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ULTRAFILTERS AND DARBOUX PROPERTY OF FINITELY ADDITIVE MEASURE

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Introduction. It is well known that if m is a σ-additive σ-finite measure defined on a δ-ring (i. e. a ring closed under countable intersections) $\mathcal{T}$ of subsets of a set Z, then the following three propositions are fulfilled:

I. If $\mathcal{T}$ is a σ-algebra (i. e. $Z \in \mathcal{T}$), then we can write $Z = A \cup B$, where A, B are disjoint and where m is purely atomic on A (i. e. A is the union of a sequence of mutually disjoint atoms) and nonatomic on B (i. e. B contains no atom). This decomposition is unique to within null sets.

II. Let $E \in \mathcal{T}$. Then $\mathcal{T}_E = \{T \cap E \colon T \in \mathcal{T}\}$ is a σ-algebra and by I, $E = A_E \cup B_E$, where m is nonatomic on B_E and A_E is the union of a sequence (finite or infinite) $\{A_i\}_{i \in I}$ of mutually disjoint atoms. We can suppose that $m(A_i) \geqq m(A_{i+1})$ whenever i, $i+1 \in I$. The measure m has the Darboux property on the set E (i. e. for every $\alpha \in (0, m(E))$ there is a measurable set $A \subset E$ such that $m(A) = \alpha$) if and only if

$$m(A_n) \leqq m(E) - \sum_{i \in I,\, i \leqq n} m(A_i)$$

for each $n \in I$.

III. The measure m has the Darboux property (i. e. m has the Darboux property on each set $E \in \mathcal{T}$) if and only if m is nonatomic.

In these propositions a set $A \in \mathcal{T}$ is called an atom (with respect to m) if and only if $m(A) > 0$ and for every measurable set $B \subset A$ we have $m(B) = 0$ or $m(B) = m(A)$.

In [2, p. 47, Example A; p. 48, Example B] and [3, 2, Theorem 1] it is shown that for a finitely additive measure analogical propositions do not hold.

In the present paper the notion of atom is generalized in such a way that some propositions about relations between the Darboux property and the properties of atoms can be formulated for a finitely additive measure analogically to the way in which they are formulated and hold for a σ-additive measure.

1. Generalization of the notion of atom

Let m be a finitely additive finite measure defined on an algebra $\mathscr{C}$ of subsets of a set X. Let us denote by $\mathbf{Q}$ the set of all ultrafilters in the algebra $\mathscr{C}$.

Definition 1. *For every ultrafilter $\mathscr{A} \in \mathbf{Q}$, the number*

$$\mathbf{m}(\mathscr{A}) = \inf \{m(A) : A \in \mathscr{A}\}$$

is called a weight of the ultrafilter $\mathscr{A}$.

Definition 2. *An ultrafilter $\mathscr{A} \in \mathbf{Q}$ with a positive weight is called an u-atom (with respect to m).*

Definition 3. *An u-atom $\mathscr{A}$ is called a trivial u-atom iff there is a set $A \in \mathscr{A}$ such that $m(A) = \mathbf{m}(\mathscr{A})$.*

The next theorem shows that a u-atom is in fact a generalization of the notion of the atom, namely that every trivial u-atom corresponds to an atom.

Theorem 1. *Let m be a finitely additive finite measure defined on an algebra $\mathscr{C}$. Then*

(i) *for each atom $A \in \mathscr{C}$ there is one and only one u-atom $\mathscr{A}_A$ such that $A \in \mathscr{A}_A$,*

(ii) *for each atom $A \in \mathscr{C}$, the u-atom $\mathscr{A}_A$ is trivial and $m(A) = \mathbf{m}(\mathscr{A}_A)$,*

(iii) *for each pair A, B of atoms $\mathscr{A}_A = \mathscr{A}_B$ if and only if $A = B \bmod m$ (i. e. $m(A + B) = 0$, where $A + B = (A - B) \cup (B - A)$),*

(iv) *for each trivial u-atom $\mathscr{A}$ the set $A \in \mathscr{A}$, for which $m(A) = \mathbf{m}(\mathscr{A})$, is an atom.*

Proof. It is easy to see that for an arbitrary atom $A \in \mathscr{C}$

$$\mathscr{A}_A = \{E \in \mathscr{C} : m(E \cap A) = m(A)\}$$

is an atom, $A \in \mathscr{A}_A$ and $m(A) = \mathbf{m}(\mathscr{A}_A)$. The properties of atom and ultrafilter imply uniqueness of $\mathscr{A}_A$ and the assertions (iii) and (iv).

Now we shall explain the relation between atom and u-atom with respect to a σ-additive measure.

Theorem 2. *If m is a σ-additive finite measure defined on a σ-algebra $\mathscr{S}$, then every u-atom with respect to m is trivial.*

Proof. Choose a sequence $\{A_n\}_{n=1}^{\infty}$ of sets belonging to a u-atom $\mathscr{A}$ such that

$$m(A_n) < \mathbf{m}(\mathscr{A}) + \frac{1}{n}$$

and put $A = \bigcap_{n=1}^{\infty} A_n$. Since $\mathscr{S}$ is a σ-algebra, $A \in \mathscr{S}$ and since m is σ-additive, $m(A) = \mathbf{m}(\mathscr{A})$.

It is necessary to prove yet that $A \in \mathscr{A}$. Choose an integer k such that

$m(A_k) < 2m(A)$. Since $m(A_k - A) < m(A) = \mathbf{m}(\mathscr{A})$, we have $A_k - A \notin \mathscr{A}$ and considering $A_k \in \mathscr{A}$, we obtain $A \in \mathscr{A}$.

2. Atoms with respect to a σ-additive extension of m to the Stone representation space

Let m be a finitely additive finite measure defined on an algebra $\mathscr{C}$ of subsets of a set X. If we denote

$$h(A) = \{\mathscr{A} \in \mathbf{Q} : A \in \mathscr{A}\}$$

for every $A \in \mathscr{C}$, then h is an isomorphism transferring $\mathscr{C}$ onto $h(\mathscr{C}) = \{h(A) : A \in \mathscr{C}\}$, where $h(\mathscr{C})$ is an algebra of all open-closed subsets with respect to the topology of the Stone representation space $\mathbf{Q}$, a base of which is $h(\mathscr{C})$. The isomorphism h transfers also the measure m to a measure m_h defined on $h(\mathscr{C})$ by the equality

$$m_h(h(A)) = m(A).$$

According to the properties of $h(\mathscr{C})$, m_h is σ-additive and it can be extended in a standard way to a σ-additive measure μ defined on the σ-algebra $\mathscr{S}(h(\mathscr{C}))$ generated by the algebra $h(\mathscr{C})$ ([4, p. 325]).

Since each u-atom with respect to m is an element of the Stone representation space $\mathbf{Q}$, there is a question of a relationship between u-atoms with respect to m and atoms with respect to μ.

Theorem 3. *Let m be a finitely additive finite measure defined on an algebra $\mathscr{C}$ and let μ be the σ-additive extension of m in the Stone representation space of the algebra $\mathscr{C}$. Then*

(i) *for each atom $\mathbf{A} \in \mathscr{S}(h(\mathscr{C}))$ with respect to μ there exists one and only one u-atom $\mathscr{A}_{\mathbf{A}} \in \mathbf{Q}$ with respect to m such that $\mathscr{A}_{\mathbf{A}} \in \mathbf{A}$ and $\mathbf{m}(\mathscr{A}_{\mathbf{A}}) = \mu(\mathbf{A})$,*

(ii) *for each u-atom $\mathscr{A} \in \mathbf{Q}$ there exists an atom $\mathbf{A} \in \mathscr{S}(h(\mathscr{C}))$ such that $\mathscr{A}_{\mathbf{A}} = \mathscr{A}$,*

(iii) *for each pair $\mathbf{A}$, $\mathbf{B}$ of atoms with respect to μ, $\mathscr{A}_{\mathbf{A}} = \mathscr{A}_{\mathbf{B}}$ if and only if $\mathbf{A} = \mathbf{B} \bmod \mu$.*

Proof. Taking an arbitrary atom $\mathbf{A}$ with respect to μ we put

$$\mathscr{A}_{\mathbf{A}} = \{A \in \mathscr{C} : \mu(h(A) \cap \mathbf{A}) = \mu(\mathbf{A})\},$$

similarly as in the proof of Theorem 1. $\mathscr{A}_{\mathbf{A}}$ is an ultrafilter and since for each set $A \in \mathscr{A}_{\mathbf{A}}$

$$m(A) = m_h(h(A)) = \mu(h(A)) \geqq \mu(\mathbf{A}),$$

we have $\mathbf{m}(\mathscr{A}_{\mathbf{A}}) \geqq \mu(\mathbf{A})$, whence $\mathscr{A}_{\mathbf{A}}$ is a u-atom with respect to m.

Since $\mathbf{A} \in \mathcal{S}(h(\mathscr{C}))$, by [1, 13.D], for each ε, $0 < \varepsilon \leqq \mu(\mathbf{A})$ there is a set $A_\varepsilon \in \mathscr{C}$ such that $\mu(h(A_\varepsilon) + \mathbf{A}) < \varepsilon$. Since $\mathbf{A}$ is an atom,

$$\mu(h(A_\varepsilon) \cap \mathbf{A}) = \mu(\mathbf{A}),$$

whence $A_\varepsilon \in \mathscr{A}_\mathbf{A}$. We have

$$\begin{aligned} \boldsymbol{m}(\mathscr{A}_\mathbf{A}) &\leqq m(A_\varepsilon) = \mu(h(A_\varepsilon)) = \mu((h(A_\varepsilon) \cap \mathbf{A}) \cup (h(A_\varepsilon) - \mathbf{A})) = \\ &= \mu(h(A_\varepsilon) \cap \mathbf{A}) + \mu(h(A_\varepsilon) - \mathbf{A}) \leqq \mu(\mathbf{A}) + \mu(h(A_\varepsilon) + \mathbf{A}) < \mu(\mathbf{A}) + \varepsilon. \end{aligned}$$

Thus $\boldsymbol{m}(\mathscr{A}_\mathbf{A}) = \mu(\mathbf{A})$.

The uniqueness of $\mathscr{A}_\mathbf{A}$ and the assertion (iii) follow from properties of ultrafilters and atoms.

According to the preceding uniqueness, to prove (ii) it suffices to show that for each u-atom $\mathscr{A} \in \mathbf{Q}$ there exists an atom $\mathbf{A} \in \mathcal{S}(h(\mathscr{C}))$ such that $\mathscr{A} \in \mathbf{A}$ and $\boldsymbol{m}(\mathscr{A}) = \mu(\mathbf{A})$. For this purpose for each integer n choose a set $B_n \in \mathscr{A}$ such that $m(B_n) < \boldsymbol{m}(\mathscr{A}) + n^{-1}$ and put $A_n = \bigcap_{i=1}^{n} B_i$. Then $\{A_n\}_{n=1}^{\infty}$ is a decreasing sequence of sets in $\mathscr{A}$ (hence $\mathscr{A} \in h(A_n)$ for each n) with the property

$$\lim_n m(A_n) = \boldsymbol{m}(\mathscr{A}).$$

From these facts, putting $\mathbf{A} = \bigcap_{n=1}^{\infty} h(A_n)$, we obtain $\mathscr{A} \in \mathbf{A}$ and

$$\boldsymbol{m}(\mathscr{A}) = \lim_n m(A_n) = \lim_n m_h(h(A_n)) = \mu(\mathbf{A}).$$

3. Decomposition of a finitely additive measure

If the measure m is not σ-additive, it is true that we can decompose the underlying set X to a set A and a set B such that m is purely atomic on A and nonatomic on B, but this decomposition is not unique ([2, p. 48, Example B]), i. e. proposition I does not hold in this case. Besides, a restriction of m to the nonatomic part of X need not have the Darboux property. Therefore it will be suitable to decompose m to a sum of a u-nonatomic measure and a purely u-atomic measure.

Definition 11. *Let m be a finitely additive finite measure defined on an algebra $\mathscr{C}$.*

We shall say that m is u-nonatomic iff $\boldsymbol{m}(\mathscr{A}) = 0$ for all $\mathscr{A} \in \mathbf{Q}$

We shall say that m is purely u-atomic iff for an arbitrary measurable set A with $m(A) > 0$ we have

$$m(A) = \Sigma\{\boldsymbol{m}(\mathscr{A}) \colon \mathscr{A} \text{ is an } u\text{-atom}, A \in \mathscr{A}\}.$$

Note that $\sum_{i \in \emptyset} c_i = 0$ and that from Theorem 3 it follows

$$\operatorname{card}\{\mathscr{A} \in \mathbf{Q}:\ \boldsymbol{m}(\mathscr{A}) > 0\} \leqq \aleph_0.$$

Theorem 4. *Let m be a finitely additive finite measure defined on an algebra $\mathscr{C}$. Then there exist measures n and p such that*

(i) *n is u-nonatomic,*

(ii) *p is purely u-atomic,*

(iii) *$m = n + p$.*

Conditions (i), (ii) *and* (iii) *determine the measures n and p uniquely.*

Proof. Denote by $\{\mathscr{A}_i\}_{i \in I}$ the set of all u-atoms with respect to m and for an arbitrary $A \in \mathscr{C}$ put

$$p(A) = \Sigma\{\boldsymbol{m}(\mathscr{A}_i):\ i \in I,\ A \in \mathscr{A}_i\}.$$

Evidently p is a finitely additive finite purely u-atomic measure. Now, if we put

$$n(A) = m(A) - p(A),$$

then n is also a finitely additive finite measure.

To show the u-nonatomicity of n let us take an arbitrary ultrafilter $\mathscr{A}$ in $\mathscr{C}$. Let $\{C_k\}_{k=1}^{\infty}$ be a decreasing sequence of sets in $\mathscr{A}$ such that $C_k \notin \mathscr{A}_i$ for each k and each $i \in I$, $i \leqq k$ for which $\mathscr{A}_i \neq \mathscr{A}$, and

$$\lim_k m(C_k) = \boldsymbol{m}(\mathscr{A}).$$

We have

$$\boldsymbol{n}(\mathscr{A}) = \inf\{n(A):\ A \in \mathscr{A}\} \leqq \inf\{n(C_k):\ k \in N\} =$$

$$= \lim_k n(C_k) = \lim_k \left(m(C_k) - \Sigma\{\boldsymbol{m}(\mathscr{A}_i):\ i \in I,\ C_k \in \mathscr{A}_i\}\right) =$$

$$= \lim_k m(C_k) - \lim_k \Sigma\{\boldsymbol{m}(\mathscr{A}_i):\ i \in I,\ C_k \in \mathscr{A}_i\} = \boldsymbol{m}(\mathscr{A}) - \boldsymbol{m}(\mathscr{A}) = 0.$$

The uniqueness of the decomposition is trivial.

4. u-nonatomicity and the Darboux property

Since the notion of the u-atom is a generalization of the notion of the atom, the condition of u-nonatomicity is stronger than the condition of the nonatomicity. We shall show now that this condition is already a necessary and sufficient one for the Darboux property of a finitely additive finite measure.

Theorem 5. *A finitely additive finite measure m defined on a σ-algebra $\mathscr{S}$ of subsets of a set X has the Darboux property if and only if it is* u-nonatomic.

Proof. We use the well-known equivalent condition of the Dabroux property of a finitely additive finite measure which is proved in [2, Theorem 2]:

A finitely additive finite measure m defined on a σ-algebra $\mathcal{S}$ of subsets of a set X has the Darboux property (i. e. m is full-valued) if and only if for each positive number ε there exists a finite measurable cover $\{A_i\}_{i=1}^n$ of the set X such that $m(A_i)<\varepsilon$ for $i=1, 2, \ldots, n$.

Let m have the Darboux property. Taking any ε, choose a finite measurable cover $\{A_i\}_{i=1}^n$ of X such that $m(A_i)<\varepsilon$ for $i=1, 2, \ldots, n$. Without lost of generality we can suppose that $A_j \cap A_k=\emptyset$ whenever $j \neq k$. If $\mathcal{A}$ is an arbitrary ultrafilter in $\mathcal{S}$, then there is a unique index p such that $A_p \in \mathcal{A}$. It follows that

$$\mathbf{m}(\mathcal{A}) \leqq m(A_p)<\varepsilon .$$

Consequently $\mathbf{m}(\mathcal{A})=0$.

Now let m be u-nonatomic and let ε be an arbitrary positive number. Since the function $\mathbf{m}$ is on $\mathbf{Q}$ identically equal to zero, for each $\mathcal{A} \in \mathbf{Q}$ there is a set $A_{\mathcal{A}} \in \mathcal{A}$ such that $m(A_{\mathcal{A}})<\varepsilon$. The class $\{A_{\mathcal{A}}: \mathcal{A} \in \mathbf{Q}\}$ is a cover of X, because for any $x \in X$ there is an ultrafilter, namely $\mathcal{A}_x=\{A \in \mathcal{S}: x \in A\}$, all elements of which contain x. Then $\{h(A_{\mathcal{A}}): \mathcal{A} \in \mathbf{Q}\}$ is an open cover of the Stone representation space $\mathbf{Q}$. Since it is compact, this cover contains a finite subcover, say $\{h(A_i)\}_{i=1}^n$. It follows that $\{A_i\}_{i=1}^n$ is the required finite measurable cover, the elements of which have their measure less than ε.

According to the proved theorem and Theorem 3 we have the following:

Theorem 6. *The finitely additive finite measure m defined on a σ-algebra $\mathcal{S}$ has the Darboux property if and only if its σ-additive extension μ to the Stone representation space has the Darboux property*

5. The Darboux property on a set

In this section, in Theorem 7, a sufficient condition is given for a finitely additive measure defined on a δ-ring to have the Darboux property on a measurable set of a finite measure.

Let m be a finitely additive measure defined on a δ-ring $\mathcal{T}$ of subsets of a set Z and let X be such a set that $X \in \mathcal{T}$ and $m(X)<\infty$. The class $\mathcal{S}=\{T \cap X: T \in \mathcal{T}\}$ is a σ-algebra and m is a finitely additive finite measure on $\mathcal{S}$. Preliminary lemmas will be used in the proof of Theorem 7.

Lemma 1. *Let $\{\mathcal{A}_i\}_{i=1}^n$ be a set of mutually different u-atoms. Then for any $\varepsilon>0$ and for $i=1, 2, \ldots, n$ there exists a set $A_i \in \mathcal{A}_i$ such that $A_j \cap A_k=\emptyset$ whenever $j \neq k$ and*

$$m\left(\bigcup_{i=1}^n A_i\right)-\sum_{i=1}^n \mathbf{m}(\mathcal{A}_i)<\varepsilon .$$

To choose A_i *such that*

$$m\left(\bigcup_{i=1}^{n} A_i\right)-\sum_{i=1}^{n} \mathbf{m}(\mathscr{A}_i)=0$$

it is possible if and only if $\mathscr{A}_i$ *is a trivial u-atom for* $i=1, 2, \ldots, n$.

Lemma 2. *Let* $Y\in\mathscr{S}$ *and* m_Y *be a restriction of* m *to the* σ*-algebra* $\mathscr{S}_Y=\{S\cap Y\colon S\in\mathscr{S}\}$. *Then* $\mathscr{B}$ *is a u-atom with respect to* m_Y *if and only if there exists a u-atom* $\mathscr{A}$ *with respect to* m *such that* $Y\in\mathscr{A}$ *and* $\mathscr{B}=\{A\cap Y\colon A\in\mathscr{A}\}$. *Moreover, writing* $\mathbf{m}_Y(\mathscr{B}) = \inf\{m(B)\colon B\in\mathscr{B}\}$ *we have* $\mathbf{m}_Y(\mathscr{B}) = \mathbf{m}(\mathscr{A})$ *in this case.*

The assertions of Lemma 1 and Lemma 2 follow from the properties of ultrafilters and from the definition of their weight.

Theorem 7. *Let* m *be a finitely additive measure defined on a* δ*-ring* $\mathscr{T}$, *let* $X\in\mathscr{T}$ *and* $m(X)<\infty$. *Let* $\{\mathscr{A}_i\}_{i\in I}$ *be the class of all u-atoms with respect to* m *restricted to the* σ*-algebra* $\mathscr{S}=\{T\cap X\colon T\in\mathscr{T}\}$. *Further, let* $\sum_{i\in I}\mathbf{m}(\mathscr{A}_i)<m(X)$ *and* $\mathbf{m}(\mathscr{A}_i)\geqq\mathbf{m}(\mathscr{A}_{i+1})$ *for* $i\in I$ *whenever* $i+1\in I$.

In order that m *may have the Darboux property on the set* X, *it is sufficient for each* $n\in I$

$$\mathbf{m}(\mathscr{A}_n)<m(X)-\sum_{i\in I,\, i\leqq n}\mathbf{m}(\mathscr{A}_i).$$

Proof. Taking any $\alpha\in(0, m(X))$ we shall find a measurable set A with $m(A)=\alpha$. We shall do it successively in three steps.

1. We assume that the class of all u-atoms in $\mathscr{S}$ is finite, i. e. $I=\{1, 2, \ldots, n\}$.

We shall take an index subset J of I as follows:

if $\alpha\leqq\mathbf{m}(\mathscr{A}_n)$, then $J=\emptyset$,
if $\alpha>\mathbf{m}(\mathscr{A}_n)$, then $J=\{k_p\}_{p=1}^{r}$,

where

$$k_1=\min\left\{i\in I\colon \mathbf{m}(\mathscr{A}_i)<\alpha\right\},$$
$$k_p=\min\left\{i\in I\colon i>k_{p-1},\ \mathbf{m}(\mathscr{A}_i)<\alpha-\sum_{j=1}^{p-1}\mathbf{m}(\mathscr{A}_{k_j})\right\}$$

for $p=2, 3, \ldots, r$.

We shall show now that

$$0<\alpha-\sum_{j\in J}\mathbf{m}(\mathscr{A}_j)<m(X)-\sum_{i\in I}\mathbf{m}(\mathscr{A}_i).$$

Note that if $J=\emptyset$, then $\sum_{j\in J}\mathbf{m}(\mathscr{A}_j)=\sum_{j\in\emptyset}c_j=0$, therefore the inequalities evidently hold.

If $J\neq\emptyset$, the left inequality follows from

$$\mathbf{m}(\mathcal{A}_{k_r})<\alpha-\sum_{j=1}^{r-1}\mathbf{m}(\mathcal{A}_{k_j}).$$

To prove the right inequality, first we suppose that $J=I$. Then

$$\alpha-\sum_{j\in J}\mathbf{m}(\mathcal{A}_j)=\alpha-\sum_{i\in I}\mathbf{m}(\mathcal{A}_i)<m(X)-\sum_{i\in I}\mathbf{m}(\mathcal{A}_i).$$

Now we suppose that $k_r<n$. Then

$$\alpha-\sum_{j\in J}\mathbf{m}(\mathcal{A}_j)\leqq\mathbf{m}(\mathcal{A}_n)<m(X)-\sum_{i\in I}\mathbf{m}(\mathcal{A}_i).$$

Finally, if $J\neq I$ and $k_r=n$, writing

$$p=\max\{j\in I\colon j\neq k_i \text{ for } i=1,2,\dots,r\},$$
$$q=\max\{j\leqq r\colon k_j<p\},$$

we obtain

$$\alpha-\sum_{i=1}^{q}\mathbf{m}(\mathcal{A}_{k_i})\leqq\mathbf{m}(\mathcal{A}_p)<m(X)-\sum_{i\in I,\, i\leq p}\mathbf{m}(\mathcal{A}_i)=$$
$$=m(X)-\sum_{i\in I}\mathbf{m}(\mathcal{A}_i)+\sum_{i=p+1}^{n}\mathbf{m}(\mathcal{A}_i)=m(X)-\sum_{i\in I}\mathbf{m}(\mathcal{A}_i)+\sum_{i=q+1}^{r}\mathbf{m}(\mathcal{A}_{k_i}),$$

whence

$$\alpha-\sum_{i=1}^{r}\mathbf{m}(\mathcal{A}_{k_i})<m(X)-\sum_{i\in I}\mathbf{m}(\mathcal{A}_i)$$

and we have proved the required inequalities.

Put now

$$\varepsilon=\min\left\{\alpha-\sum_{j\in J}\mathbf{m}(\mathcal{A}_j),\ m(X)-\sum_{i\in I}\mathbf{m}(\mathcal{A}_i)-\alpha+\sum_{j\in J}\mathbf{m}(\mathcal{A}_j)\right.\!.$$

By Lemma 1 choose mutually disjoint sets $A_i\in\mathcal{A}_i$ such that

$$m\Big(\bigcup_{i\in I}A_i\Big)-\sum_{i\in I}\mathbf{m}(\mathcal{A}_i)<\varepsilon.$$

If we denote

$$\delta=m\Big(\bigcup_{j\in J}A_j\Big)-\sum_{j\in J}\mathbf{m}(\mathcal{A}_j),$$

we have

$$0\leqq\alpha-\sum_{j\in J}\mathbf{m}(\mathcal{A}_j)-\varepsilon\leqq\alpha-\sum_{j\in J}\mathbf{m}(\mathcal{A}_j)-\delta=\alpha-m\Big(\bigcup_{j\in J}A_j\Big)$$

and further

$$\alpha - m\Big(\bigcup_{j\in J} A_j\Big) \leqq \alpha - m\Big(\bigcup_{j\in J} A_j\Big) + \delta = \alpha - \sum_{j\in J} \mathbf{m}(\mathscr{A}_j) \leqq$$
$$\leqq m(X) - \sum_{i\in I} \mathbf{m}(\mathscr{A}_i) - \varepsilon < m(X) - m\Big(\bigcup_{i\in I} A_i\Big) = m\Big(X - \bigcup_{i\in I} A_i\Big).$$

Since the set $X - \bigcup_{i\ I} A_i$ does not belong to any of the u-atoms $\mathscr{A}_i$, $i = 1, 2, \dots, n$, by Lemma 5, m is u-nonatomic on the set $X - \bigcup_{i\in I} A_i$. Consequently, m has the Darboux property on $X - \bigcup_{i\in I} A_i$, therefore there is a set $B \in \mathscr{S}$, $B \subset X - \bigcup_{i\in I} A_i$ such that

$$m(B) = \alpha - m\Big(\bigcup_{j\in J} A_j\Big).$$

The set $A = B \cup \bigcup_{j\in J} A_j$ is the required set, satisfying $m(A) = \alpha$.

2. In the second case we shall assume that the set $\{\mathscr{A}_i\}_{i\in I}$ of all u-atoms is infinite, i. e. $I = \{1, 2, \dots\}$ (note that by Theorem 3 I is at most countable) and

$$\alpha < m(X) - \sum_{i\in I} \mathbf{m}(\mathscr{A}_i).$$

By means of recurrence we shall construct sequences $\{K_n\}_{n=1}^{\infty}$ and $\{L_n\}_{n=1}^{\infty}$ of measurable sets and a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ of positive numbers such that $\lim_n \varepsilon_n = 0$ and for all n $K_n \subset K_{n+1} \subset L_{n+1} \subset L_n$ and

$$\alpha - \varepsilon_n \leqq m(K_n) < \alpha < m(L_n) < \alpha + 2\varepsilon_n.$$

By Theorem 4 decompose the measure m on $\mathscr{S}$ to the sum of the u-nonatomic part n and the purely u-atomic part p. Take an arbitrary positive $\varepsilon_1 < \min\{\alpha, n(X) - \alpha\}$. Since n is u-nonatomic, it has the Darboux property, therefore there is a set $F_1' \in \mathscr{S}$ such that $n(F_1') = \alpha + \varepsilon_1$. Further, we can choose an integer k_1 such that $\sum_{i>k_1} \mathbf{m}(\mathscr{A}_i) < \varepsilon_1$ and by Lemma 1 for each $i \leqq k_1$ we can choose a set $A_i \in \mathscr{A}_i$ such that $A_j \cap A_k = \emptyset$ whenever $j \neq k$ and

$$m\Big(\bigcup_{i=1}^{k_1} A_i\Big) - \sum_{i=1}^{k_1} \mathbf{m}(\mathscr{A}_i) < \varepsilon_1.$$

If we put $F_1 = F_1' - \bigcup_{i=1}^{k_1} A_i$, we have

$$m(F_1)=n(F_1)+p(F_1)=n(F_1)+p\Big(F_1'-\bigcup_{i\leqq k_1}A_i\Big)\leqq$$
$$\leqq n(F_1')+\sum_{i>k_1}\boldsymbol{m}(\mathscr{A}_i)<\alpha+2\varepsilon_1$$

and on the other hand

$$m(F_1)\geqq n(F_1)=n\Big(F_1'-\bigcup_{i\leqq k_1}A_i\Big)\geqq$$
$$\geqq n(F_1')-n\Big(\bigcup_{i\leqq k_1}A_i\Big)=n(F_1')-\Big(m\Big(\bigcup_{i\leqq k_1}A_i\Big)-p\Big(\bigcup_{i\leqq k_1}A_i\Big)\Big)\geqq$$
$$\geqq n(F_1')-m\Big(\bigcup_{i\leqq k_1}A_i\Big)+\sum_{i\leqq k_1}\boldsymbol{m}(\mathscr{A}_i)>n(F_1')-\varepsilon_1=\alpha.$$

Since $0<\varepsilon_1<\alpha<n(F_1)$ and n has the Darboux property, there is a measurable set $E_1\subset F_1$ such that $n(E_1)=\alpha-\varepsilon_1$. Then

$$\alpha-\varepsilon_1=n(E_1)\leqq m(E_1)=n(E_1)+p(E_1)\leqq n(E_1)+p(F_1)\leqq$$
$$\leqq n(E_1)+\sum_{i>k_1}\boldsymbol{m}(\mathscr{A}_i)<\alpha-\varepsilon_1+\varepsilon_1=\alpha.$$

Now we put $K_1=E_1$ and $L_1=F_1$. For the sets K_1, L_1 and the number ε_1 we obtain

$$K_1\subset L_1,$$
$$\alpha-\varepsilon_1\leqq m(K_1)<\alpha<m(L_1)<\alpha+2\varepsilon_1$$

and

$$n(L_1-K_1)>\alpha-\alpha+\varepsilon_1=\varepsilon_1.$$

Thus we have the first step of the construction by means of recurrence. Let us suppose now that there are the sets $K_1\subset K_2\subset\ldots\subset K_{n-1}$, $L_1\supset L_2\supset\ldots\supset L_{n-1}$ and a positive number ε_{n-1} such that

$$K_{n-1}\subset L_{n-1},$$
$$n(L_{n-1}-K_{n-1})>\varepsilon_{n-1},$$
$$\alpha-\varepsilon_{n-1}\leqq m(K_{n-1})<\alpha<m(L_{n-1})<\alpha+2\varepsilon_{n-1}.$$

Take any positive ε_n such that

$$\varepsilon_n<\min\{\alpha-m(K_{n-1}),\ n(L_{n-1}-K_{n-1})-\alpha+m(K_{n-1})\}.$$

Since

$$0<\alpha-m(K_{n-1})+\varepsilon_n<n(L_{n-1}-K_{n-1})$$

and n has the Darboux property, there is a set $F_n'\subset L_{n-1}-K_{n-1}$ such that

$$n(F_n')=\alpha-m(K_{n-1})+\varepsilon_n.$$

Choose an integer k_n such that $\sum_{i>k_n} \mathbf{m}(\mathcal{A}_i)<\varepsilon_n$ and by Lemma 1 for each $i\leqq k_n$ take a set $A_i\in\mathcal{A}_i$ such that $A_j\cap A_k=\emptyset$ whenever $j\neq k$ and

$$m\Big(\bigcup_{i\leqq k_n} A_i\Big)-\sum_{i\leqq k_n}\mathbf{m}(\mathcal{A}_i)<\varepsilon_n .$$

If we put $F_n=F_n'-\bigcup_{i\leqq k_n} A_i$, we have

$$m(F_n)=n(F_n)+p(F_n)=n(F_n)+p\Big(F_n'-\bigcup_{i\leqq k_n}A_i\Big)\leqq$$
$$\leqq n(F_n')+\sum_{i>k_n}\mathbf{m}(\mathcal{A}_i)<\alpha-m(K_{n-1})+2\varepsilon_n$$

and on the other hand

$$m(F_n)\geqq n(F_n)=n\Big(F_n'-\bigcup_{i\leqq k_n}A_i\Big)\geqq n(F_n')-n\Big(\bigcup_{i\leqq k_n}A_i\Big)=$$
$$=n(F_n')-\Big(m\Big(\bigcup_{i\leqq k_n}A_i\Big)-p\Big(\bigcup_{i\leqq k_n}A_i\Big)\Big)\geqq$$
$$\geqq n(F_n')-m\Big(\bigcup_{i\leqq k_n}A_i\Big)+\sum_{i\leqq k_n}\mathbf{m}(\mathcal{A}_i)>n(F_n')-\varepsilon_n=\alpha-m(K_{n-1}).$$

Since $0<\varepsilon_n<\alpha-m(K_{n-1})<n(F_n)$ and n has the Darboux property, there is a measurable set $E_n\subset F_n$ such that $n(E_n)=\alpha-m(K_{n-1})-\varepsilon_n$. Then

$$\alpha-m(K_{n-1})-\varepsilon_n=n(E_n)\leqq m(E_n)=n(E_n)+p(E_n)\leqq n(E_n)+p(F_n)\leqq$$
$$\leqq n(E_n)+\sum_{i>k_n}\mathbf{m}(\mathcal{A}_i)<\alpha-m(K_{n-1})-\varepsilon_n+\varepsilon_n=\alpha-m(K_{n-1}).$$

If we put $K_n=K_{n-1}\cup E_n$ and $L_n=K_{n-1}\cup F_n$, then

$$K_{n-1}\subset K_n\subset L_n\subset L_{n-1},$$
$$n(L_n-K_n)>\alpha-m(K_{n-1})-\alpha+m(K_{n-1})+\varepsilon_n=\varepsilon_n,$$
$$\alpha-\varepsilon_n\leqq m(K_n)<\alpha<m(L_n)<\alpha+2\varepsilon_n.$$

Finally, putting $A=\bigcup_{n=1}^{\infty}K_n$ $\Big($or $A=\bigcap_{n=1}^{\infty}L_n\Big)$ we obtain $m(A)=\alpha$.

3. There remains to find a set A with $m(A)=\alpha$ by the assumption $I=\{1,2,\ldots,n,\ldots\}$ and $\alpha\geqq n(X)$. Similarly as in the first case we shall choose an index subset $J=\{k_p\}_{p=1}^{\infty}$ of I as follows

$$k_1=\min\{i\in I:\ \mathbf{m}(\mathcal{A}_i)<\alpha\},$$
$$k_p=\min\Big\{i\in I:\ i>k_{p-1},\ \mathbf{m}(\mathcal{A}_i)<\alpha-\sum_{j=1}^{p-1}\mathbf{m}(\mathcal{A}_{k_j})$$

for $p=2, 3, \ldots$. Since I is infinite, J is infinite too. We shall show that there exists an index $q \in J$ such that

$$\alpha - \sum_{i \in J,\, i \leqq q} \boldsymbol{m}(\mathscr{A}_i) < n(X).$$

For this purpose we first assume $I-J$ to be infinite. Then for any positive ε there is an index $p \in I-J$ such that $\boldsymbol{m}(\mathscr{A}_p) < \varepsilon$. If we put $\varepsilon = n(X)$, we obtain

$$n(X) > \boldsymbol{m}(\mathscr{A}_p) \geqq \alpha - \sum_{i \in J,\, i < p} \boldsymbol{m}(\mathscr{A}_i).$$

Let q be the last index in J less than p (q exists because $\alpha \geqq n(X)$). Then q satisfies our condition.

Further, we assume $I=J$. Then

$$\alpha - \sum_{i \in J} \boldsymbol{m}(\mathscr{A}_i) < m(X) - \sum_{i \in I} \boldsymbol{m}(\mathscr{A}_i) = n(X)$$

and there follows the existence of the required q.

Finally, let $J \neq I$ and $I-J$ be a finite set. If we denote by r the last index of $I-J$, we have

$$\alpha - \sum_{i \in J,\, i < r} \boldsymbol{m}(\mathscr{A}_i) \leqq \boldsymbol{m}(\mathscr{A}_r) < m(X) - \sum_{i \in I,\, i \leqq r} \boldsymbol{m}(\mathscr{A}_i) =$$
$$= \sum_{i \in I,\, i > r} \boldsymbol{m}(\mathscr{A}_i) + n(X) = \sum_{i \in J,\, i > r} \boldsymbol{m}(\mathscr{A}_i) + n(X),$$

whence

$$\alpha - \sum_{i \in J} \boldsymbol{m}(\mathscr{A}_i) < n(X).$$

The last inequality implies the existence of the required q also in this case.

Now put

$$\varepsilon = \min\left\{\alpha - \sum_{i \in J,\, i \leqq q} \boldsymbol{m}(\mathscr{A}_i),\ n(X) - \left(\alpha - \sum_{i \in J,\, i \leqq q} \boldsymbol{m}(\mathscr{A}_i)\right)\right\}$$

and by Lemma 1 for each $i \in J$, $i \leqq q$ choose $A_i \in \mathscr{A}_i$ such that $A_j \cap A_k = 0$ whenever $j \neq k$ and

$$m\left(\bigcup_{i \in J,\, i \leqq q} A_i\right) - \sum_{i \in J,\, i \leqq q} \boldsymbol{m}(\mathscr{A}_i) < \varepsilon.$$

Let us denote by $Y = X - \bigcup_{i \in J,\, i \leqq q} A_i$, $\beta = \alpha - m\left(\bigcup_{i \in J,\, i \leqq q} A_i\right)$, m_Y the restriction of m to $\mathscr{S}_Y = \{S \cap Y \colon S \in \mathscr{S}\}$, $\boldsymbol{m}_Y(\mathscr{B})$ the weight of a u-atom $\mathscr{B}$ with respect to m_Y, n_Y the u-nonatomic part of m_Y and $\{\mathscr{B}_k\}_{k \in K}$ the set of all u-atoms with respect to m_Y. By Lemma 2

$$\sum_{i \in J,\, i > q} \boldsymbol{m}(\mathscr{A}_i) + \sum_{i \in I-J} \boldsymbol{m}(\mathscr{A}_i) \geqq \sum_{k \in K} \boldsymbol{m}_Y(\mathscr{B}_k),$$

whence

$$\beta \leqq \alpha - \sum_{i \in J, i \leqq q} \mathbf{m}(\mathcal{A}_i) \leqq n(X) - \varepsilon = m(X) - \sum_{i \in I} \mathbf{m}(\mathcal{A}_i) - \varepsilon = m(Y) +$$

$$+ m\Big(\bigcup_{i \in J, i < q} A_i\Big) - \sum_{i \in J, i \leqq q} \mathbf{m}(\mathcal{A}_i) - \sum_{i \in J, i > q} \mathbf{m}(\mathcal{A}_i) - \sum_{i \in I - J} \mathbf{m}(\mathcal{A}_i) - \varepsilon < m(Y) + \varepsilon -$$

$$- \sum_{i \in J, i > q} \mathbf{m}(\mathcal{A}_i) - \sum_{i \in I - J} \mathbf{m}(\mathcal{A}_i) - \varepsilon \leqq m(Y) - \sum_{k \in K} \mathbf{m}_Y(\mathcal{B}_k) = n_Y(Y).$$

The measure m_Y and the number β satisfy now the conditions of the preceding case, therefore there is a set $B \in \mathcal{S}_Y$ such that $m_Y(B) = m(B) = \beta$. Putting $A = B \cup \bigcup_{i \in J, i \leqq q} A_i$ we have $m(A) = \alpha$.

Note. In general we cannot give a necessary and sufficient condition in order that a finitely additive measure defined on a δ-ring $\mathcal{T}$ may have the Darboux property on a set $X \in \mathcal{T}$. It is possible if the number of u-atoms in $\mathcal{T} = \{T \cap X \colon T \in \mathcal{T}\}$ is finite, but a formulation of this necessary and sufficient condition is too complicated and therefore it is not effective.

6. The Darboux property in the sense of Radakovič

Let us consider a measure m defined on a ring $\mathcal{R}$. For a σ-additive σ-finite measure m the propositions II and III remain valid if the Darboux property is weakened in the following sense:

Definition 12. *We say that a finitely additive measure m defined on a ring $\mathcal{R}$ has the Darboux property in the sense of Radakovič on a set $E \in \mathcal{R}$ iff for every $\alpha \in (0, m(E))$ and for every $\varepsilon > 0$ there is a measurable set $A \subset E$ such that*

$$|m(A) - \alpha| < \varepsilon.$$

Definition 13. *We say that a finitely additive measure m defined on a ring $\mathcal{R}$ has the Darboux property in the sense of Radakovič iff m has the Darboux property in the sense of Radakovič on the set E for every $E \in \mathcal{R}$.*

We shall show now that analogical propositions hold for a finitely additive measure if the notion of the atom will be replaced by the notion of the u-atom.

Theorem 8. *Let m be a finitely additive measure defined on a ring $\mathcal{R}$ and let $X \in \mathcal{R}$ with $m(X) < \infty$. Let $\{\mathcal{A}_i\}_{i \in I}$ be the set of all u-atoms of the algebra $\mathcal{C} = \{E \cap X \colon E \in \mathcal{R}\}$, indexed such that $\mathbf{m}(\mathcal{A}_i) \geqq \mathbf{m}(\mathcal{A}_{i+1})$ for all i, $i + 1 \in I$. The measure m has the Darboux property in the sense of Radakovič on the set X if and only if for all $n \in I$*

$$\mathbf{m}(\mathcal{A}_n) \leqq m(X) - \sum_{i \in I, i \leqq n} \mathbf{m}(\mathcal{A}_i).$$

Proof. Let us consider the Stone representation space $\mathbf{Q}$ of the algebra $\mathscr{C}$ and the σ-additive extension μ of m to $\mathscr{S}(h(\mathscr{C}))$. According to Theorem 3 there is a class $\{\mathbf{A}_i\}_{i\in I}$ of all atoms in $\mathscr{S}(h(\mathscr{C}))$ such that $\mathscr{A}_i \in \mathbf{A}_i$, $\mu(\mathbf{A}_i)=\boldsymbol{m}(\mathscr{A}_i)$ for all $i\in I$ and (since μ is σ-additive and $\mathscr{S}(h(\mathscr{C}))$ is a σ-algebra) $\mathbf{A}_j \cap \mathbf{A}_k = \emptyset$ whenever $j\neq k$. It follows that μ has the Darboux property on $\mathbf{Q}=h(X)$, whence we obtain the proof of sufficiency using [1, 13.D]. The necessity of the condition is obvious.

Corollary. *If m is finitely additive finite measure defined on an algebra $\mathscr{C}$ and μ is the σ-additive extension of m to the Stone representation space of the algebra $\mathscr{C}$, then the following assertions are equivalent:*

(i) *m has the Darboux property in the sense of Radakovič,*

(ii) *m is u-nonatomic,*

(iii) *μ has the Darboux property.*

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МАКСИМАЛЬНЫЕ ФИЛЬТРЫ И СВОЙСТВО ДАРБУ КОНЕЧНО АДДИТИВНОЙ МЕРЫ

Владимир Олейчек

Резюме

Обобщается понятие атома меры таким образом, чтобы некоторые утверждения, касающиеся взаимосвязи свойства Дарбу и свойств атомов, было можно переформулировать для конечно аддитивной меры аналогично тому, как это сделано для σ-аддитивной меры. Для обобщенного понятия атома, названного u-атомом, справедливо, что конечно аддитивная мера обладает свойством Дарбу тогда и только тогда, когда она u-неатомическая. При помощи этого доказываются утверждения, в которых применяется понятие u-атома для приведения какогото достаточного и, в частности, необходимого и достаточного условия свойства Дарбу на множестве.