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ON THE PRODUCT TOPOLOGY ASSOCIATED WITH SEMI-CLOSED SETS

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Introduction

In 1963 N. Levine introduced the idea of semi-open sets in a topological space. Later S. G. Crossley and S. K. Hildebrand in their paper [3] introduced the idea of semiclosed sets and semi-closure of a set in a topological space $(X, \mathcal{T})$. In the same paper they proved the existence of a minimal set D_A (with respect to set inclusion) for each set $A \subset X$ such that scl $(A \cup D_A \cup B) = A \cup D_A \cup \mathrm{scl}\, B$ for all subsets $B \subset X$. Also by defining $c\colon \mathcal{P}(X) \to \mathcal{P}(X)$ by the rule: $cA = A \cup D_A$ for all $A \in \mathcal{P}(X)$ where $\mathcal{P}(X)$ denotes the class of all subsets of X, it has been shown that c is a Kuratowski closure operator on X. The topology induced by the Kuratowski closure operator c on X is denoted by $\mathcal{F}(\mathcal{T})$. In [3] it has been shown that $\mathcal{F}(\mathcal{T})$ is finer than $\mathcal{T}$ on X. The characterization of the set D_A for any set $A \subset R$, where R denotes the set of reals with usual topology can be found in [4]. This characterization has also been extended to a first countable topological space by C. Bandyapadhyay in her Ph. D. Thesis [1].

In this paper we consider the spaces $(X, \mathcal{T})$ to be first countable. In [2] we have proved that the class of realvalued continuous functions on $(X, \mathcal{T})$ and the class of real-valued continuous functions on $(X, \mathcal{F}(\mathcal{T}))$ are identical. In this paper, taking $(Y, \mathcal{U})$ to be a regular space, we have proved that the classes of continuous functions from $(X, \mathcal{T})$ to $(Y, \mathcal{U})$ and from $(X, \mathcal{F}(\mathcal{T}))$ to $(Y, \mathcal{U})$ are identical. Now we consider a finite number of spaces $\{(X_i, \mathcal{T}_i)\colon i = 1, 2, \ldots, n\}$. Hence there are associated the two product topologies for the Cartesian product $\prod_{i=1}^{n} X_i$: One is $\mathcal{F}(\mathcal{T}_1 \times \mathcal{T}_2 \times \ldots \times \mathcal{T}_n)$ and the other is $\mathcal{F}(\mathcal{T}_1) \times \mathcal{F}(\mathcal{T}_2) \times \ldots \times \mathcal{F}(\mathcal{T}_n)$. Questions naturally arise about the usual partial order relation viz, the relation of inclusion, between these two topologies. It has been shown that $\mathcal{F}(\mathcal{T}_1) \times \mathcal{F}(\mathcal{T}_2) \times \ldots \times \mathcal{F}(\mathcal{T}_n) \subset \mathcal{F}(\mathcal{T}_1 \times \mathcal{T}_2 \times \ldots \times \mathcal{T}_n)$. An example has been cited to show that there are spaces where the inclusion is proper. As regards the classes of continuous functions, it has been shown that $\mathscr{C}\left(\left(\prod_{i=1}^{n} X_i, \mathcal{U}_j\right), (Y, \mathcal{U})\right)$ for $j = 1, 2, 3$ are identical, where

$\mathscr{U}_1=\prod_{i=1}^{n}\mathscr{T}_i$, $\mathscr{U}_2=\prod_{i=1}^{n}\mathscr{F}(\mathscr{T}_i)$ and $\mathscr{U}_3=\mathscr{F}\left(\prod_{i=1}^{n}\mathscr{T}_i\right)$ and where $\mathscr{C}((X, \mathscr{T}), (Y, \mathscr{U}))$ denotes the class of continuous functions from $(X, \mathscr{T})$ to $(Y, \mathscr{U})$.

Definition 1. *Let $(X, \mathscr{T})$ be a space and $A\subset X$. Then A is said to be semi-open if there exists an open set O in X such that $O\subset A\subset \overline{O}$, where $(\overline{\ \ })$ denotes the $\mathscr{T}$ — closure* [5].

Definition 2. *Let $(X, \mathscr{T})$ be a space and $A, B\subset X$. Then A is semi-closed iff $X-A$ is semi-open and the semi-closure of B denoted by* scl B *is the intersection of all semi-closed sets of X containing B* [3].

Theorem 1. *For a topological space $(X, \mathscr{T})$ a subset $G\subset X$ belongs to $\mathscr{F}(\mathscr{T})$ iff for each $x\in G$ there is an $\mathscr{T}$ — open neighbourhood N_x of x such that $(\overline{G^0})\supset N_x$ where $(\ )^0$ denotes the $\mathscr{T}$ — interior* [2].

Theorem 2. *Every real-valued continuous function on $(X, \mathscr{F}(\mathscr{T}))$ is continuous on $(X, \mathscr{T})$* [2].

Theorem 2. *Every real-valued continuous function on $(X, \mathscr{F}(\mathscr{T}))$ is continuous on $(X, \mathscr{T})$* [2].

Theorem 3. *Let $(Y, \mathscr{U})$ be a regular space. $f\colon (X, \mathscr{F}(\mathscr{T}))\to(Y, \mathscr{U})$ is continuous iff $f\colon (X, \mathscr{T})\to(Y, \mathscr{U})$ is continuous.*

Proof. Sufficiency. Since $\mathscr{T}\subset\mathscr{F}(\mathscr{T})$ it follows immediately that $f\colon (X, \mathscr{F}(\mathscr{T}))\to (Y, \mathscr{U})$ is continuous whenever $f\colon (X, \mathscr{T})\to(Y, \mathscr{U})$ is continuous.

Necessity. The proof follows by using the lines of proof of Theorem 1.2 [2] where we have only utilized the regularity property of the set of reals with the usual topology.

Example 1. Let $X=\{a, b, c\}$, $\mathscr{T}=\{\emptyset, X, \{a\}\}$. By Theorem 1, $\mathscr{F}(\mathscr{T})=\{\emptyset, X, \{a\}, \{a, c\}, \{a, b\}\}$.

Let $Y=\{x, y, z\}$, $\mathscr{U}=\{\emptyset, Y, \{x\}\}$ Clearly $(Y, \mathscr{U})$ is not regular. Let us define $f\colon (X, \mathscr{F}(\mathscr{T}))\to(Y, \mathscr{U})$ By the rule $f(a)=f(b)=x$, $f(c)=y$. We see that $f\colon (X, \mathscr{F}(\mathscr{T}))\to(Y, \mathscr{U})$ is continuous but $f\colon (X, \mathscr{T})\to(Y, \mathscr{U})$ is not continuous.

Remark 1. By virtue of the above example 1 we see that the condition $(Y, \mathscr{U})$ to be regular in Theorem 3 is not redundant. Now we extend our discussion to product spaces.

Theorem 4. *Let $\{(X_i, \mathscr{T}_i)\colon i=1, 2\}$ be two spaces. Then $\mathscr{F}(\mathscr{T}_1)\times\mathscr{F}(\mathscr{T}_2)\subset\mathscr{F}(\mathscr{T}_1\times\mathscr{T}_2)$.*

Proof. Take $G\in\mathscr{F}(\mathscr{T}_1)\times\mathscr{F}(\mathscr{T}_2)$. Let $(x, y)\in G$. Thus there are $\mathscr{F}(\mathscr{T}_1)$-open U of x and $\mathscr{F}(\mathscr{T}_2)$-open V of y such that $(x, y)\in U\times V\subset G$. Since $U\in\mathscr{F}(\mathscr{T}_1)$ and $V\in\mathscr{F}(\mathscr{T}_2)$ it follows from Theorem 1 that there are a $\mathscr{T}_1$-open neighbourhood N_x of x and a $\mathscr{T}_2$-open neighbourhood N_y of y such that $\mathscr{T}_1-\mathrm{Int}\,U$ is $\mathscr{T}_1$-everywhere dense in N_x and $\mathscr{T}_2 - \text{Int } V$ is $\mathscr{T}_2$-everywhere dense in N_y. Clearly $(x, y) \in N_x \times N_y$. We claim that $(\mathscr{T}_1 \times \mathscr{T}_2) - \text{Int } G$ is $(\mathscr{T}_1 \times \mathscr{T}_2)$-everywhere dense in $N_x \times N_y$. Choose W to be any non-empty open subset in $N_x \times N_y$. So there are non-empty $R \in \mathscr{T}$, and $S \in \mathscr{T}_2$ such that $R \times S \subset W \subset N_x \times N_y$. By Theorem 1 it follows that $R \cap (\mathscr{T}_1 - \text{Int } U) \neq \emptyset$ and $S \cap (\mathscr{T}_2 - \text{Int } V) \neq \emptyset$. Choose $\xi \in R \cap (\mathscr{T}_1 - \text{Int } U)$ and $\eta \in S \cap (\mathscr{T}_2 - \text{Int } V)$. Thus, $(\xi, \eta) \in (\mathscr{T}_1 - \text{Int } U) \times (\mathscr{T}_2 - \text{Int } V) = (\mathscr{T}_1 \times \mathscr{T}_2) - \text{Int } (U \times V) \subset (\mathscr{T}_1 \times \mathscr{T}_2) - \text{Int } G$. Hence $(\xi, \eta) \in W$ and $W \cap (\mathscr{T}_1 \times \mathscr{T}_2) - \text{Int } G \neq \emptyset$. Thus according to Theorem 1 it follows that $G \in \mathscr{F}(\mathscr{T}_1 \times \mathscr{T}_2)$.

Corollary 1. *For a family of spaces* $\{(X_i, \mathscr{T}_i): i = 1, 2, \ldots, n\}$ *we have* $\prod_{i=1}^{n} \mathscr{T}_i \subset \prod_{i=1}^{n} \mathscr{F}(\mathscr{T}_i) \subset \mathscr{F}\left(\prod_{i=1}^{n} \mathscr{T}_i\right)$. *The converse of the Theorem 4 is not necessarily true.*

Example 2. Let $X_1 = \{a, b\}$, $\mathscr{T}_1 = \{\emptyset, X_1, \{a\}\}$ *and* $X_2 = \{x, y\}$, $\mathscr{T}_2 = \{\emptyset, X_2, \{x\}\}$. Now $\mathscr{T}_1 \times \mathscr{T}_2 = [\emptyset, \{(a, x), (b, x), (a, y), (b, y)\}, \{(a, x), (a, y)\}, \{(a, x), (b, x)\}, \{(a, x)\}, \{(a, x), (a, y), (b, x)\}]$.

By Theorem 1, we have $\mathscr{F}(\mathscr{T}_1) = \{\emptyset, X_1, \{a\}\}$ and $\mathscr{F}(\mathscr{T}_2) = \{\emptyset, X_2, \{x\}\}$. Clearly $\mathscr{T}_1 \times \mathscr{T}_2 = \mathscr{F}(\mathscr{T}_1) \times \mathscr{F}(\mathscr{T}_2)$. Now, $\mathscr{F}(\mathscr{T}_1 \times \mathscr{T}_2) = [\emptyset, \{(a, x), (b, x), (a, y), (b, y)\}, \{(a, x), (a, y), (b, x)\}, \{(a, x), (a, y)\}, \{(a, x), (b, x)\}, \{(a, x), (b, y)\}, \{a, x)\}, \{(a, x), (a, y), (b, y)\}, \{(a, x), (b, x), (b, y)\}]$.

Hence, $\mathscr{F}(\mathscr{T}_1) \times \mathscr{F}(\mathscr{T}_2) \subsetneqq \mathscr{F}(\mathscr{T}_1 \times \mathscr{T}_2)$

Theorem 5. Let $\{(X_i, \mathscr{T}_i): i = 1, 2, \ldots, n\}$ *be a family of spaces and* $(Y, \mathscr{U})$ *be a regular space. Then the following statements are equivalent.*

(i) $f\colon \left(\prod_{i=1}^{n} X_i, \prod_{i=1}^{n} \mathscr{T}_i\right) \to (Y, \mathscr{U})$ *is continuous.*

(ii) $F\colon \left(\prod_{i=1}^{n} X_i, \prod_{i=1}^{n} \mathscr{F}(\mathscr{T}_i)\right) \to (Y, \mathscr{U})$ *is continuous.*

(iii) $f\colon \left(\prod_{i=1}^{n} X_i, \mathscr{F}\left(\prod_{i=1}^{n} \mathscr{T}_i\right)\right) \to (Y, \mathscr{U})$ *is continuous.*

Proof. (i) implies (ii) and (ii) implies (iii) from Corollary 1. Since $\left(\prod_{i=1}^{n} X_i, \prod_{i=1}^{n} \mathscr{T}_i\right)$ satisfies the first axiom of countability, it follows from Theorem 3 that (iii) implies (i). This completes the proof of the theorem.

We state below the well-known result.

Lemma 1. *A topological space X is disconnected iff there exists a continuous mapping of X onto the discrete two-point space* $\{0, 1\}$ [6 p. 144].

The following theorem is an easy consequence of Lemma 1 and Theorem 5.

Theorem 6. *Let* $\{(X_i, \mathcal{T}_i) : i = 1, 2, \ldots, n\}$ *be a family of spaces. Then the following statements are equivalent.*

(i) $\left(\prod_{i=1}^{n} X_i, \prod_{i=1}^{n} \mathcal{T}_i\right)$ *is connected.*

(ii) $\left(\prod_{i=1}^{n} X_i, \prod_{i=1}^{n} \mathcal{F}(\mathcal{T}_i)\right)$ *is connected.*

(iii) $\left(\prod_{i=1}^{n} X_i, \mathcal{F}\left(\prod_{i=1}^{n} \mathcal{T}_i\right)\right)$ *is connected.*

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О ТОПОЛОГИИ ПРОИЗВЕДЕНИЯ СВЯЗАННОЙ С ПОЛУЗАМКНУТЫМИ МНОЖЕСТВАМИ

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Резюме

В работе рассматривается конечное семейство топологических пространств удовлетворяющих первой аксиоме счетности. Кроме топологии произведения на декартовом произведении этих пространств тоже изучаются две других топологий соответствующих полузамкнутым множествам. Показано, что эти три топологии различные, но семейства всех непрерывных отображений из каждого с этих пространств в любое регулярное пространство, совпадают. Даны также соотношения между связностью этих пространств.