can be found in [13], [9].

The above remarks show that standard random variables can be identified with a particular class of o.r.variables, that of strict ones; their characteristic feature is that they map Dirac measures into Dirac measures, while a general o.r.variable could assign nontrivial probability distibutions to Dirac measures on Ω. Following common habits of SPT, where usually one does not worry about distinguishing between random variables and their formal representants, measurable functions, we will identify o.r.variables with the corresponding statistical maps. Nevertheless, we should mention that there are various formal representations of the concept of o.r.variable, formally equivalent to the one we adopt here (see [12]). In particular, a natural one-to-one correspondence between statistical maps and effect valued measures ([12; Theorem of 4.5]) makes it possible to represent o.r.variables just by effect valued measures; that has been proposed recently by S. Gudder.

2. Operational probability theory

2.1. Let us recall what is stated in numerous textbooks and monographs (here we quote [16]): *Probability theory will provide us with mathematical models for describing experiments with random outcomes.* A short reflection suffices to realize that general statistical maps are the most natural formal representants for such experiments. We restrict our attention to a special (nevertheless sufficiently large) class of experiments with random outcomes called *measurements* (see, for instance, [17], [18], or a clear and concise description in [19]).

It is commonly accepted that any measurement has to consists of three stages:

- a preparation of the object to be examined,
- bringing it in contact with another object, the measuring device,
- an observation of the reaction of the latter.

In general, a measurement is performed on a statistical ensemble of identically prepared objects, so it results in a probability distribution on the space of outcomes of single individual measurements. It can happen that two different preparation procedures produce the same result for every measuring device, also two different measuring devices could show the same final response at every preparation. That leads to natural equivalence relations on the set of all preparation procedures and on the set of all measuring devices; equivalence classes defined by them are called respectively *states* and *observables* (measurable properties). Now, an abstract description of an observable (or: of a measurement of an observable, if we would wish to distinguish between measurable properties and the way they are measured) is a map which transforms states into probability distributions on the space of outcomes of the observable.

We should take into account that various statistical ensembles can be mixed together; that induces a convex structure on the set S of states with extreme points of S representing most precise preparations (physical: *pure states*). All observables we are going to consider react linearly on the mixing of ensembles; thus, an observable is an affine (i.e. convexity preserving) map of a convex set S of states into a convex set $M_1^+(\Xi)$ of probability measures on the outcome space Ξ.

Finally, we assume that the "experiments with random outcomes" we are interested in are performed on "classical" (as opposed to "quantal") objects. It is commonly accepted that the "classical" nature of an object manifests formally as $S = M_1^+(\Omega)$ for some measurable space Ω. Now, if we extend the property of preserving convexity over "uncountable mixtures", we end with the concept of statistical map.

Thus we have found that a natural mathematical model of an "experiment with random outcomes" performed on a "classical" object is provided by a statistical map $A\colon M_1^+(\Omega) \to M_1^+(\Xi)$. The remarks of Section 1 indicate that some special statistical maps provide (*via* the concept of the distribution functional) an equivalent description of traditional random variables; nevertheless, there are statistical maps which do not correspond to standard random variables. The above discussion indicates that the latter have the same operational interpretation as the former, so there is no *a priori* reason to deny them as possible models of "experiments with random outcomes". Hence, we admit all statistical maps as representing generalized random variables (o.r.variables); o.r.variables corresponding to standard random variables will be called *strict*.

Separating for a moment the operational concept of an o.r.variable from its formal model, one can say that an o.r.variable is an experimentally measurable property of a "classical" statistical system which, when tested at a fixed most precisely prepared state, produces a (nontrivial, in general) probability distribution on its outcome space. A direct inspiration comes, of course, from quantum mechanics, because quantal "observables" usually generate nontrivial probability distributions at pure states.

The distinction between standard and nonstandard o.r.variables is in fact of a deep nature. The traditional notion of random variable is based on the deterministic paradigm: given cause (a state prepared in the most precise way) implies always the same well defined result (the value of a classical random variable at that state). The operational notion of random variable is intrinsically indeterministic: given cause (a state prepared in the most precise way) implies in general a nontrivial statistical scatter of possible results (as the probability measure attached by the operational random variable to that state does not have to be concentrated at one point). Thus the classical probability theory describes merely a lack of a complete information in a deterministic world, while the operational probability theory is able to describe an essential randomness — either inherent or induced by an uncontrolled outer influence.

It would be worth to mention here that o.r.variables cannot be identified as fuzzy random variables (in spite of the original misinterpretation, [8]). A fuzzy random variable (f.r.variable for short) would be described by a function F: $\Omega \to \mathcal{E}(\Lambda)$ for some measurable spaces Ω and Λ, comp. [14], [15]. Now the remarks of [12; Subsection 4.7] show that o.r.variables and f.r.variables are represented by essentially different formal concepts. In view of the discussion above, one can say that f.r.variables belong to the class of standard (i.e. deterministic) random variables; namely, they are just these standard random variables which have outcomes in families of fuzzy sets.

One could ask why the concept of o.r.variable, being so evident and natural from the general point of view of the theory of measurement, is ignored by SPT. One of possible reasons could be the classical deterministic paradigm which seems to underlay SPT and makes it to accept only these uncertainties which results from an incomplete knowledge of causes. Another hypothetical reason would be a rather unexpected fact: one can easily show that o.r.variables can be well represented by standard random variables. We will show that in Section 3, now we have to introduce some basic concepts and facts of OPT.

2.2. We start with reminding the concept of a product of statistical maps (hence, of o.r.variables) introduced in [12].

Let $\{A_i : i = 1,2,\dots,n\}$ be a finite set of o.r.variables, $A_i\colon M_1^+(\Omega) \to M_1^+(\Xi_i)$, let $\bigtimes_{i=1}^{n} \Xi_i$ denote the Cartesian product $\Xi_1 \times \Xi_2 \times \cdots \times \Xi_m$, which carry the measurable structure generated in the known way by the ones of $\Xi_1, \Xi_2, \dots, \Xi_m$. To any $\omega \in \Omega$ one assigns the product measure

$$A_1\delta_\omega \otimes A_2\delta_\omega \otimes \cdots \otimes A_n\delta_\omega \in M_1^+\Big(\bigtimes_{i=1}^{n} \Xi_i\Big).$$

The obtained map $\Omega \to M_1^+\Big(\bigtimes_{i=1}^{n} \Xi_i\Big)$ defines the unique o.r.variable $M_1^+(\Omega) \to M_1^+\Big(\bigtimes_{i=1}^{n} \Xi_i\Big)$, denoted $\bigotimes_{i=1}^{n} A_i$ and called the *product* of o.r.variables $A_1, \dots, A_n$.

It is easy to show that the product of o.r.variables generalizes the corresponding concept of SPT (see [12; Subsection 5.7]).

LEMMA. *Let $\{A_1, A_2, \dots, A_m\}$ be a finite collection of strict o.r.variables, $A_i\colon M_1^+(\Omega) \to M_1^+(\Xi_i)$, so every A_i is the distribution functional of a standard random variable $F_i\colon \Omega \to \Xi_i$, $i = 1,2,\dots,m$. Then the product o.r.variable $\bigotimes_{i=1}^{n} A_i$ is the strict o.r.variable which is the distribution functional of the standard random variable $F\colon \Omega \to \bigtimes_{i=1}^{n} \Xi_i$,*

$$\Omega \ni \omega \to F(\omega) = \big(F_1(\omega), F_2(\omega), \dots, F_m(\omega)\big) \in \bigtimes_{i=1}^{n} \Xi_i .$$

P r o o f. $\Big(\bigotimes_{i=1}^{n} A_i\Big)\delta_\omega = \bigotimes_{i=1}^{n}(A_i\delta_\omega) = \bigotimes_{i=1}^{n} \delta_{F_i(\omega)} = \delta_{F(\omega)}$. The measurability of the function $\omega \to F(\omega) = \big(F_1(\omega), F_2(\omega), \dots, F_m(\omega)\big)$ is a standard fact of SPT, see [16] for instance. Now,

$$\begin{aligned}\Big(\bigotimes_{i=1}^{n} A_i\Big)\mu(X) &= \int_\Omega \Big(\bigotimes_{i=1}^{n} A_i\Big)\delta_\omega(X)\,\mu(\mathrm{d}\omega)\\ &= \int_\Omega \delta_{F(\omega)}(X)\,\mu(\mathrm{d}\omega) = \mu\big(F^{-1}(X)\big)\end{aligned}$$

for any $X \in \mathcal{B}\Big(\bigtimes_{i=1}^{n} \Xi_i\Big)$. □

2.3. A remarkable property of the product $\bigotimes_{i=1}^{n} A_i \colon M_1^+(\Omega) \to M_1^+\Big(\bigtimes_{i=1}^{n} \Xi_i\Big)$ of a family $\{A_i : i = 1, 2, \ldots, n\}$ of o.r.variables is that one can get back all original o.r.variables A_i, $i = 1, 2, \ldots, n$, by composing $\bigotimes_{i=1}^{n} A_i$ with marginal projections $\Pi_i \colon M_1^+\Big(\bigtimes_{i=1}^{n} \Xi_j\Big) \to M_1^+(\Xi_i)$:

$$A_i = \Pi_i \circ \bigotimes_{i=1}^{n} A_j\,.$$

This property of the product of o.r.variables leads to the concept of a joint o.r.variable.

Definition. A *joint o.r.variable* for a family $\{A_i : i = 1, 2, \ldots, n\}$ of o.r.variables $A_i \colon M_1^+(\Omega) \to M_1^+(\Xi_i)$ is an o.r.variable $A \colon M_1^+(\Omega) \to M_1^+\Big(\bigtimes_{i=1}^{n} \Xi_i\Big)$ such that $A_i = \Pi_i \circ A$ for every $i = 1, \ldots, n$.

2.4. Any finite collection $\{A_i : i = 1, 2, \ldots, n\}$ of o.r.variables has at least one joint o.r.variable: the product $\bigotimes_{i=1}^{n} A_i$. It is interesting to observe that if A_i are not sharp, then there are many joint o.r.variables for $\{A_i : i = 1, 2, \ldots, n\}$, see below. That does not occur if all A_i are strict.

Lemma. *Let $\{A_i : i = 1, 2, \ldots, n\}$ be a family of strict o.r.variables. Then the product $\bigotimes_{i=1}^{n} A_i$ is the unique joint o.r.variable for $\{A_i : i = 1, 2, \ldots, n\}$.*

Proof. If $B \colon M_1^+(\Omega) \to M_1^+\Big(\bigtimes_{i=1}^{n} \Xi_i\Big)$ would be another joint o.r.variable for the family $\{A_i : i = 1, 2, \ldots, n\}$, then

$$(\Pi_i \circ B)\delta_\omega = \delta_{F_i(\omega)}$$

for all $i = 1,\dots,n$ and $\omega \in \Omega$, where $F_i\colon \Omega \to \Xi_i$ are the standard random variables corresponding to the deterministic o.r.variables A_i, $i = 1,\dots,n$. This implies that

$$B\delta_\omega = \delta_{F(\omega)} = \Big(\bigotimes_{i=1}^{n} A_i\Big)\delta_\omega ,$$

where $F\colon \Omega \to \bigtimes_{i=1}^{n} \Xi_i$ is defined in Lemma of 2.2 above. Hence, B is a strict o.r.variable generated by the same standard random variable as $\bigotimes_{i=1}^{n} A_i$, what means that $B = \bigotimes_{i=1}^{n} A_i$. □

2.5. The fact that a genuine collection of o.r.variables admits many different joint o.r.variables can be easily demonstrated on elementary examples, one coming from quantum mechanics can be found in [8], another is provided below [10]:

EXAMPLE. Consider two o.r.variables $A_1\colon M_1^+(\Omega) \to M_1^+(\mathbb{R})$, $A_2\colon M_1^+(\Omega) \to M_1^+(\mathbb{R})$ such that $A_1\mu = A_2\mu$ for all $\mu \in M_1^+(\Omega)$. The product o.r.variable $A_1 \otimes A_2\colon M_1^+(\Omega) \to M_1^+(\mathbb{R}^2)$ is one of their joint o.r.variables, another joint o.r.variable for A_1 and A_2 can be constructed as follows.

Let $\Lambda := \{(\lambda,\lambda) :\ \lambda \in \mathbb{R}\} \subset \mathbb{R}^2$ denote the diagonal of the direct product $\mathbb{R}^2$; clearly $\Lambda \in \mathcal{B}(\mathbb{R}^2)$. For arbitrary $\nu \in M_1^+(\mathbb{R})$ we define $\nu \odot \nu \in M_1^+(\mathbb{R}^2)$ by

$$(\nu \odot \nu)(X) := \nu\big(\pi_1(X \cap \Lambda)\big),$$

where π_1 is the coordinate projection $\mathbb{R}^2 \ni (\lambda_1,\lambda_2) \to \lambda_1 \in \mathbb{R}$. Then we define map $\varphi\colon \Omega \to M_1^+(\mathbb{R}^2)$, $\varphi(\omega) := A_1\delta_\omega \odot A_2\delta_\omega$ for all $\omega \in \Omega$. It is easy to show that φ is a statistical function (see [12]). Indeed, for a fixed $X \in \mathcal{B}(\mathbb{R}^2)$ we have

$$\varphi(\omega)(X) = (A_1\delta_\omega \odot A_2\delta_\omega)(X) = (A_1\delta_\omega)\big(\pi_1(X \cap \Lambda)\big),$$

and the resulting function on Ω is measurable because A_1 is an o.r.variable.

Now we take the o.r.variable generated by the statistical function φ (see [12]) and call it $A_1 \odot A_2$; clearly, $A_1 \odot A_2$ is a joint o.r.variable for A_1 and A_2. The uniqueness of joint o.r.variable for strict o.r.variables implies that $A_1 \odot A_2 = A_1 \otimes A_2$ for strict A_1 (and A_2, as we have assumed $A_1 = A_2$). It is equally evident that $A_1 \odot A_2 \neq A_1 \otimes A_2$ for genuine o.r.variable A_1 $(= A_2)$. Indeed, $(A_1 \odot A_2)\delta_\omega$ is concentrated on Λ, while $(A_1 \otimes A_2)\delta_\omega$ is concentrated on $\mathrm{Supp}(A_1\delta_\omega) \times \mathrm{Supp}(A_2\delta_\omega)$ where $\mathrm{Supp}(\nu)$ denotes the support of the measure ν. Hence $(A_1 \odot A_2)\delta_\omega \neq (A_1 \otimes A_2)\delta_\omega$ except the case of $A_1\delta_\omega = \delta_\lambda$ for some $\lambda \in \mathbb{R}$.

2.6. Let us notice a "global" attitude of OPT as opposed to the "local" one assumed by SPT: a majority of traditional concepts and results of SPT refers to a fixed probability measure on Ω, while, in the framework of OPT, one is interested rather in concepts and results referring to all probability measures on $(\Omega, \mathcal{B}(\Omega))$. Thus OPT is founded on the convex set $M_1^+(\Omega)$ instead on a probability space $(\Omega, \mathcal{B}(\Omega), \mu)$. That "global" point of view can be traced back to the operational approach to statistical physical theories (see [7] and references quoted therein), which is the birth-place of OPT. In fact it is not a necessary feature, nevertheless it would be rather hard to distinguish OPT from SPT if the former, like the latter, would be considered "locally", i.e. for a fixed measure on the space of elementary events. It seems that the only "local" effect of OPT which cannot be obtained from SPT is the Bell phenomenon (see Section 4).

3. Representing o.r.variables by standard random variables

3.1. Let $A\colon M_1^+(\Omega) \to M_1^+(\Xi)$ be an o.r.variable on Ω. Take the identity o.r.variable on Ω, $I_\Omega\colon M_1^+(\Omega) \leftrightarrow M_1^+(\Omega)$, and consider the product o.r.variable $A \otimes I_\Omega\colon M_1^+(\Omega) \to M_1^+(\Xi \times \Omega)$. It is easy to show that the composed map

$$\Pi_1 \circ (A \otimes I_\Omega)\colon M_1^+(\Omega) \to M_1^+(\Xi \times \Omega) \to M_1^+(\Xi),$$

where $\Pi_1 : M_1^+(\Xi \times \Omega) \to M_1^+(\Xi)$ is the marginal projection, returns the original o.r.variable:

Lemma 1. *For every o.r.variable $A\colon M_1^+(\Omega) \to M_1^+(\Xi)$,*

$$A = \Pi_1 \circ (A \otimes I_\Omega).$$

P r o o f. As every statistical map is determined by its restriction to Dirac measures (comp. formula (2) above), it suffices to show that $A\delta_\omega = (\Pi_1 \circ (A \otimes I_\Omega))\delta_\omega$ for every $\omega \in \Omega$. Now $(\Pi_1 \circ (A \otimes I_\Omega))\delta_\omega = \Pi_1(A\delta_\omega \otimes I_\Omega \delta_\omega) = A\delta_\omega$, because $\Pi_1(\mu \otimes \nu) = \mu$. □

Lemma 2. *For an arbitrary o.r.variable $A\colon M_1^+(\Omega) \to M_1^+(\Xi)$, the product o.r.variable $A \otimes I_\Omega\colon$ $M_1^+(\Omega) \to M_1^+(\Xi \times \Omega)$ is injective.*

P r o o f. Taking into account the condition (2) we get:

$$((A \otimes I_\Omega)\mu)(X \times Y) = \int_\Omega ((A \otimes I_\Omega)\delta_\omega)(X \times Y)\, \mu(\mathrm{d}\omega) = \int_Y (A\delta_\omega)(X)\, \mu(\mathrm{d}\omega)$$

for any $X \in \mathcal{B}(\Xi)$, $Y \in \mathcal{B}(\Omega)$, $\mu \in M_1^+(\Omega)$. If $X = \Xi$, then

$$((A \otimes I_\Omega)\mu)(\Xi \times Y) = \int_Y \mu(\mathrm{d}\omega) = \mu(Y).$$

Now it is evident that $\mu_1 \neq \mu_2$ implies $(A \otimes I_\Omega)\mu_1 \neq (A \otimes I_\Omega)\mu_2$ for any $\mu_1, \mu_2 \in M_1^+(\Omega)$. □

3.2. The marginal projection $\Pi_1 \colon M_1^+(\Xi \times \Omega) \twoheadrightarrow M_1^+(\Xi)$ can be seen as the strict o.r.variable defined by the measurable coordinate function $\pi_1 \colon \Xi \times \Omega \twoheadrightarrow \Xi$. Thus, the original o.r.variable A is the composition of the injective o.r.variable $A \otimes I_\Omega \colon M_1^+(\Omega) \to M_1^+(\Xi \times \Omega)$ and the strict o.r.variable Π_1 corresponding to the standard random variable π_1. That proves the following statement (a similar result has been obtained recently by S. Gudder [11]):

Proposition. *For any o.r.variable $A \colon M_1^+(\Omega) \to M_1^+(\Xi)$ there are: a measurable space Ω', an injective o.r.variable $H \colon M_1^+(\Omega) \to M_1^+(\Omega')$, and a standard random variable $F \colon \Omega' \to \Xi$ such that*

$$A = D_F \circ H,$$

where $D_F \colon M_1^+(\Omega') \to M_1^+(\Xi)$ is the distribution functional of F.

That result admits the following interpretation. The standard random variable F returns all probability distributions produced by the original o.r.variable A on its outcome space Ξ, so could be considered as a faithful representative of A. Assume now that for some $\omega_0 \in \Omega$ the resulting distribution $A\delta_{\omega_0}$ is not concentrated at one point of the value space Ξ, what is the indication of an inherent indeterminism of the o.r.variable A. The injective map $H \colon M_1^+(\Omega) \to M_1^+(\Omega')$ can be understood as resulting from a kind of "hidden variables theory", which provides a deeper and more detailed description of the set of states of the object under examination. Former most precise preparations symbolized by Dirac measures δ_ω, $\omega \in \Omega$, or just by points of Ω, disclose under H their subtle structure: in the new description, they are represented by nontrivial, in general, probability measures $H\delta_\omega$ on the new space Ω'. The equality

$$A\delta_{\omega_0} = D_F(H\delta_{\omega_0})$$

means now that the new extended description of states provided by H removes the original indeterminism demonstrated by A at ω_0, because the nontrivial probability distribution $A\delta_{\omega_0}$ appears "in fact" generated in a "classical" way by the standard random variable F at the state of incomplete information $H\delta_{\omega_0} \in M_1^+(\Omega')$. It is remarkable that this reinterpretation works simultaneously for all original pure states, so the "hidden variables theory" symbolized by H together with the standard random variable F representing A eliminate all instances of indeterminism which could be showed by the original o.r.variable A. In this way the original "indeterministic" o.r.variable become safely reinterpreted and absorbed by the standard formalism of SPT.

3.3. The above method of representing o.r.variables by standard random variables works also for finite families of o.r.variables (if only we ignore their eventual mutual connections, see the next section).

Let us consider a family $\{A_i : i = 1, \ldots, n\}$ of o.r.variables, $A_i \colon M_1^+(\Omega) \to M_1^+(\Xi_i)$. Let $A \colon M_1^+(\Omega) \to M_1^+\left(\bigtimes_{i=1}^{n} \Xi_i\right)$ be a joint o.r.variable of theirs. The preceeding considerations show that the o.r.variable $A \colon M_1^+(\Omega) \to M_1^+\left(\bigtimes_{i=1}^{n} \Xi_i\right)$ is well represented by the marginal projection

$$\Pi_{(1,2,\ldots,n)} \colon M_1^+\left(\left(\bigtimes_{i=1}^{n} \Xi_i\right) \times \Omega\right) \twoheadrightarrow M_1^+\left(\bigtimes_{i=1}^{n} \Xi_i\right),$$

what is symbolized by the equality

$$A = \Pi_{(1,2,\ldots,n)} \circ (A \otimes I_\Omega).$$

Taking into account that $A_j = \Pi_j \circ A$, $j = 1, \ldots, n$, (see 2.3), we get

$$A_j = \Pi_j \circ \Pi_{(1,2,\ldots,n)} \circ (A \otimes I_\Omega).$$

As $\Pi_j \circ \Pi_{(1,2,\ldots,n)}$ is the marginal projection $M_1^+\left(\left(\bigtimes_{i=1}^{n} \Xi_i\right) \times \Omega\right) \twoheadrightarrow M_1^+(\Xi_j)$ we find that every o.r.variable of the original set $\{A_i : i = 1, \ldots, n\}$ gets the faithful and strict representation on the extended space $\left(\bigtimes_{i=1}^{n} \Xi_i\right) \times \Omega$.

Thus we can simultaneously obtain standard representants for any finite family of o.r.variables. In fact, that holds also for an arbitrary family of o.r.variables, because the product o.r.variable can be defined for arbitrary sets of o.r.variables (comp. [12]).

The elimination of o.r.variables like that described above explains, perhaps why SPT could for so long time ignore o.r.variables. A decisive argument in favour of the proposed extension of the standard concept of random variabl comes from quantum physics: we demonstrate below that SPT (contrary to OPT) is not able to describe the Bell phenomenon.

4. Bell phenomenon

4.1. Families of o.r.variables considered up to now were of a rather particular form: outcome spaces of o.r.variables they consisted of showed no mutual connections. We are going to consider finite collections of o.r.variables which are free of that limitation.

DEFINITION. Let $(\Xi_1, \Xi_2, \dots, \Xi_n)$ be a finite ordered collection of measurable spaces. A *semi-projective family* of o.r.variables on Ω with outcome spaces $(\Xi_1, \Xi_2, \dots, \Xi_n)$ is a family $\mathcal{A}$ of o.r.variables on $M_1^+(\Omega)$ such that:

(i) for every $(i_1, i_2, \dots, i_r) \subseteq (1, 2, \dots, n)$ there is in $\mathcal{A}$ at most one (i.e. one or none) o.r.variable

$$A_{(i_1,i_2,\dots,i_r)} \colon M_1^+(\Omega) \to M_1^+(\Xi_{i_1} \times \Xi_{i_2} \times \dots \times \Xi_{i_r})\,;$$

(ii) if

$$A_{(i_1,i_2,\dots,i_r)} \colon M_1^+(\Omega) \to M_1^+(\Xi_{i_1} \times \Xi_{i_2} \times \dots \times \Xi_{i_r})\,,$$
$$A_{(j_1,j_2\dots,j_s)} \colon M_1^+(\Omega) \to M_1^+(\Xi_{j_1} \times \Xi_{j_2} \times \dots \times \Xi_{j_s})$$

are two members of $\mathcal{A}$ and if

$$(i_1, i_2, \dots, i_r) \cap (j_1, j_2 \dots, j_s) = (k_1, k_2, \dots, k_t) \neq \emptyset\,,$$

then there is in $\mathcal{A}$ a statistical map $A_{(k_1,k_2,\dots,k_t)} \colon M_1^+(\Omega) \to M_1^+(\Xi_{k_1} \times \Xi_{k_2} \times \dots \times \Xi_{k_t})$ such that

$$A_{(k_1,k_2,\dots,k_t)} = \Pi^{(i_1,i_2,\dots,i_r)}_{(k_1,k_2,\dots,k_t)} \circ A_{(i_1,i_2,\dots,i_r)} = \Pi^{(j_1,j_2,\dots,j_s)}_{(k_1,k_2,\dots,k_t)} \circ A_{(j_1,j_2\dots,j_s)}\,,$$

where

$$\Pi^{(i_1,i_2,\dots,i_r)}_{(k_1,k_2,\dots,k_t)} \colon M_1^+(\Xi_{i_1} \times \Xi_{i_2} \times \dots \times \Xi_{i_r}) \to M_1^+(\Xi_{k_1} \times \Xi_{k_2} \times \dots \times \Xi_{k_t})\,,$$
$$\Pi^{(j_1,j_2,\dots,j_s)}_{(k_1,k_2,\dots,k_t)} \colon M_1^+(\Xi_{j_1} \times \Xi_{j_2} \times \dots \times \Xi_{j_s}) \to M_1^+(\Xi_{k_1} \times \Xi_{k_2} \times \dots \times \Xi_{k_t})$$

are the marginal projections;

(iii) for every $i \in \{1, 2, \dots, n\}$ there is $A_i \in \mathcal{A}$ such that $A_i \colon M_1^+(\Omega) \to M_1^+(\Xi_i)$.

The o.r.variables $A_i \colon M_1^+(\Omega) \to M_1^+(\Xi_i)$ are called *basic* o.r.variables for $\mathcal{A}$.

The concept of a semi-projective family of o.r.variables has been introduced in [3] and called there a consistent family of observables.

4.2. Notice two special kinds of semi-projective families of o.r.variables:

Considered up to now families of o.r.variables like $\{A_i : i = 1, \dots, n\}$, $A_i \colon M_1^+(\Omega) \to M_1^+(\Xi_i)$, satisfy the above definition: they are semi-projective families as "empty" as possible — containing no o.r.variables besides the basic ones. The other extreme form these semi-projective families which have no "free places" at all. They are called projective families.

DEFINITION 2. A semi-projective family $\mathcal{A}$ of o.r.variables on Ω with outcome spaces $(\Xi_1, \Xi_2, \ldots, \Xi_n)$ is *projective* if instead of (i) we have

(i)' for every $(i_1, i_2, \ldots, i_r) \subseteq (1, 2, \ldots, n)$ there is in $\mathcal{A}$ exactly one o.r.variable

$$A_{(i_1,i_2,\ldots,i_r)}: M_1^+(\Omega) \to M_1^+(\Xi_{i_1} \times \Xi_{i_2} \times \cdots \times \Xi_{i_r}).$$

Let $\mathcal{A}$ be a semi-projective family of o.r.variables on Ω with outcome spaces $(\Xi_1, \Xi_2, \ldots, \Xi_n)$, let $\mu \in M_1^+(\Omega)$. It is evident that o.r.variables of $\mathcal{A}$ generate, acting on μ, a family of probability measures on spaces $\Xi_1, \Xi_2, \ldots, \Xi_n$ and eventually on some of their Cartesian products. The obtained family of measures, denoted $\mathcal{A}\mu$, will be called a *semi-projective family of probability measures* on $(\Xi_1, \Xi_2, \ldots, \Xi_n)$. If $\mathcal{A}$ is projective, then $\mathcal{A}\mu$ is a *projective family of probability measures in the standard sense* (see e.g. [16]).

4.3. A projective family of o.r.variables has a very special structure: all its members must have the form

$$A_{(i_1,i_2,\ldots,i_r)} = \Pi^{(1,2,\ldots,n)}_{(i_1,i_2,\ldots,i_r)} \circ A_{(1,2,\ldots,n)}, \tag{3}$$

so the "top" member $A_{(1,2,\ldots,n)}$ of such a family generates all other its members. That leads to a generalization of the concept of joint o.r.variable.

DEFINITION. Let $\mathcal{A}$ be a semi-projective family of o.r.variables on Ω with outcome spaces $(\Xi_1, \Xi_2, \ldots, \Xi_n)$. An o.r.variable $B: M_1^+(\Omega) \to M_1^+\left(\bigtimes_{i=1}^{n} \Xi_i\right)$ such that for every $A_{(i_1,i_2,\ldots,i_r)} \in \mathcal{A}$ the equality

$$A_{(i_1,i_2,\ldots,i_r)} = \Pi^{(1,2,\ldots,n)}_{(i_1,i_2,\ldots,i_r)} \circ B$$

holds will be called a *joint o.r.variable for* $\mathcal{A}$.

Thus, if a semi-projective family $\mathcal{A}$ possesses the "top" o.r.variable $A_{(1,2,\ldots,n)}$, then $A_{(1,2,\ldots,n)}$ is a joint o.r.variable for $\mathcal{A}$ (see formula (3)). One can ask if a semi-projective family of o.r.variables which does not possess the "top" member can have a joint o.r.variable, or equivalently: if a semi-projective family can be extended to a projective one.

The answer is obviously affirmative if a semi-projective family $\mathcal{A}$ contains only basic o.r.variables, because then the considerations of 2.4 apply. That observation generalizes [3]:

LEMMA. *If a semi-projective family $\mathcal{A}$ contains only product o.r.variables of its basic o.r.variables $A_1, A_2, \ldots, A_n$, then $A_1 \otimes A_2 \otimes \cdots \otimes A_n$ is a joint o.r.variable for $\mathcal{A}$.*

Proof. It suffices to realize that a composition of a marginal projection and a product o.r.variable is again a product o.r.variable. □

Now it is evident that the uniqueness of joint o.r.variables for sets of strict o.r.variables implies (see Lemma of 2.4):

COROLLARY. *If all basic o.r.variables of a semi-projective family $\mathcal{A}$ are strict, then there is a unique joint o.r.variable for $\mathcal{A}$, equal to the product of its basic o.r.variables.*

Proof. These o.r.variables of $\mathcal{A}$ which are not basic must be joint o.r.variables of some of basic ones. In fact, they must be product ones because the basic o.r.variables are strict (see Lemma of 2.4). Hence we can apply the last lemma. □

4.4. It can be demonstrated that there are semi-projective families of o.r.variables which do not have any joint o.r.variable. The simplest example [8] is quoted below, another one can be found in [3].

EXAMPLE. Let Ω be a finite set, consider a fixed point $\omega_0 \in \Omega$ and a family of o.r.variables $\mathcal{A} = \{A_1, A_2, A_3, A_{12}, A_{23}, A_{13}\}$ on Ω. Assume that A_1, A_2, A_3 are basic o.r.variables with two-point outcome spaces

$$\begin{aligned} A_1 &: M_1^+(\Omega) \to M_1^+(\{\xi_1, \xi_2\}), \\ A_2 &: M_1^+(\Omega) \to M_1^+(\{\eta_1, \eta_2\}), \\ A_3 &: M_1^+(\Omega) \to M_1^+(\{\zeta_1, \zeta_2\}), \end{aligned}$$

while

$$\begin{aligned} A_{12} &: M_1^+(\Omega) \to M_1^+(\{\xi_1, \xi_2\} \times \{\eta_1, \eta_2\}), \\ A_{23} &: M_1^+(\Omega) \to M_1^+(\{\eta_1, \eta_2\} \times \{\zeta_1, \zeta_2\}), \\ A_{13} &: M_1^+(\Omega) \to M_1^+(\{\xi_1, \xi_2\} \times \{\zeta_1, \zeta_2\}). \end{aligned}$$

Assume further that

$$\begin{aligned} A_1 \delta_{\omega_0} &= \frac{1}{2}\delta_{\xi_1} + \frac{1}{2}\delta_{\xi_2}, \\ A_2 \delta_{\omega_0} &= \frac{1}{2}\delta_{\eta_1} + \frac{1}{2}\delta_{\eta_2}, \\ A_3 \delta_{\omega_0} &= \frac{1}{2}\delta_{\zeta_1} + \frac{1}{2}\delta_{\zeta_2}, \end{aligned}$$

and choose the o.r.variables A_{12}, A_{23}, A_{13} in such a way that:

$$\begin{aligned} A_{12} \delta_{\omega_0} &= \frac{1}{2}\delta_{(\xi_1, \eta_1)} + \frac{1}{2}\delta_{(\xi_2, \eta_2)}, \\ A_{23} \delta_{\omega_0} &= \frac{1}{2}\delta_{(\eta_1, \zeta_1)} + \frac{1}{2}\delta_{(\eta_2, \zeta_2)}, \\ A_{13} \delta_{\omega_0} &= \frac{1}{2}\delta_{(\xi_1, \zeta_2)} + \frac{1}{2}\delta_{(\xi_2, \zeta_1)}. \end{aligned}$$

The measures generated by A_{12} and A_{23} at δ_{ω_0} show a strong correlation, while the measure $A_{13}\delta_{\omega_0}$ shows a strong anticorrelation of the corresponding marginal measures.

So far we have defined o.r.variables $A_1, A_2, A_3, A_{12}, A_{23}, A_{13}$ at one point of Ω. The assumed finiteness of Ω implies (see 1.4) that the o.r.variables in question can be defined at other points of Ω in an arbitrary way. Hence, without providing an explicit construction for $A_1, A_2, A_3, A_{12}, A_{23}, A_{13}$, we state that there exist o.r.variables having the listed above properties.

Now we easily find that there is no probability measure on the 8-point set $\{\xi_1, \xi_2\} \times \{\eta_1, \eta_2\} \times \{\zeta_1, \zeta_2\}$ which would return the three measures $A_{12}\delta_{\omega_0}$, $A_{23}\delta_{\omega_0}$, $A_{13}\delta_{\omega_0}$ (hence the all six measures $\mathcal{A}\delta_{\omega_0}$) as its marginals. Indeed, if such a global joint measure did exist, then, for instance, the two measures $A_{12}\delta_{\omega_0}$ and $A_{23}\delta_{\omega_0}$ would force the third one to be $\frac{1}{2}\delta_{(\xi_1,\zeta_1)} + \frac{1}{2}\delta_{(\xi_2,\zeta_2)}$ instead of $\frac{1}{2}\delta_{(\xi_1,\zeta_2)} + \frac{1}{2}\delta_{(\xi_2,\zeta_1)}$.

4.5. The appearance of semi-projective families of o.r.variables which do not have any joint o.r.variable is called the *Bell phenomenon* ([3]). Corollary of 4.3 shows that the Bell phenomenon does not appear in SPT, while the above example shows that it does in OPT. It is clear that the occurrence of Bell phenomenon in OPT and its non-occurrence in SPT are implied by the uniqueness of joint random variables in SPT and the non-uniqueness in OPT.

Finally, let us notice the following simple fact. If $\mathcal{A}$ is a semi-projective family of o.r.variables on Ω which has a joint o.r.variable, then for every $\mu \in M_1^+(\Omega)$ the semi-projective family of measures $\mathcal{A}\mu$ has a joint probability measure. Thus, if there exists $\mu \in M_1^+(\Omega)$ such that the family of measures $\mathcal{A}\mu$ does not have any joint probability measure, then the family $\mathcal{A}$ does not have a joint o.r.variable (hence shows the Bell phenomenon). This is a typical situation when we observe the Bell phenomenon.

5. The Bell phenomenon in quantum mechanics

5.1. In order to connect the above defined concept of Bell phenomenon with the research field initiated by the work of Bell we need some new notions. In what follows "quantum mechanics" means the standard version of quantum theory as described for instance in the monograph of von Neumann [20] as well as its modern extension called operational quantum mechanics as described in [7].

Let $S_{\mathcal{H}}$ denote the convex set of von Neumann's density operators on a complex separable Hilbert space $\mathcal{H}$, let $\Omega_{\mathcal{H}}$ denote its set of extreme points $S_{\mathcal{H}}$, which formally consists of all trace class self-adjoint operators of trace 1, represents the set of states of a quantum-mechanical system, while $\Omega_{\mathcal{H}}$ is the set of pure states. All physical quantities ("observables") pertinent to the quantum-mechanical system are represented by affine maps $S_{\mathcal{H}} \to M_1^+(\Xi)$ with Ξ — an arbitrary measurable space of interest. If we represent an observable in the traditional way as a self-adjoint operator on $\mathcal{H}$, the corresponding affine map is defined *via* the spectral decomposition.

One can construct (see [21], [22]) an affine surjection $R_M : M_1^+(\Omega_{\mathcal{H}}) \twoheadrightarrow S_{\mathcal{H}}$, called the *canonical classical extension* of quantum mechanics, which provides a natural representation for all quantum observables: an observable $A\colon S_{\mathcal{H}} \to M_1^+(\Xi)$ is represented simply by the o.r.variable

$$A \circ R_M : M_1^+(\Omega_{\mathcal{H}}) \twoheadrightarrow S_{\mathcal{H}} \to M_1^+(\Xi).$$

The Misra map R_M is the connecting bridge between quantum mechanics and OPT.

5.2. Let then $\mathcal{A}'$ be a finite family of quantum observables, it can happen that the set $\mathcal{A} := \{A \circ R_M\colon\ A \in \mathcal{A}'\}$ of o.r.variables on $\Omega_{\mathcal{H}}$ is an instance of the Bell phenomenon. That means that there is $\mu \in M_1^+(\Omega_{\mathcal{H}})$ such that the family of measures $\mathcal{A}\mu = \{(A \circ R_M)\mu :\ A \in \mathcal{A}'\}$ does not have a joint probability measure. The semi-projective family of measures $\{(A \circ R_M)\mu :\ A \in \mathcal{A}'\}$ can be seen as $\{A(R_M\mu) :\ A \in \mathcal{A}'\}$ i.e. as generated directly by observables of $\mathcal{A}'$ from the quantal state $\alpha := R_M\mu \in S_{\mathcal{H}}$. The nonexistence of a joint probability measure for the semi-projective family of measures $\{A\alpha :\ A \in \mathcal{A}'\}$ can be expressed as a violation of Bell-type inequalities. In this way all instances of the Bell theorem find faithful representations in the framework of OPT by means of the canonical classical extension of SQM.

Now it is easy to realize that Bell inequalities of all kinds as well as all Bell-type theorems in SQM can be reduced to the following observation: there exists a family $\mathcal{A}'$ of observables of SQM such that the family $\mathcal{A} := \{A \circ R_M :\ A \in \mathcal{A}'\}$ is a semi-projective family of o.r.variables on $\Omega_{\mathcal{H}}$ which does not have any joint o.r.variable. A family of quantal observables having that property has been for the first time constructed by D. Bohm in his discussion of the EPR example [23], J. S. Bell [1] was the first who proved that Bohm's family of observables shows what we call the Bell phenomenon. Then it has been noticed by Fine [2] that the violation of Bell inequalities means nothing more as the nonexistence of a joint probability measure, that idea has been developed above following [3].

As the Bell phenomenon cannot occur in SPT (see Corollary of 4.3), all known experimental confirmations of violations of Bell's inequalities support the idea of generalizing SPT into OPT.

6. Concluding remarks

We have demonstrated that operational probability theory (OPT) is an essential and nontrivial extension of the traditional probability theory. Its ability to adopt the typically quantal Bell phenomenon shows, together with remarks of Subsection 2.1 and Section 5, that OPT could provide an underlying probabilistic framework for both classical and quantal statistical physical theories and, possibly, for hypothetical theories of mesoscopic objects. The concept of operational random variable applies to essentially indeterministic systems in general what suggests that OPT should have interesting applications outside physics as well.

REFERENCES

[1] BELL, J. S.: *On the Einstein-Podolsky-Rosen paradox*, Physics **1** (1964), 195–200.

[2] FINE, A.: *Hidden variables, joint probability, and the Bell inequalities*, Phys. Rev. Lett. **48** (1982), 291–295.

Joint distributions, quantum correlations, and commuting observables, J. Math. Phys. **23** (1982), 1306–1310.

[3] BELTRAMETTI, E. G.—BUGAJSKI, S.: *The Bell phenomenon in classical frameworks*, J. Phys. A **29** (1996), 247–261.

[4] ACCARDI, L.—CECCHINI, C.: *Conditional expectations on von Neumann algebras and a theorem of Takesaki*, J. Funct. Anal. **45** (1982), 245–273.

[5] ACCARDI, L.—FRIGERIO, A.—LEWIS, J. T.: *Quantum stochastic processes*, Publ. Res. Inst. Math. Sci. **18** (1982), 97–133.

[6] STREATER, R. F.: *Classical and quantum probability*. arXiv:math-ph/0002029 (27 Feb 2000).

[7] BUSCH, P.—GRABOWSKI, M.—LAHTI, P. J.: *Operational Quantum Physics*, Springer-Verlag, Berlin, 1995.

[8] BUGAJSKI, S.: *Fundamentals of fuzzy probability theory*, Internat. J. Theoret. Phys. **35** (1996), 2229–2244.

[9] BUGAJSKI, S.—HELLWIG, K.-E.—STULPE, W.: *On fuzzy random variables and statistical maps*, Rep. Math. Phys. **41** (1998), 1–11.

[10] BUGAJSKI, S.: *Fuzzy stochastic processes*, Open Syst. Inf. Dyn. **5** (1998), 169–185.

[11] GUDDER, S.: *Fuzzy probability theory*, Demonstratio Math. **31** (1998), 235–254.

[12] BUGAJSKI, S.: *Statistical maps I. Basic properties*, Math. Slovaca **51** (2001), 321 342.

[13] BUGAJSKI, S.: *Classical and quantal in one or How to describe mesoscopic systems*, Molecular Phys. Rep. **11** (1995), 161–171.

[14] PURI, M. L.—RALESCU, D. A.: *Fuzzy random variables*, J. Math. Anal. Appl. **114** (1986), 409–422.

[15] RIEČAN, B.—NEUBRUNN, T.: *Integral, Measure, and Ordering*. Math. Appl. 411, Kluwer, Dordrecht, 1997.

[16] BAUER, H.: *Probability Theory and Elements of Measure Theory*, Academic Press, London, 1981.

[17] HOLEVO, A. S.: *Probabilistic and Statistical Aspects of Quantum Theory*, North-Holland, Amsterdam, 1982.

[18] BUSCH, P.—LAHTI, P. J.—MITTELSTAEDT, P.: *The Quantum Theory of Measurement* (2nd ed.), Springer-Verlag, Berlin, 1996.

[19] ARAKI, H.: *A remark on Machida-Namiki theory of measurement*, Progr. Theoret. Phys. **64** (1980), 719–730.

[20] VON NEUMANN, J.: *Mathematical Foundations of Quantum Mechanics*, Princeton University Press, Princeton, N.J., 1955.

[21] MISRA, B.: *On a new definition of quantal states.* In: Physical Reality and Mathematical Description (C. P. Enz, J. Mehra, eds.), D. Reidel Publishing Company, Dordrecht-Holland, 1974, pp. 455–476.

[22] BELTRAMETTI, E. G.—BUGAJSKI, S.: *Quantum observables in classical frameworks*, Internat. J. Theoret. Phys. **34** (1995), 1221–1229.
A classical extension of quantum mechanics, J. Phys. A **28** (1995), 3329–3343.

[23] BOHM, D.: *Quantum Theory*, Prentice-Hall, Inc., Englewood Cliffs, NJ., 1951.

Received July 17, 1999
Revised May 4, 2000

Institute of Physics
University of Silesia
ul. Uniwersytecka 4
PL 40-007 Katowice
POLAND
E-mail: bugajski@us.edu.pl