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NONLINEAR BOUNDARY VALUE PROBLEMS AT RESONANCE FOR DIFFERENTIAL EQUATIONS IN BANACH SPACES

BOGDAN PRZERADZKI[1]

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ABSTRACT. The perturbation method developed in [12] – [16] is applied to nonlinear BVP's $x' - A(t)x = f(t,x)$, $B_1x(0) + B_2x(1) = B_3(x)$, in a Banach space, where the linear homogeneous problem possesses nontrivial solutions and the nonlinearities f, B_3 have at most linear growth. Examples of such problems are given.

1. Introduction

The question of the solvability of boundary value problems $Lx = N(x)$, where L is a linear differential operator with nontrivial kernel and N is a superposition operator, has a long history. The first remarkable result was obtained in 1969 by Landesman and Lazer [10] for the zero-data Dirichlet BVP for a second order elliptic equation in a bounded domain Ω with $N(x)(t) = \lambda_0 x + f(t,x)$, where λ_0 is a simple eigenvalue of the elliptic operator and $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is bounded. The authors used the well-known alternative method (see[1]), which was also applied by Williams [17] to generalize this result for an arbitrary eigenvalue (this means that the dimension of the linear space of solutions to $Lx = 0$ may be greater than 1 but finite; we shall say that the resonance is multidimensional). This and other methods were then used to get existence for many similar problems such as:

$$x'' + m^2 x = f(t,x)\,, \qquad x(0) = x(\pi) = 0\,,$$
$$x'' = f(t,x,x')\,, \qquad x(0) = x(T)\,,\ \ x'(0) = x'(T)$$
$$x' = f(t,x)\,, \qquad x(0) = x(T)\,,$$

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(see [6], [8], [11], for example). The almost complete list of references can be found in [4].

The perturbation method (this name was proposed by Kannan [9]) is based on the observation that if one perturbs the linear operator L by λI (I is the identity map, and λ, a small parameter), then it becomes invertible, solutions can be found and the only problem is to prove a compactness of the set of solutions to perturbed equations. Obviously, the nonlinearity N should be bounded or, at least, sublinear as it is in the paper by de Figueiredo [3]. He obtained an abstract result for the equation $Lx = N(x)$, using the perturbation method, but his proof does not involve a form of the inverse operators $(L-\lambda I)^{-1}$. The present author has studied the abstract problem, taking into account a family of equations $L(\lambda)x = N(x)$ with $L(\lambda)$ invertible for $\lambda \neq \lambda_0$ and $L(\lambda_0)$, a Fredholm linear operator. The inverse operators are supposed to have the special form

$$L(\lambda)^{-1} = G_0(\lambda) + \sum_{j=1}^{n} c_j(\lambda)\langle u_j(\lambda), \cdot\, \rangle w_j(\lambda)\,,$$

where all terms except $c_j(\lambda)$ have continuous extensions to λ_0, $|c_j(\lambda)| \to \infty$, $w_j(\lambda_0)$, $j = 1, \ldots, n$, span the kernel of $L(\lambda_0)$, and the common part of $\ker u_j(\lambda_0)$, $j = 1, \ldots, n$, equals the range of $L(\lambda_0)$. This generalization of $L-\lambda I$ to $L(\lambda)$ enables us to study equations depending explicitly on a real parameter (for instance, the bifurcation problems). On the other hand, the form of $L(\lambda)^{-1}$ is natural from point of view of applications: Green operators for ordinary differential equations have this form, and if the Hilbert-Schmidt theory is applicable, then $L(\lambda)^{-1}$ is a sum of a series built of eigenvalues and eigenfunctions, $c_j(\lambda) = (\lambda_0 - \lambda)^{-1}$, and $G_0(\lambda)$ is the rest of this series in whose terms λ_0 does not occur. The method is useful not only for sublinear nonlinearities. They may have a linear growth at infinity or even be superlinear. A lot of theoretical results based on the topological degree theory and similar techniques are given in [12] – [16]. Below, we shall show that this method (with some improvements) can be applied to BVP's in a Banach space E. All difficulties connected with a partition of a function space into a topological sum of its subspaces are reduced to the same (but easier) problem for underlying space E. We shall also consider problems with a nonlinear boundary condition, using the observation of Furi and Pera [7]. We refer the reader to [2] for information on differential equations in infinite dimensional spaces.

2. General problem

Let E be a Banach space, $A\colon \langle 0,1\rangle \to L(E)$, a continuous function taking values in the space of bounded linear operators of E, $f\colon \langle 0,1\rangle \times E \to E$, a continuous function, $B_1, B_2 \in L(E)$, and let $B_3\colon C\big(\langle 0,1\rangle, E\big) \to E$ be a nonlinear continuous mapping defined on the Banach space of continuous functions $\langle 0,1\rangle \to E$. We look for a solution of the first order differential equation

$$x' - A(t)x = f(t,x) \tag{2.1}$$

satisfying the boundary condition

$$B_1x(0) + B_2x(1) = B_3(x)\,. \tag{2.2}$$

System (2.1)–(2.2) is at resonance, which means that the linear homogeneous problem

$$x' - A(t)x = 0\,, \qquad B_1x(0) + B_2x(1) = 0\,,$$

has a nonzero solution. We shall assume that there exists an operator $A_0 \in L(E)$ commuting with the resolvent $U\colon \langle 0,1\rangle \to L(E)$ of the operator $x' - A(t)x = 0$, such that $B_1 + B_2\exp(\lambda A_0)U(1)$ is an automorphism of E for λ from a neighbourhood (nhbd) of $0 \in \mathbb{R}$. Usually, $A_0 = I$ is the identity operator. Moreover, let $B_1 + B_2U(1)$ be a linear Fredholm operator (its index must be 0, by the above). Our assumptions mean that the problems

$$x' - A(t)x - \lambda A_0x = 0\,, \qquad B_1x(0) + B_2x(1) = 0\,, \tag{2.3}$$

have only the zero-solution for $\lambda \neq 0$ belonging to the nhbd of 0, the subspace of initial points of solutions to (2.3), with $\lambda = 0$, is finite dimensional, and the range of the operator $B_1 + B_2U(1)$ has a finite codimension.

Take any basis $x_1,\dots,x_n$ in $\ker\big(B_1+B_2U(1)\big)$ and suppose that the following limits

$$\lim_{\lambda\to 0} B(\lambda)x_j/\|B(\lambda)x_j\| =: h_j\,, \qquad j = 1,\dots,n\,, \tag{2.4}$$

where $B(\lambda) = B_1 + B_2U(1)\exp\lambda A_0$, exist and constitute a linearly independent system such that

$$\operatorname{Lin}\{h_1,\dots,h_n\} \oplus \operatorname{Im} B(0) = E\,.$$

Then, of course, this condition is satisfied for each basis.

Let $E_1 = \ker B(0)$, and let E_0 be its topological complement:

$$E_1 \oplus E_0 = E\,.$$

We have $B(\lambda)E_1 \oplus B(\lambda)E_0 = E$ for $\lambda \neq 0$ sufficiently close to 0. Moreover,

$$\operatorname{Lin}\{h_1,\dots,h_n\} \oplus B(0)E_0 = E\,.$$

Define the system of linear bounded functionals on E: $v_j(\lambda)$, $j = 1,\dots,n$, for $\lambda \neq 0$ by the formulae

$$\big\langle v_j(\lambda), B(\lambda)x_i\big\rangle = \delta_{ij}\|B(\lambda)x_i\|\,, \qquad i = 1,\dots,n\,,$$
$$v_j(\lambda) \mid B(\lambda)E_0 = 0\,.$$

Obviously, v_j are continuous functions of λ and have continuous extensions to 0 such that

$$\langle v_j(0), h_i\rangle = \delta_{ij}\,, \qquad v_j(0) \mid B(0)E_0 = 0\,.$$

If we denote by $P_1(\lambda)$ (resp. $P_0(\lambda)$) the projectors on $B(\lambda)E_1$ (resp. $B(\lambda)E_0$) along $B(\lambda)E_0$ (resp. $B(\lambda)E_1$) for $\lambda \neq 0$ and similarly for $\lambda = 0$ with natural changes, then we can find the representation of $B(\lambda)^{-1}$:

$$B(\lambda)^{-1} = B(\lambda)^{-1}P_0(\lambda) + \sum_{j=1}^{n} \|B(\lambda)x_j\|^{-1}\big\langle v_j(\lambda), \cdot\,\big\rangle x_j\,,$$

where the first summand has a continuous extension to 0: $(B(0) \mid E_0)^{-1}P_0(0)$. We shall denote this summand by $R(\lambda)$, and

$$c_j(\lambda) := \|B(\lambda)x_j\|^{-1}\,, \qquad j = 1,\dots,n\,,$$

are the only parts which make $\lambda = 0$ a singular point of $B(\lambda)^{-1}$. We have

$$B(\lambda)^{-1} = R(\lambda) + \sum_{j=1}^{n} c_j(\lambda)\big\langle v_j(\lambda), \cdot\,\big\rangle x_j\,, \tag{2.5}$$

which is similar to the corresponding formula from the previous papers [12] – [16].

It is easy to see that $V_\lambda(t) = \exp(\lambda t A_0)U(t)$ is the resolvent for the operator $x' - A(t)x - \lambda A_0 x$. This implies that the unique solution to the BVP

$$x' - A(t)x - \lambda A_0 x = b(t)\,, \qquad B_1x(0) + B_2x(1) = 0\,,$$

is the function

$$x(t) = V_\lambda(t)x_0 + V_\lambda(t)\int_0^t V_\lambda^{-1}(s)b(s)\,\mathrm{d}s \tag{2.6}$$

with the initial vector x_0 for which

$$B(\lambda)x_0 = -B_2\exp(\lambda A_0)U(1)\int_0^1 \exp(-\lambda s A_0)U^{-1}(s)b(s)\,\mathrm{d}s\,. \tag{2.7}$$

We shall denote the right-hand side of the last equality by $C(\lambda, b)$, where $b \in C\big(\langle 0,1\rangle, E\big)$. Applying (2.5), we get, for $\lambda \neq 0$,

$$x_0 = R(\lambda)C(\lambda,b) + \sum_{j=1}^{n} c_j(\lambda)\big\langle v_j(\lambda), C(\lambda,b)\big\rangle x_j$$

and

$$x(t) = \exp(\lambda t A_0)U(t)R(\lambda)C(\lambda, b) + \exp(\lambda t A_0)U(t)\int_0^t \exp(\lambda s A_0)U^{-1}(s)b(s)\ \mathrm{d}s$$
$$+\sum_{j=1}^{n} c_j(\lambda)\big\langle v_j(\lambda), C(\lambda, b)\big\rangle \exp(\lambda t A_0)U(t)x_j\,.$$

Now, we are able to write down the system equivalent to the BVP

$$x' - A(t)x - \lambda A_0 x = f(t,x)\,, \qquad B_1 x(0) + B_2 x(1) = B_3 x\,,$$

for $\lambda \neq 0$:

$$x_0 = R(\lambda)\Big(C\big(\lambda, N(x)\big) + B_3(x)\Big) + \sum_{j=1}^{n} c_j(\lambda)\big\langle v_j(\lambda), C\big(\lambda, N(x)\big) + B_3(x)\big\rangle x_j\,, \tag{2.8}$$

$$x(t) = V_\lambda(t)R(\lambda)\Big(C\big(\lambda, N(x)\big) + B_3(x)\Big) + V_\lambda(t)\int_0^t V_\lambda^{-1}(s)N(x)(s)\ \mathrm{d}s$$
$$+\sum_{j=1}^{n} c_j(\lambda)\big\langle v_j(\lambda), C\big(\lambda, N(x)\big) + B_3(x)\big\rangle V_\lambda(t)x_j\,, \tag{2.9}$$

where $N(x)(t) = f\big(t, x(t)\big)$. The scheme of our considerations is the following. First, we shall show that the operator defined by the right-hand sides of (2.8), (2.9) on $E \times C\big(\langle 0,1\rangle, E\big)$ is completely continuous (under some assumptions on f and B_3). Then we can find solutions to (2.8)–(2.9) for $\lambda \neq 0$ if f and B_3 are sublinear, by the Rothe fixed point theorem [5], and prove that the existence of a bounded sequence of solutions for $\lambda_m \to 0$ implies the solvability of the studied resonance problem (2.1)–(2.2). Next, we should find conditions excluding the existence of unbounded sequence of solutions (they correspond to the well-known Landesman-Lazer condition). The case of nonlinearities with a linear growth will be examined separately by homotopy arguments.

3. Compactness

We shall assume that f is more than continuous: $f(t,\cdot\,)$ is completely continuous, i.e. it maps bounded sets into compact ones, for any $t \in \langle 0,1\rangle$, and functions $f_x \in C\big(\langle 0,1\rangle, E\big)$, $f_x(t) = f(t,x)$, are equicontinuous for x belonging to every bounded set. Moreover, let B_3 be completely continuous. Fix $\lambda \neq 0$. The following lemma is essential for the proof of the complete continuity of (2.8)–(2.9).

Lemma 1. *If f is as above and $K\colon \langle 0,1\rangle \to L(E)$ is continuous, then $F\colon C\big(\langle 0,1\rangle, E\big) \to C\big(\langle 0,1\rangle, E\big)$, given by*

$$F(x)(t) = \int_0^t K(s) f\big(s, x(s)\big)\, \mathrm{d}s\,,$$

is completely continuous.

Proof. The continuity of F is obvious. If we take all continuous functions $x \in C\big(\langle 0,1\rangle, E\big)$, $\|x\|_\infty := \sup_t \|x(t)\| \le M$, then

$$\|F(x)(t)\| \le \sup_s \|K(s)\| \cdot \sup_{s,\|x\|\le M} \|f(s,x)\| < \infty\,,$$

so, the functions $F(x)$ are equibounded. Similarly, they are equicontinuous. In order to apply the Generalized Ascoli-Arzelà theorem, we should show that the sets

$$\left\{ \int_0^t K(s) f\big(s, x(s)\big)\, \mathrm{d}s :\ \|x\|_\infty \le M \right\} \tag{3.1}$$

are relatively compact in E, $t \in \langle 0,1\rangle$. Take $\varepsilon > 0$ and $\delta > 0$, such that $|t_2 - t_1| < \delta$ implies $\|f(t_2,x) - f(t_1,x)\| < \frac{\varepsilon}{3} \sup \|K(s)\|$ if $\|x\| \le M$ and $\|K(t_2) - K(t_1)\| < \frac{\varepsilon}{3} \sup \|f(t,x)\|$. Divide the interval $\langle 0,1\rangle$ into subintervals of length less than δ: $0 < t_1 < t_2 < \dots < t_k = 1$, and choose finite $\frac{\varepsilon}{3}$-nets for $K(t_j) f\big(t_j, \bar{B}(0,M)\big)$: $K(t_j) f(t_j, x_i)$, $j = 1, \dots, k$, $i = 1, \dots, l(j)$. We get

$$\big\|K(s) f\big(s, x(s)\big) - K(t_j) f(t_j, x_i)\big\| < \varepsilon$$

for any $\|x\|_\infty \le M$ and $s \in \langle 0,1\rangle$, where we take t_j such that $|t_j - s| < \delta$, and x_i such that $\big\|K(t_j) f\big(t_j, x(s)\big) - K(t_j) f(t_j, x_i)\big\| < \frac{\varepsilon}{3}$. Hence the set $\big\{K(s) f\big(s, x(s)\big) :\ s \in \langle 0,1\rangle,\ \|x\|_\infty \le M\big\}$ is relatively compact, and so is its closed convex hull (the Mazur theorem). But (3.1) are contained in this hull, which ends the proof. □

Notice that $C\big(\lambda, N(\cdot)\big)$ and the second summand in (2.9) are completely continuous by Lemma 1, and that the remaining terms involve B_3 or are finite dimensional. Therefore the right-hand sides of (2.8), (2.9) define a completely continuous operator on $E \times C\big(\langle 0,1\rangle, E\big)$.

Suppose that

$$\lim_{\|x\|\to\infty} \|f(t,x)\|/\|x\| = \lim_{\|x\|_\infty\to\infty} \|B_3(x)\|/\|x\|_\infty = 0 \tag{3.2}$$

(the nonlinearity is sublinear). It is easy to see that the boundary of a sufficiently large ball centred at 0 is mapped by the above mentioned operator into this ball. Due to the Rothe fixed point theorem [5], we have a solution $(x_0^\lambda, x^\lambda) \in E \times C(\langle 0,1\rangle, E)$ to system (2.8)–(2.9) for any $\lambda \neq 0$. However, the radius of the ball tends to infinity as $\lambda \to 0$, and the assumption of the following lemma is not unconditionally satisfied.

LEMMA 2. *If $\lambda_m \to 0$ and $(x_0^m, x^m)_m$ is a bounded sequence of solutions to (2.8)–(2.9) for $\lambda = \lambda_m$, $m \in \mathbb{N}$, then problem (2.1)–(2.2) is solvable.*

Proof. Passing to convergent subsequences, we may assume without loss of generality that

$$C(\lambda_m, N(x^m)) \to y_1\,, \qquad B_3(x^m) \to y_2\,,$$

$$V_{\lambda_m}(t)\int_0^t V_{\lambda_m}^{-1} N(x^m)(s)\,\mathrm{d}s \rightrightarrows y(t)\,.$$

By the linear independence of x_j, $j = 1,\dots,n$, and $V_{\lambda_m}(\cdot)x_j$, $j = 1,\dots,n$, the scalar sequences contain the convergent subsequences

$$c_j(\lambda_m)\big\langle v_j(\lambda_m), C(\lambda_m, N(x^m)) + B_3(x^m)\big\rangle \to d_j$$

for $j = 1,\dots,n$; thus

$$x_0^m \to R(0)(y_1+y_2) + \sum d_j x_j =: x_0\,,$$

$$x^m(t) \rightrightarrows U(t)\Big(R(0)(y_1+y_2) + \sum d_j x_j\Big) + y(t) =: x(t)\,.$$

Therefore

$$y(t) = U(t)\int_0^t U^{-1}(s) f\big(s, x(s)\big)\,\mathrm{d}s\,,$$

which implies that the function x satisfies equation (2.1). Since $c_j(\lambda_m) \to \infty$, we have

$$\big\langle v_j(0), C(0, N(x)) + B_3(x)\big\rangle = 0\,, \qquad j = 1,\dots,n\,,$$

which means that

$$y_1 + y_2 = C\big(0, N(x)\big) + B_3(x) \in B(0)E_0 = B(0)E\,;$$

so

$$B(0)x_0 = C\big(0, N(x)\big) + B_3(x)\,.$$

The last equality is equivalent to boundary condition (2.2). □

4. The Landesman-Lazer condition

Suppose that the sequence (x_0^m, x^m) from Lemma 2 is unbounded. Then (x^m) is unbounded and one may assume that $\|x^m\|_\infty \to \infty$. Dividing both sides of (2.9), for $\lambda = \lambda_m$, by $\|x^m\|_\infty$, we find that the first and second summands on the right tend to 0, hence the sequence

$$\|x^m\|_\infty^{-1} \sum_{j=1}^{n} c_j(\lambda_m)\big\langle v_j(\lambda_m), C\big(\lambda_m, N(x^m)\big) + B_3(x^m)\big\rangle V_{\lambda_m}(t)x_j$$

is bounded and, as in the proof of Lemma 2, one can choose convergent scalar subsequences

$$\|x^m\|_\infty^{-1} c_j(\lambda_m)\big\langle v_j(\lambda_m), C\big(\lambda_m, N(x^m)\big) + B_3(x^m)\big\rangle \to d_j\,,$$

$j = 1, \ldots, n$, and obtain

$$\|x^m\|_\infty^{-1} x^m(t) \;\rightrightarrows\; \sum_{j=1}^{n} d_j U(t) x_j\,.$$

Thus $\big\langle v_j(\lambda_m), C\big(\lambda_m, N(x^m)\big) + B_3(x^m)\big\rangle$ has the same sign as d_j for large m and each j. Introduce the following condition:

for any $(x^m) \subset C\big(\langle 0,1\rangle, E\big)$ with the properties $\|x^m\|_\infty \to \infty$, $\|x^m\|_\infty^{-1} x^m \to \sum d_j U(\cdot) x_j$ for some $(d_1, \ldots, d_n) \in \mathbb{R}^n$, there exists $j \in \{1, \ldots, n\}$ such that

$$\limsup_{m\to\infty} d_j \big\langle v_j(0), D(x^m)\big\rangle < 0\,,$$

where

$$D(x) = -B_2 U(1) \int_0^1 U^{-1}(s) f\big(s, x(s)\big)\, \mathrm{d}s + B_3(x)\,.$$

From the above arguments, this condition (referred to as the L-L condition) implies that the assumption of Lemma 2 holds. We have proved

Theorem 1. *Under the assumptions of Sections* 2, 3*, if the L-L condition is satisfied, then boundary value problem* (2.1)–(2.2) *has a solution.*

If there exist limits $D(d_1, \ldots, d_n) = \lim_{m\to\infty} D(x^m)$ independently of (x^m) such that $\|x^m\|_\infty \to \infty$, $\|x^m\|_\infty^{-1} x^m \to \sum d_j U(\cdot) x_j$, then the L-L condition has the form: for each $(d_1, \ldots, d_n) \in \mathbb{R}^n \setminus \{0\}$, there exists j such that

$$d_j \big\langle v_j(0), D(d_1, \ldots, d_n)\big\rangle < 0\,.$$

5. The nonlinearity with linear growth

Keep all the assumptions and notations of Sections 2 and 3 in mind, except (3.2), which we replace by

$$\beta(s) := \limsup_{\|x\|\to\infty} \|f(s,x)\|/\|x\| < \infty, \tag{5.1}$$

$$\gamma_2 := \limsup_{\|x\|_\infty\to\infty} \|B_3(x)\|/\|x\|_\infty < \infty. \tag{5.2}$$

Let

$$\begin{cases} \hat\beta &:= \int\limits_0^1 \beta(s)\|U^{-1}(s)\|\,\mathrm{d}s\,, \\ \gamma_1 &:= \|B_2U(1)\|\hat\beta\,, \\ \gamma &:= \gamma_1+\gamma_2\,, \end{cases} \tag{5.3}$$

and suppose that

$$\hat\beta \sup_t \|U(t)\| < 1\,. \tag{5.4}$$

Theorem 2. *Assume that there exist $\sigma_1 > 0$ and $r > 0$ such that, for any $j\in\{1,\dots,n\}$,*

$$\sup d_j\bigl\langle v_j(0), C\bigl(0,N(x)\bigr)+B_3(x)\bigr\rangle < 0 \tag{5.5}$$

over the set of all $x(t) = U(t)\bigl(\tilde x_0 + \sum d_i x_i + y(t)\bigr)$ with $|d_j|\ge r$, $\tilde x_0\in E_0$, $|d_i|\le|d_j|$, $\|\tilde x_0\|\le\sigma_1\bigl\|\sum d_i x_i\bigr\|$, $\|y\|_\infty \le \hat\beta\sigma_2\bigl\|\tilde x_0+\sum d_ix_i\bigr\|$, $y(0)=0$, where

$$\sigma_2 = \sup_t \|U(t)\|\Bigl(1-\hat\beta\sup_t\|U(t)\|\Bigr)^{-1}. \tag{5.6}$$

If

$$\gamma\|R(0)\|\sigma_2(1+\sigma_1)\sigma_1^{-1} < 1\,, \tag{5.7}$$

then BVP (2.1)-(2.2) has a solution.

Proof. Define a homotopy $H=(H_0,H_1)\colon E\times C\bigl(\langle 0,1\rangle,E\bigr)\times\langle 0,1\rangle \to E\times C\bigl(\langle 0,1\rangle,E\bigr)$ by the formulae

$$\begin{aligned} H_0(x_0,x,\alpha) &= (1-\alpha)R(\alpha\lambda_1)\Bigl(C\bigl(\alpha\lambda_1,N(x)\bigr)+B_3(x)\Bigr) \\ &\quad + \sum_j c_j(\alpha\lambda_1)\bigl\langle v_j(\alpha\lambda_1), C\bigl(\alpha\lambda_1,N(x)\bigr)+B_3(x)\bigr\rangle x_j\,, \end{aligned}$$

$$H_1(x_0,x,\alpha) = V_{\alpha\lambda_1}(t)H_0(x_0,x,\alpha) + V_{\alpha\lambda_1}(t)\int\limits_0^t V_{\alpha\lambda_1}^{-1}(s)N(x)(s)\,\mathrm{d}s\,,$$

where λ_1 is a positive number sufficiently close to 0. We shall show that the homotopy H has fixed points (if they exist) in a bounded set.

First of all, notice that (5.5) is satisfied for 0 replaced by $\lambda \in \langle 0, \lambda_1 \rangle$, and $x(t) = V_\lambda(t)\big(\tilde{x}_0 + \sum d_i x_i + y(t)\big)$ with $\tilde{x}_0$, d_i, y as above (in definition (5.6) of σ_2, U is replaced by V_λ), but σ_1 and σ_2 satisfying

$$\gamma \|R(\lambda)\| \sigma_2 (1 + \sigma_1) \sigma_1^{-1} < 1 .$$

If $x = H_1(x_0, x, \alpha)$, $x_0 = H_0(x_0, x, \alpha)$, then

$$\|x\|_\infty \le \big(1 - \hat{\beta} \sup \|V_{\alpha\lambda_1}(t)\|\big)^{-1} \sup \|V_{\alpha\lambda_1}(t)\| \|x_0\| \tag{5.8}$$

and

$$\begin{aligned} \|\tilde{x}_0\| &= \big\|(1-\alpha) R(\alpha\lambda_1)\big(C(\alpha\lambda_1, N(x)) + B_3(x)\big)\big\| \\ &\le \|R(\alpha\lambda_1)\| \left(\|B_2 V_{\alpha\lambda_1}(1)\| \int_0^1 \|V_{\alpha\lambda_1}^{-1}(s)\| \|N(x)(s)\| \, \mathrm{d}s + \|B_3(x)\| \right). \end{aligned}$$

Enlarging γ_1, γ_2 with (5.7) kept, we can estimate this norm for large $\|x\|_\infty$:

$$\|\tilde{x}_0\| \le \gamma \|R(0)\| \|x\|_\infty \le \gamma \|R(0)\| \sigma_2 \|x_0\| \le \gamma \|R(0)\| \sigma_2 \Big(\|\tilde{x}_0\| + \Big\| \sum d_i x_i \Big\| \Big),$$

thus

$$\|\tilde{x}_0\| \le \gamma \|R(0)\| \sigma_2 \big(1 - \gamma \|R(0)\| \sigma_2\big)^{-1} \Big\| \sum d_i x_i \Big\| < \sigma_1 \Big\| \sum d_i x_i \Big\| .$$

For such fixed points, we have

$$\|y\|_\infty = \sup_t \left\| \int_0^t V_{\alpha\lambda_1}^{-1}(s) N(x)(s) \, \mathrm{d}s \right\| \le \hat{\beta} \|x\|_\infty \le \hat{\beta} \sigma_2 \|x_0\| ,$$

which is needed to apply (5.5).

Take any solution $x_0 = \tilde{x}_0 + \sum d_i x_i$, $d \in \mathbb{R}^n$ and d_j with the maximal modulus. Obviously, $|d_j| < r$ by (5.5) (with $\alpha\lambda_1$ instead of 0). So, $\big\|\sum d_i x_i\big\|$ is bounded, which gives an estimate on $\|\tilde{x}_0\|$, then on $\|x_0\|$ and, at last, on $\|y\|_\infty$. Due to (5.8), we have an upper bound for the norms of solutions x. Denote by Ω_0 (resp. Ω_1) a ball containing fixed points of H_0 (resp. H_1). The Leray-Schauder degree

$$\deg_{LS}\big((I - H_0) \times (I - H_1), \Omega_0 \times \Omega_1, 0\big) \tag{5.9}$$

does not depend on $\alpha \in (0, 1\rangle$. We can deform $H(\cdot, 1)$ by $H(x_0, x, \mu) = \mu H(x_0, x, 1)$ which is fixed point free on the boundary of $\Omega_0 \times \Omega_1$ by similar (but simpler) arguments. For $\mu = 1$, we have degree (5.9) and, for $\mu = 0$, $\deg_{LS}(I, \Omega_0 \times \Omega_1, 0)$. Therefore, H has a fixed point in $\Omega_0 \times \Omega_1$ for any $\alpha > 0$. Repeating the arguments from the proof of Lemma 2 with a slight change, we get the assertion. □

6. Examples

Let us consider the BVP:

$$x' = f(t,x)\,, \tag{6.1}$$
$$x(1) = Bx(0) \tag{6.2}$$

in a Hilbert space E with B being a linear self-adjoint completely continuous operator in E and f satisfying the continuity assumptions from Section 3. Since we are interested in resonance problems, $1 \in \operatorname{Sp} B$, and we can take $A_0 = I$, $B(\lambda) = B - e^{\lambda} I$. Obviously, $E_1 = \ker(B - I)$ is finite dimensional. Take any orthonormal set $\{x_1, \dots, x_n\}$ spanning $\ker(B - I)$. We have $h_j = -x_j$ for $\lambda \to 0^+$, $j = 1, \dots, n$, and

$$\langle v_j(0), x \rangle = -(x_j, x)\,.$$

The L-L condition, in the sublinear case, has the form:
for any sequence (x^m) such that $\|x^m\|_\infty \to \infty$, $\|x^m\|_\infty^{-1} x^m \rightrightarrows \sum d_i x_i$, there exists $j \in \{1, \dots, n\}$ such that

$$\liminf_{m\to\infty} d_j \left(x_j, \int_0^1 f\big(s, x^m(s)\big)\, \mathrm{d}s \right) > 0\,. \tag{6.3}$$

One can use the weaker condition summing up (6.3) over the numbers j:

$$\liminf_{m\to\infty} \int_0^1 \Big(x^m(s), f\big(s, x^m(s)\big)\Big)\, \mathrm{d}s > 0 \tag{6.4}$$

or even

$$\liminf_{\substack{\|x\|\to\infty,\\ x\in G}} \int_0^1 \big(x, f(s,x)\big)\, \mathrm{d}s > 0\,,$$

where $G = \{\lambda x :\ \lambda \in \mathbb{R},\ x \in W\}$ and W is a nhbd of $\{x \in \ker(B - I) : \|x\| = 1\}$.

One can examine a more general boundary condition

$$x(1) = Bx(0) + B_3(x) \tag{6.5}$$

with B_3 sublinear. Assumption (6.4) should be replaced by

$$\liminf_{m\to\infty} \int_0^1 \Big(x^m(s), f\big(s, x^m(s)\big) + B_3(x^m)\Big)\, \mathrm{d}s > 0$$

or

$$\liminf_{\substack{\|x\|\to\infty,\\ x\in G}} \int_0^1 \big(x, f(s,x) + B_3(x)\big)\ \mathrm{d}s > 0\,,$$

where $B_3(x)$ means the value of B_3 on the constant function equal to x. If

$$B_3(x) = \int_0^1 \|x(t)\|^\rho\ \mathrm{d}t \cdot x_0\,,$$

where $\rho \in (0,1)$ and x_0 is a fixed vector orthogonal to $\ker(B-I)$, then (6.4) is still a sufficient condition for the solvability of (6.1), (6.5).

Now, we consider BVP (6.1), (6.2) with the nonlinearity f of a linear growth. In the notations of Section 5,

$$\gamma = \gamma_1 = \hat\beta = \int_0^1 \beta(s)\ \mathrm{d}s\,.$$

Let $f_j(t,x) = \big(f(t,x), x_j\big)$, $j = 1,\dots,n$, and let f_0 be the orthogonal projection of f onto $E_0 = \mathrm{Im}(B-I)$. We shall assume that

$$\limsup_{d_j\to-\infty} \int_0^1 f_j\Big(s,\ x_0+\sum d_i x_i\Big)\ \mathrm{d}s < 0 < \liminf_{d_j\to+\infty} \int_0^1 f_j\Big(s,\ x_0+\sum d_i x_i\Big)\ \mathrm{d}s \quad (6.6)$$

for any $j\in\{1,\dots,n\}$, $d = (d_1,\dots,d_n)\in\mathbb{R}^n$ and $x_0\in E_0$, and that the limits are separated from 0 uniformly on bounded sets. Moreover, let

$$\hat\beta < \big(\sqrt{n}+1\big)^{-1} \qquad (6.7)$$

and

$$\frac{\hat\beta}{1-\big(\sqrt{n}+1\big)\hat\beta} \max_{\substack{\lambda\in\mathrm{Sp}\,B,\\ \lambda\neq1}} |1-\lambda|^{-1} < 1 \qquad (6.8)$$

(the maximum in this inequality equals $\|R(0)\|$). It is easy to calculate that there exists $\sigma_1>0$ satisfying inequality (5.7) ($\sigma_2 = (1-\hat\beta)^{-1}$) and such that

$$\sigma_1 < \frac{1-\big(\sqrt{n}+1\big)\hat\beta}{\sqrt{n}\hat\beta}\,.$$

Hence, if we take $\tilde x_0 \in E_0$, $d\in\mathbb{R}^n$, $y\in C\big(\langle 0,1\rangle, E\big)$, such that $\|\tilde x_0\| \le \sigma_1\big\|\sum d_i x_i\big\|$, $\|y\|_\infty \le \hat\beta\sigma_2\big\|\tilde x_0 + \sum d_i x_i\big\|$, then if $|d_j| = \max_i |d_i|$ and we have the worst case: "y takes values in $\mathrm{Lin}\{x_1,\dots,x_n\}$"; then we obtain

$$\|y\|_\infty < (1-\varepsilon)|d_j|$$

for some positive ε. Thus, we can make the coefficient standing with x_j in the projection of $\tilde{x}_0 + \sum d_i x_i + y(t)$ onto $\mathrm{Lin}\{x_1, \dots, x_n\}$ arbitrarily large if $|d_j| \to \infty$. By (6.6), this means that the assumption (5.5) of Theorem 2 holds and, therefore, BVP (6.1), (6.2) has a solution provided that the projections of f onto $\mathrm{Lin}\{x_1, \dots, x_n\}$ satisfy (6.6), the constant $\hat{\beta}$ describing a linear growth of f satisfies (6.7), and $\hat{\beta}\big(1 - (\sqrt{n}+1)\hat{\beta}\big)^{-1}$ is less than the distance between 1 and the nearest eigenvalue of B.

Remark. Since, for $\lambda \to 0^-$, we have $h_j = x_j$ instead of $-x_j$, we can reverse inequalities (6.3) and (6.4) replacing "lim inf" by "lim sup". Moreover, we can change mutually $\pm\infty$ in (6.6) and the solvability does not fail.

Now, we shall study a BVP for second order differential equations in the Banach space $\mathbf{l}^\infty$ of bounded sequences:

$$x'' + m^2 x = f(t, x, x'), \tag{6.9}$$

where $x = (x_j)_{j\in\mathbb{N}}$, $f = (f_j)_{j\in\mathbb{N}}$ and m is an odd integer. The boundary condition is partially periodic and partially antiperiodic:

$$\begin{aligned} &x_j(0) = x_j(\pi), \quad {x_j}'(0) = -{x_j}'(\pi), \qquad j \le n, \\ &x_j(0) = x_j(\pi), \quad {x_j}'(0) = {x_j}'(\pi), \qquad j > n. \end{aligned} \tag{6.10}$$

It is easily seen that the corresponding homogeneous linear problem has non-trivial solutions $\sin mt \sum_{i=1}^{n} c_i e_i$, where e_i are elements of the standard basis in $\mathbf{l}^\infty$. Consider the equivalent first order system and perturb it by λI:

$$x' = y + \lambda x, \quad y' = -m^2 x + \lambda y.$$

We can put $E = \mathbf{l}^\infty \oplus \mathbf{l}^\infty$, $U(t) = \cos mt\, I + \dfrac{1}{m} \sin mt\, \mathbf{A}$, where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -m^2 & 0 \end{pmatrix},$$

$x_j = (0, e_j)$, $j = 1, \dots, n$. Then $h_j = -x_j$ if $\lambda \to 0^+$, and $-\langle v_j(0), z\rangle$ is the jth coordinate in the second summand of $z \in E = \mathbf{l}^\infty \oplus \mathbf{l}^\infty$.

The nonlinearity has the form $(0, f)$, where f should satisfy the following conditions (see Section 3): f_j are equi-uniformly continuous on bounded sets and, for any $\varepsilon > 0$ and $M > 0$, there exists $k \in \mathbb{N}$ such that

$$|f_j(t, x, y)| < \varepsilon, \qquad \text{for} \quad \|x\|, \|y\| \le M, \;\; t \in \langle 0, \pi\rangle, \;\; j > k.$$

There are conditions less restrictive than the last one, but very complicated guaranteeing the compactness of $f(t, \cdot)$. Moreover, let f be sublinear. It is easy to calculate the L-L condition for BVP (6.9), (6.10):

for $(x^k) \subset X$, $a_k = \max(\|x^k\|_\infty, \|x^{k'}\|_\infty) \to \infty$, $a_k^{-1}x^k(t) \rightrightarrows \sin mt\,(d_1, \dots, d_n, 0)$, $a_k^{-1}x^{k'}(t) \rightrightarrows m\cos mt\,(d_1,\dots,d_n,0)$, there exists $j \in \{1,\dots,n\}$ such that

$$\liminf_{k\to\infty} d_j \int_0^\pi \cos ms\, f_j\big(s, x^k(s), x^{k'}(s)\big)\,\mathrm{d}s > 0\,,$$

where 0's stand for the jth coordinates of x and y with $j > n$. The inequality can be reversed $(\lambda \to 0^-)$ with replacing "lim inf" by "lim sup". If f does not depend on derivative x', $n = 1$, and if there exist uniform limits

$$\lim_{d\to\pm\infty} f_1(s,d,x) = f_1^{\pm}(s)$$

independent of $x = (x_2, x_3, \dots)$, we can simplify this condition, as the numbers

$$\int\limits_{\sin mt>0} f_1^+(s)\cos ms\,\mathrm{d}s + \int\limits_{\sin mt<0} f_1^-(s)\cos ms\,\mathrm{d}s\,,$$

$$\int\limits_{\sin mt<0} f_1^+(s)\cos ms\,\mathrm{d}s + \int\limits_{\sin mt>0} f_1^-(s)\cos ms\,\mathrm{d}s\,,$$

have opposite signs.

It is interesting that the last condition differs from the classical Landesman-Lazer condition only by the kernel function $\cos ms$. This is a consequence of the fact that the BVP is not self-adjoint as

$$x'' + m^2x = 0\,, \qquad x(0) = x(\pi) = 0$$

is. Similarly, one can introduce a nonlinearity B_3 to boundary condition (6.10) and study BVP (6.9), (6.10) with nonlinearities having linear growth.

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Inst. of Math. UŁ
Banacha 22
PL - 90-238 Łódź
Poland