Mathematica Slovaca

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Mathematica Slovaca, Vol. 53 (2003), No. 3, 215--232

Persistent URL: http://dml.cz/dmlcz/130474

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AUTOMORPHISM GROUPS OF CERTAIN UNSTABLE GRAPHS

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(Communicated by Martin Škoviera)

ABSTRACT. The automorphism groups of the canonical double covers of the unstable graphs in two infinite families are considered. These graphs are important from the point of view of algebraic map theory as they can be embedded into regular oriented maps.

0. Introduction

For a simple graph $\Gamma = (V, E)$, the *canonical double cover* $\widetilde{\Gamma}$ of Γ (as introduced by D. Marušič in [4]) is the graph $\widetilde{\Gamma} = \Gamma(V \times \{\pm 1\}, \widetilde{E})$, where $\{(v, \zeta), (v', \zeta')\} \in \widetilde{E}$ if and only if $\{v, v'\} \in E$ and $\zeta' = -\zeta$, $\zeta = \pm 1$. Note that when Γ is connected, $\widetilde{\Gamma}$ is disconnected if and only if Γ is bipartite. In any case, $\widetilde{\Gamma}$ is bipartite and so it is of little interest to iterate the $\Gamma \rightsquigarrow \widetilde{\Gamma}$ construction.

The automorphism group $\mathrm{Aut}(\Gamma)$ acts as a group of automorphisms of $\widetilde{\Gamma}$ in the obvious way, stabilizing the partite classes $V \times \{1\}$ and $V \times \{-1\}$. Also there is the involutory "deck transformation" $(v, 1) \leftrightarrow (v, -1)$, which together with $\mathrm{Aut}(\Gamma)$ gives an embedding of $Z_2 \times \mathrm{Aut}(\Gamma)$ into $\mathrm{Aut}(\widetilde{\Gamma})$. When this subgroup is the full automorphism of $\widetilde{\Gamma}$, we call the graph *stable*. The property of stability is interesting only in the case when Γ is *vertex-determining*, i.e., when every vertex is uniquely determined by its set of neighbours (see [4; (4.1)]). As $Z_2 \times \mathrm{Aut}(\Gamma)$ embeds into $\mathrm{Aut}(\widetilde{\Gamma})$ with equality if and only if Γ is stable, it is reasonable to call the group index $[\mathrm{Aut}(\widetilde{\Gamma}) : Z_2 \times \mathrm{Aut}(\Gamma)]$ the *index of instability* of the graph Γ. Thus, index of instability 1 is tantamount to stability.

In [5], Nedela and Škoviera showed that if all embeddings of a stable graph into an orientable regular map are known, then all embeddings of the

2000 Mathematics Subject Classification: Primary 05C25.
Keywords: graph, stable graph, unstable graph, Paley graph, regular oriented graph, orientable regular map, canonical double cover, automorphism group.

canonical double cover of this graph into orientable regular maps are also known. At the time of the writing of their paper, the only known unstable (vertex-determining) arc-transitive graph was the underlying graph of the dodecahedron. In [7], however, I gave three infinite families of arc-transitive, vertex-determining unstable graphs. By the results of [2], not all of the graphs considered in [7] can admit embeddings into regular maps; a necessary condition is that a vertex stabilizer has a cyclic subgroup acting regularly on the vertex neighbours. It is for the canonical double covers of these graphs that the full automorphism groups are computed; the remaining cases shall be considered elsewhere.

Thus, in Section 1, I compute the automorphism group of $\widetilde{\Gamma}$ where Γ is the "Shult-Taylor" double of the Paley graph $QR(q)$, where q is a prime power congruent to 1 modulo 4. The result is that the automorphism group has the structure of $\mathrm{P\Gamma L}_2(q)$ acting on a Klein four-group, with resulting index of instability 2. In Section 2, I compute $\mathrm{Aut}(\widetilde{\Gamma})$, where Γ is the graph whose vertices comprise a conjugacy class of elements of order p in $\mathrm{PSL}_2(p)$, where p is prime and $p \equiv 5, 7 \pmod{12}$, and adjacency is where the product of the elements has order 3. In this case, $\mathrm{Aut}(\widetilde{\Gamma})$ has the structure of $\mathrm{PGL}_2(p)$ acting on a dihedral group of order $p-1$, giving index of instability $(p-1)/2$. Thus, in this case we see that the graphs in question become increasingly unstable as $p \to \infty$.

1. Graphs related to the Paley graphs

Let $\mathbb{F} = \mathbb{F}_q$ be the finite field of q elements, where $q \equiv 1 \pmod 4$. The *Paley graph*, $QR(q)$ is the graph whose vertices are the elements of $\mathbb{F}$, and whose edges are of the form $\{a, b\}$, where $a - b$ is a non-zero square in $\mathbb{F}$.

The following is, of course, well known:

LEMMA 1.1. *$QR(q)$ is a strongly regular graph with parameters $\lambda = (q-5)/4$ and $\mu = (q-1)/4$.*

Proof. Let S and NS denote the non-zero squares and non-squares in $\mathbb{F}$, respectively, and note that the stabilizer of 0 in $\mathrm{Aut}(QR(q))$ acts regularly on the sets S and NS via $x \mapsto \alpha x$, where $\alpha \in S$, $x \in \mathbb{F}$. Thus it follows that $QR(q)$ is strongly regular. Next, note that the mapping $x \mapsto 1/x$ interchanges the edges and non-edges between vertices of S and vertices of NS. Since there are $(q-1)/2$ vertices in each of S, NS, we infer that there are altogether $(q-1)^2/8$ edges joining vertices of S to vertices of NS. Using the transitivity mentioned above, this already implies that $\mu = (q-1)/4$. By the same token, for each vertex $x \in S$, we infer that x is adjacent to exactly $(q-1)/4$ vertices of NS, from which it follows immediately that $\lambda = (q-1)/2 - (q-1)/4 - 1 = (q-5)/4$. □

From [7; Proposition 2] it follows immediately that:

COROLLARY 1.2. *The Paley graphs $QR(q)$ are stable.*

Next, we review the so-called Shult-Taylor construction (see, e.g. [6]). If Γ is a graph, form the new graph Γ^* as follows. First let Γ' be an isomorphic copy of the graph Γ with isomorphism $\Gamma \xrightarrow{(\cdot)'} \Gamma'$. If V, V' are the vertices of Γ, Γ' respectively, then Γ^* has vertices $\{\infty, \infty'\} \cup V \cup V'$. The edge relations in Γ^* are as follows:

(i) ∞ is adjacent to every element of V;
(ii) ∞' is adjacent to every element of V';
(iii) the subgraphs induced by the vertices V, V' give Γ, Γ' as subgraphs of Γ^*;
(iv) if $v \in V$, $w' \in V'$, then $\{v, w'\}$ is an edge of Γ^* if and only if $\{v, w\}$ is *not* an edge of Γ.

As proved in [7; Sec. 3], if $\Gamma = QR(q)$, $q \equiv 1 \pmod 4$, then $\widetilde{\Gamma^*}$ is unstable. Let $A = \mathrm{Aut}(\Gamma)$, $A^* = \mathrm{Aut}(\Gamma^*)$, and set $\widetilde{A^*} = \mathrm{Aut}(\widetilde{\Gamma^*})$. If Δ is the involution on $\widetilde{\Gamma^*}$ given by $(v^*, \delta)\Delta = (v^*, -\delta)$, $v^* \in V^*$, $\delta = \pm 1$, then we have that $\langle \Delta \rangle \times A^*$ is embedded as a subgroup of $\widetilde{A}$. For convenience we shall write the vertices of $\widetilde{\Gamma^*}$ as $\pm v^*$, $v^* \in V^*$; thus the "deck transformation" is given by $\Delta\colon \pm v^* \leftrightarrow \mp v^*$, $v^* \in V^*$.

For each positive integer i, denote by $\widetilde{\Gamma_i^*}$ the vertices in $\widetilde{\Gamma^*}$ at distance i from the vertex $+\infty$. One easily checks that

$$\widetilde{\Gamma_1^*} = -V, \quad \widetilde{\Gamma_2^*} = +V \cup +V', \quad \widetilde{\Gamma_3^*} = -V' \cup \{-\infty, -\infty'\}, \quad \widetilde{\Gamma_4^*} = \{+\infty'\}.$$

Furthermore, since $-\infty$ is joined to every vertex in $+V$, and since $-\infty'$ is joined to every vertex of $+V'$, we conclude that if $\widetilde{A^*_{+\infty}}$ is the stabilizer in $\widetilde{A^*}$ of the vertex $+\infty$, then $\widetilde{A^*_{+\infty}}$ must act imprimitively on the set $\widetilde{\Gamma_2^*} = +V \cup +V'$, with $+V$ and $+V'$ being sets of imprimitivity.

Next, note that the subgraph of $\widetilde{\Gamma^*}$ induced by the vertices $-V \cup +V$ is precisely the canonical double cover $\widetilde{\Gamma}$ of Γ. As noted above, Γ is stable where $\Gamma = QR(q)$, and so the stabilizer $H \leq \widetilde{A^*_{+\infty}}$ of the set $+V$ maps into the group $Z_2 \times \mathrm{Aut}(\Gamma)$. However, since $\widetilde{A^*_{+\infty}}$ cannot interchange the sets $+V$ and $-V$, we conclude that, in fact, H must map into the group $\mathrm{Aut}(\Gamma)$. If $h \in H$ is in the kernel of this map, then h fixes pointwise the sets $\pm V$. By assumption h acts on the subgraph of $\widetilde{\Gamma^*}$ induced by the vertex set $-V \cup +V'$. However, this subgraph is the canonical double cover of the complementary graph $\overline{\Gamma}$ of Γ. Since Γ is self-complementary, we infer that $\overline{\Gamma}$ is stable and so it follows that h must induce the identity on $-V \cup +V'$, as well. Finally, the subgraph induced by $+V' \cup -V'$ is the canonical double cover of $\Gamma' \cong \Gamma$ and so h induces the identity on $-V'$, as well and hence h induces the identity on all of $\widetilde{\Gamma^*}$. Therefore H embeds into $\mathrm{Aut}(\Gamma)$, and so

$$\left|\widetilde{A^*_{+\infty}}\right| \leq 2 \times |H| \leq 2 \times |\mathrm{Aut}(\Gamma)|,$$

from which we infer that

$$\left|\widetilde{A^*}\right| \leq 2 \times |\mathrm{Aut}(\Gamma)| \times 2 \times |V^*|.$$

Since we may identify $\mathrm{Aut}(\Gamma)$ with the stabilizer in $A^* = \mathrm{Aut}(\Gamma^*)$ of ∞, we conclude that $|\mathrm{Aut}(\Gamma)| \times |V^*| = |A^*| = |\mathrm{Aut}(\Gamma^*)|$; therefore,

$$\left|\widetilde{A^*}\right| \leq 2 \times 2 \times |A^*|.$$

Since the reverse inequality is the point of the graph $\widetilde{\Gamma^*}$ not being stable, we conclude that equality obtains:

$$\left|\widetilde{A^*}\right| = 2 \times 2 \times |A^*|.$$

Therefore, the canonical double cover $\widetilde{\Gamma^*}$ of the Shult-Taylor double of the Paley graph $QR(q)$, $q \equiv 1 \pmod 4$, has index of instability 2, as claimed.

Having established the automorphism group order, it remains to give the group explicitly. First of all, note that the Paley graph Γ admits automorphisms of the form $x \mapsto \alpha x + \beta$, where $x, \beta \in \mathbb{F} = \mathbb{F}_q$, and where $\alpha \in S = \mathbb{F}^{\times 2}$. When $\mathbb{F}$ is not a prime field, there are more automorphisms: if $q = p^n$, where p is prime, and if $G = \mathrm{Gal}(\mathbb{F}/\mathbb{F}_p)$, then mappings of the form $x \mapsto \alpha x^\sigma + \beta$, $x, \beta \in \mathbb{F}$, $\alpha \in S$, $\sigma \in G$, all determine automorphisms of Γ.

The following is an immediate consequence of the work in [1]:

PROPOSITION 1.3. $\mathrm{Aut}(\Gamma)$ *consists precisely of the above automorphisms.*

Next, we proceed to compute the automorphism group of the Shult-Taylor graph Γ^* associated with the Paley graph Γ. First of all, if A^*_∞ is the stabilizer of ∞ in $A^* = \mathrm{Aut}(\Gamma^*)$, then $A^*_\infty \cong A$, where $A = \mathrm{Aut}(\Gamma)$. Therefore, the task remains to find a group of order $2(q-1)|A|$ acting on Γ^*. Such a group is at hand: we shall show that, in fact, $A^* \cong Z_2 \times \mathrm{P\Sigma L}_2(q)$, where $\mathrm{P\Sigma L}_2(q) = G \ltimes \mathrm{PSL}_2(q)$ ($G = \mathrm{Gal}(\mathbb{F}/\mathbb{F}_p)$, as above) by giving an action of $Z_2 \times \mathrm{P\Sigma L}_2(q)$ on Γ^*. We note here that $\mathrm{P\Sigma L}_2(q)$ is a normal subgroup of the larger group $\mathrm{P\Gamma L}_2(q) = G \ltimes \mathrm{PGL}_2(q)$. This larger group will appear in Theorem 1.5 below.

In order to obtain an action of $\mathrm{P\Sigma L}_2(q)$ on Γ^*, it is convenient to alter some of the notation. To this end, we first identify the set $\mathbb{F} \cup \{\infty\}$ with the "projective line" $\mathbb{FP}^1$ over $\mathbb{F}$:

$$\mathbb{FP}^1 = \left\{[a, b] : a, b \in \mathbb{F},\ \text{not both zero}\right\},$$

where, as usual, $[a,b] = [\alpha a, \alpha b]$ for all $0 \neq \alpha \in \mathbb{F}$. In this way, we identify elements of $\mathbb{F}$ with corresponding elements of $\mathbb{FP}^1$ via $a \leftrightarrow [a,1]$ and identify ∞ with the point $[1,0]$. Finally, we identify the vertex set of Γ^* with the set $\mathbb{FP}^1 \times \{\pm 1\}$.

Next, let $\eta\colon \mathbb{F}^\times \to \{\pm 1\}$ be the homomorphism having kernel $\mathbb{F}^{\times 2}$:

$$\eta(x) = \begin{cases} 1 & \text{if } x \in \mathbb{F}^{\times 2}, \\ -1 & \text{if } x \notin \mathbb{F}^{\times 2}. \end{cases}$$

Note that since $q \equiv 1 \pmod 4$, $\eta(x) = \eta(-x) = \eta(x^{-1})$ for all $x \in \mathbb{F}^\times$. Next, we *define* $\eta(0) = 1$ and extend η to a mapping $\eta\colon \mathbb{FP}^1 \to \{\pm 1\}$ via

$$\eta[a,b] = \begin{cases} \eta(a/b) & \text{if } b \neq 0, \\ \eta(b/a) & \text{if } a \neq 0. \end{cases}$$

Thus, it is clear that for all $[a,b] \in \mathbb{FP}^1$, $\eta[a,b] = \eta[b,a]$; furthermore, if $a,b,c,d \neq 0$, then $\eta[a,b]\eta[c,d] = \eta[ab,cd]$, which implies in particular that $\eta[a,c]\eta[c,b] = \eta[a,b]$ $(a,b,c \neq 0)$.

In terms of η, we now can describe the edges of Γ^* as those pairs

$$\{([a,b],\varepsilon), ([a',b'],\varepsilon')\}$$

where

$$\eta[ab'-a'b, bb'] = \varepsilon\varepsilon'.$$

It is routine to check that this gives the Shult-Taylor construction Γ^* of the Paley graph Γ.

The action of $\mathrm{P\Sigma L}_2(q)$ on Γ^* is as follows (in the sequel, all matrices are to be regarded modulo scalar matrices):

$$\begin{aligned} &([a,b],\varepsilon)\left(\sigma, \begin{bmatrix} x & y \\ z & w \end{bmatrix}\right) \\ &\quad = \begin{cases} ([a^\sigma x + b^\sigma z, a^\sigma y + b^\sigma w],\ \varepsilon\eta[a^\sigma y + b^\sigma w, b^\sigma y]\eta[y,1]) & \text{if } y \neq 0 \\ ([a^\sigma x + b^\sigma z, b^\sigma w],\ \varepsilon\eta[w,1]) & \text{if } y = 0. \end{cases} \end{aligned} \tag{$*$}$$

Note that in the first coordinate we have the usual action of $\mathrm{P\Sigma L}_2(q)$ on the projective line $\mathbb{FP}^1$ by semilinear transformations.

One must show that $(*)$

(1) defines an automorphism of the graph Γ^* for each $(\sigma, T) \in \mathrm{P\Sigma L}_2(q)$, and

(2) defines an action of the group $\mathrm{P\Sigma L}_2(q)$ on the vertices $\mathbb{FP}^1 \times \{\pm 1\}$ of Γ^*.

We first show (1), viz., that the above defines a graph automorphism. Thus, assume that $\{([a,b],\varepsilon), ([a',b'],\varepsilon')\}$ is an edge of Γ^*, and that

$$(\sigma, T) \in \mathrm{P\Sigma L}_2(q), \quad T = \begin{bmatrix} x & y \\ z & w \end{bmatrix}.$$

Therefore,

$$\eta[ab'-a'b, bb'] = \varepsilon\varepsilon' ;$$

we shall consider first the case in which $y = 0$. To show that $([a,b],\varepsilon)(\sigma,T) \sim ([a',b'],\varepsilon)(\sigma,T)$, one must show that

$$\eta\big[(a^\sigma x + b^\sigma z)b'^\sigma w - (a'^\sigma x + b'^\sigma z)b^\sigma w,\ b^\sigma b'^\sigma w^2\big] = \varepsilon\varepsilon' .$$

Since $xw = 1$, this reduces to showing that

$$\eta[a^\sigma b'^\sigma - a'^\sigma b^\sigma,\ b^\sigma b'^\sigma w^2] = \varepsilon\varepsilon' .$$

However, it is clear that $\eta[a^\sigma b'^\sigma - a'^\sigma b^\sigma,\ b^\sigma b'^\sigma w^2] = \eta[ab' - a'b,\ bb']$, and so we are finished in this case.

Next assume that $y \neq 0$; we need to conclude that

$$\begin{aligned}\eta\big[(a^\sigma x + b^\sigma z)(a'^\sigma y + b'^\sigma w) - (a'^\sigma x + b'^\sigma z)(a^\sigma y + b^\sigma w),\\ (a^\sigma y + b^\sigma w)(a'^\sigma y + b'^\sigma w)\big]\\ = \varepsilon\varepsilon'\eta[a^\sigma y + b^\sigma w,\ b^\sigma y]\eta[a'^\sigma y + b'^\sigma w,\ b'^\sigma y] .\end{aligned}$$

Since $xw - ya = 1$, the above condition reduces to checking that

$$\begin{aligned}\eta\big[a^\sigma b'^\sigma - a'^\sigma b^\sigma,\ (a^\sigma y + b^\sigma w)(a'^\sigma y + b'^\sigma w)\big]\\ = \varepsilon\varepsilon'\eta[a^\sigma y + b^\sigma w,\ b^\sigma y]\eta[a'^\sigma y + b'^\sigma w,\ b'^\sigma y] .\end{aligned}$$

If $bb', (a^\sigma y + b^\sigma w)(a'^\sigma y + b'^\sigma w) \neq 0$, then by what was noted above,

$$\begin{aligned}\eta\big[a^\sigma b'^\sigma - a'^\sigma b^\sigma,\ (a^\sigma y + b^\sigma w)(a'^\sigma y + b'^\sigma w)\big]\\ = \eta[a^\sigma b'^\sigma - a'^\sigma b^\sigma,\ b^\sigma b'^\sigma]\eta\big[b^\sigma b'^\sigma,\ (a^\sigma y + b^\sigma w)(a'^\sigma y + b'^\sigma w)\big] .\end{aligned}$$

Since $\eta[a^\sigma b'^\sigma - a'^\sigma b^\sigma,\ b^\sigma b'^\sigma] = \eta[ab'-a'b,\ bb']^\sigma = \eta[ab'-a'b,\ bb'] = \varepsilon\varepsilon'$, we are reduced (modulo the assumption $bb' \neq 0$) to checking that

$$\eta\big[b^\sigma b'^\sigma,\ (a^\sigma y + b^\sigma w)(a'^\sigma y + b'^\sigma w)\big] = \eta\big[a^\sigma y + b^\sigma w,\ b^\sigma y\big]\eta\big[a'^\sigma y + b'^\sigma w,\ b'^\sigma y\big] .$$

But then

$$\begin{aligned}\eta\big[b^\sigma b'^\sigma, (a^\sigma y + b^\sigma w)(a'^\sigma y + b'^\sigma w)\big] &= \eta\big[b^\sigma,\ (a^\sigma y + b^\sigma w)\big]\eta\big[b'^\sigma,\ (a'^\sigma y + b'^\sigma w)\big]\\ &= \eta\big[b^\sigma y, (a^\sigma y + b^\sigma w)\big]\eta\big[b'^\sigma y, (a'^\sigma y + b'^\sigma w)\big]\\ &= \eta\big[a^\sigma y + b^\sigma w,\ b^\sigma y\big]\eta\big[a'^\sigma y + b'^\sigma w,\ b'^\sigma y\big] ,\end{aligned}$$

as required. The excluded cases are also easily handled. Therefore the permutation given in $(*)$ defines an automorphism of Γ^*.

We turn next to the verification that $(*)$ defines an action of $\mathrm{P\Sigma L}_2(q)$ on the vertices of Γ^*.

Thus, let

$$g = \left(\sigma, \begin{bmatrix} x & y \\ z & w \end{bmatrix}\right) = (\sigma, T),\ g' = \left(\nu, \begin{bmatrix} x' & y' \\ z' & w' \end{bmatrix}\right) = (\nu, T') \in \mathrm{P\Sigma L}_2(q)\,.$$

Therefore,

$$gg' = (\sigma\nu, T^\nu T') = \left(\sigma\nu, \begin{bmatrix} x^\nu x' + y^\nu z' & x^\nu y' + y^\nu w' \\ z^\nu x' + w^\nu z' & z^\nu y' + w^\nu w' \end{bmatrix}\right).$$

In order to show that for every vertex $v^* \in V^*$, $v^*(TS) = (v^*T)S$, there are five cases to consider:

1. $y, y' = 0$ (and so $x^\nu y' + y^\nu w' = 0$);
2. $y = 0$, $y' \neq 0$ (and so $x^\nu y' + y^\nu w' \neq 0$);
3. $y \neq 0$, $y' = 0$ (and so $x^\nu y' + y^\nu w' \neq 0$);
4. $y, y' \neq 0$, $x^\nu y' + y^\nu w' = 0$;
5. $y, y', x^\nu y' + y^\nu w' \neq 0$.

We shall be content to investigate case 1 (the easiest) and case 5 (the generic). Thus, assume that $y, y' = 0$; we have

$$\begin{aligned}
\big(([a,b],\varepsilon)g\big)g' &= \big([a^\sigma, b^\sigma]T, \varepsilon\eta[w,1]\big)(\nu, T') \\
&= \big([a^{\sigma\nu}, b^{\sigma\nu}]T^\nu T', \varepsilon\eta[w,1]\eta[w',1]\big) \\
&= \big([a^{\sigma\nu}, b^{\sigma\nu}]T^\nu T', \varepsilon\eta[w^\nu,1]\eta[w',1]\big) \\
&= \big([a^{\sigma\nu}, b^{\sigma\nu}]T^\nu T', \varepsilon\eta[w^\nu w',1]\big) \\
&= \big([a,b],\varepsilon\big)(\sigma\nu, T^\nu T') \\
&= \big([a,b],\varepsilon\big)(gg')\,.
\end{aligned}$$

We turn now to case 5. In this case, we have

$$\begin{aligned}
&\big(([a,b],\varepsilon)g\big)g' \\
&\quad= \big([a^\sigma, b^\sigma]T, \varepsilon\eta[a^\sigma y + b^\sigma w, b^\sigma y]\eta[y,1]\big)g' \\
&\quad= \big([a^{\sigma\nu}, b^{\sigma\nu}]T^\nu T', \varepsilon\eta[a^\sigma y + b^\sigma w, b^\sigma y]\eta[y,1] \\
&\qquad\quad \times \eta\big[(a^{\sigma\nu}x^\nu + b^{\sigma\nu}z^\nu)y' + (a^{\sigma\nu}y^\nu + b^{\sigma\nu}w^\nu)w', (a^{\sigma\nu}y^\nu + b^{\sigma\nu}w^\nu)y'\big]\eta[y',1]\big)\,.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\big([a,b],\varepsilon\big)(\sigma,T)(\nu,T') \\
&\quad= \big([a,b],\varepsilon\big)(\sigma\nu, T^\nu T') \\
&\quad= \big([a^{\sigma\nu}, b^{\sigma\nu}]T^\nu T', \varepsilon\eta\big[a^{\sigma\nu}(x^\nu y' + y^\nu w') + b^{\sigma\nu}(z^\nu y' + w^\nu w'), \\
&\qquad\qquad b^{\sigma\nu}(x^\nu y' + y^\nu w')\big]\eta[x^\nu y' + y^\nu w', 1]\big)\,.
\end{aligned}$$

There are two subcases here: $b=0$ and $b\neq 0$. If $b=0$, then $b^\sigma=b^{\sigma\nu}=0$ and $a^{\sigma\nu}\neq 0$. In this case the above computation affords

$$\begin{aligned}
(([a,b],\varepsilon)g)g' &= (([1,0],\varepsilon)g)g' \\
&= ([1,0]T^\nu T', \varepsilon\eta[y,1]\eta[x^\nu y'+y^\nu w',\, y^\nu y']\eta[y',1]) \\
&= ([1,0]T^\nu T', \varepsilon\eta[y^\nu,1]\eta[x^\nu y'+y^\nu w',\, y^\nu y']\eta[y',1]) \\
&= ([1,0]T^\nu T', \varepsilon\eta[x^\nu y'+y^\nu w',1]) \\
&= ([1,0],\varepsilon)gg' \\
&= ([a,b],\varepsilon)gg'.
\end{aligned}$$

If $b\neq 0$, there are two further subcases within this case to consider, viz., $a^\sigma y+b^\sigma w=0$ and $a^\sigma y+b^\sigma w\neq 0$. In the first case we also must have $a^{\sigma\nu}y^\nu+b^{\sigma\nu}w^\nu=0$, and so it follows easily that

$$(([a,b],\varepsilon)g)g' = ([a^{\sigma\nu},b^{\sigma\nu}]T^\nu T', \varepsilon\eta[y,1]\eta[y',1]).$$

On the other hand, one computes

$$\begin{aligned}
&([a,b],\varepsilon)(gg') \\
&= ([a^{\sigma\nu},b^{\sigma\nu}],\ \varepsilon\eta\big[a^{\sigma\nu}x^\nu y'+b^{\sigma\nu}z^\nu y',\ b^{\sigma\nu}(x^\nu y'+y^\nu w')\big]\eta[x^\nu y'+y^\nu w',1]) \\
&= ([a^{\sigma\nu},b^{\sigma\nu}]T^\nu T',\ \varepsilon\eta[a^{\sigma\nu}x^\nu y'+b^{\sigma\nu}z^\nu y',b^{\sigma\nu}]) \\
&= ([a^{\sigma\nu},b^{\sigma\nu}]T^\nu T',\ \varepsilon\eta[a^{\sigma\nu}x^\nu+b^{\sigma\nu}z^\nu,b^{\sigma\nu}]\eta[y',1]) \\
&= ([a^{\sigma\nu},b^{\sigma\nu}]T^\nu T',\ \varepsilon\eta[a^\sigma x+b^\sigma z,b^\sigma]\eta[y',1]).
\end{aligned}$$

However, from $a^\sigma y+b^\sigma w=0$ and $b^\sigma\neq 0$ we obtain $a^\sigma/b^\sigma=-w/y$ and so

$$[a^\sigma x+b^\sigma z,b^\sigma]=[-xw+zy,y]=[-1,y]\in\mathbb{FP}^1.$$

Therefore,

$$\begin{aligned}
([a,b],\varepsilon)(gg') &= ([a^{\sigma\nu},b^{\sigma\nu}]T^\nu T',\ \varepsilon\eta[a^\sigma x+b^\sigma z,b^\sigma]\eta[y',1]) \\
&= ([a^{\sigma\nu},b^{\sigma\nu}]T^\nu T',\ \varepsilon\eta[-1,y]\eta[y',1]) \\
&= ([a^{\sigma\nu},b^{\sigma\nu}]T^\nu T',\ \varepsilon\eta[y,1]\eta[y',1]) \qquad (\text{since } -1\in\mathbb{F}^{\times 2}) \\
&= (([a,b],\varepsilon)g)g',
\end{aligned}$$

proving the result in this case. Finally, we consider the case $a^\sigma y+b^\sigma w\neq 0$. Noting that $\eta[c^\tau,d^\mu]=\eta[c,d]$ for any $a,b\in\mathbb{F}$, $\tau,\mu\in G$, we easily obtain that

$$\begin{aligned}
&(([a,b],\varepsilon)g)g' \\
&= ([a^{\sigma\nu},b^{\sigma\nu}]T^\nu T',\ \varepsilon\eta\big[(a^{\sigma\nu}x^\nu+b^{\sigma\nu}z^\nu)y'+(a^{\sigma\nu}y^\nu+b^{\sigma\nu}w^\nu)w',\ b^\sigma\big]) \\
&= ([a^{\sigma\nu},b^{\sigma\nu}]T^\nu T',\ \varepsilon\eta\big[a^{\sigma\nu}(x^\nu y'+y^\nu w')+b^{\sigma\nu}(z^\nu y'+w^\nu w'),\ b^{\sigma\nu}\big]) \\
&= ([a^{\sigma\nu},b^{\sigma\nu}]T^\nu T',\ \varepsilon\eta\big[a^{\sigma\nu}(x^\nu y'+y^\nu w')+b^{\sigma\nu}(z^\nu y'+w^\nu w'),\ b^{\sigma\nu}(x^\nu y'+y^\nu w')\big] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times\eta[x^\nu y'+y^\nu w',1]) \\
&= ([a,b],\varepsilon)(\sigma,T)(\nu,T') = ([a,b],\varepsilon)(gg'),
\end{aligned}$$

proving that $(*)$ does indeed define an action of $\mathrm{P\Sigma L}_2(q)$ on V^*.

Finally, note that we have the involutory automorphism Δ^* of Γ^* interchanging antipodal vertices: $([a,b],\varepsilon)\Delta^* = ([a,b],-\varepsilon)$. Note that Δ^* is clearly centralized by $\mathrm{P\Sigma L}_2(q)$, and so we have an action of $Z_2 \times \mathrm{P\Sigma L}_2(q)$ on Γ^*, proving that

$$A^* = \mathrm{Aut}(\Gamma^*) \cong Z_2 \times \mathrm{P\Sigma L}_2(q).$$

Having computed $A^* = \mathrm{Aut}(\Gamma^*)$, where Γ is the Paley graph, it remains to compute the structure of $\widetilde{A^*} = \mathrm{Aut}(\widetilde{\Gamma^*})$. The above work already shows that $|\widetilde{A^*}| = 8 \times |\mathrm{P\Sigma L}_2(q)| = 4nq(q^2-1)$, where $q = p^n$, and where p is prime. Thus, the canonical double cover $\widetilde{\Gamma^*}$ has vertices $\mathbb{FP}^1 \times \{\pm 1\} \times \{\pm 1\}$ and incidence given by $([a,b],\varepsilon,\delta) \sim ([a',b'],\varepsilon',\delta')$ if and only if $([a,b],\varepsilon) \sim ([a',b'],\varepsilon')$ in Γ^* and $\delta\delta' = -1$.

CLAIM. *Let $t \in \mathbb{F}^\times \setminus \mathbb{F}^{\times 2}$ be a non-square. Then the permutation $c_t\colon \widetilde{V^*} \to \widetilde{V^*}$ given by*

$$([1,0],\varepsilon,\delta)c_t = ([1,0],-\varepsilon\delta,-\delta);$$
$$([a,1],\varepsilon,\delta)c_t = ([at,1],\varepsilon\delta,-\delta)$$

determines an automorphism of $\widetilde{\Gamma^*}$. In terms of the mapping $\eta\colon \mathbb{FP}^1 \to \{\pm 1\}$ this can be written as

$$([a,b],\varepsilon,\delta)c_t = ([at,b],-\varepsilon\delta\eta[t,b]\eta[1,b],-\delta).$$

We proceed to verify that c_t determines an automorphism of $\widetilde{\Gamma^*}$. Thus, let $([a,b],\varepsilon,\delta) \sim ([a',b'],\varepsilon',\delta')$. Therefore, $ab' - a'b \neq 0$ and

$$\eta[ab' - a'b, bb'] = \varepsilon\varepsilon', \quad \text{and} \quad \delta\delta' = -1.$$

We must show that

$$\eta[ab't - a'bt, bb'] = \varepsilon\varepsilon'\delta\delta'\eta[t,b]\eta[1,b]\eta[t,b']\eta[1,b'].$$

If $bb' \neq 0$, we have

$$\begin{aligned}
\eta[ab't - a'bt, bb'] &= \eta[ab' - a'b, bb']\eta[t,1] \\
&= -\eta[ab' - a'b, bb'] \\
&= -\varepsilon\varepsilon' \\
&= -\varepsilon\varepsilon'\eta[t,b]\eta[1,b]\eta[t,b']\eta[1,b'] \\
&= \varepsilon\varepsilon'\delta\delta'\eta[t,b]\eta[1,b]\eta[t,b']\eta[1,b'],
\end{aligned}$$

as required. The remaining cases are easier (note that we cannot have both $b, b' = 0$).

If $(\sigma, T) \in \mathrm{P\Sigma L}_2(q)$, let $g_{\sigma,T} \in \mathrm{Aut}(\widetilde{\Gamma^*})$ be the action of $\widetilde{\Gamma^*}$ induced by the action of $\mathrm{P\Sigma L}_2(q)$ on Γ^* as given in $(*)$. Therefore, if $T = \left[\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right]$, then

$$([a,b], \varepsilon, \delta)\, g_{\sigma,T} = \begin{cases} ([a^\sigma x + b^\sigma z, a^\sigma y + b^\sigma w], \varepsilon\eta[a^\sigma y + b^\sigma w, b^\sigma y]\eta[y,1], \delta) & \text{if } y \neq 0, \\ ([a^\sigma x + b^\sigma z, b^\sigma w], \varepsilon\eta[w,1], \delta) & \text{if } y = 0. \end{cases}$$

Therefore, from the above discussion, we conclude that if t is a fixed non-square in $\mathbb{F}$, then

$$\mathrm{Aut}(\widetilde{G^*}) = \langle \Delta, \Delta^*, c_t, g_{\sigma,T} \mid (\sigma, T) \in \mathrm{P\Sigma L}_2(q) \rangle .$$

The following are easy calculations:

LEMMA 1.4. *Let t is a fixed non-square in $\mathbb{F}$, and set $g_T = g_{1,T} \in \mathrm{PSL}_2(q)$.*

(i) $c_t g_T c_t^{-1} = g_{T'}$, *where*

$$T' = \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} T \begin{bmatrix} t^{-1} & 0 \\ 0 & t \end{bmatrix}.$$

(ii) $c_t \Delta c_t^{-1} = \Delta\Delta^*$.

From the above, we can elucidate the structure of $\mathrm{Aut}(\widetilde{\Gamma^*})$, as follows.

THEOREM 1.5. *The automorphism group of the canonical double cover $\widetilde{\Gamma^*}$ of the Shult-Taylor double Γ^* of the Paley graph Γ has structure*

$$\mathrm{Aut}(\widetilde{\Gamma^*}) \cong (Z_2 \times Z_2) \rtimes \mathrm{P\Gamma L}_2(q),$$

where $\mathrm{P\Sigma L}_2(q)$ centralizes $Z_2 \times Z_2$ and where any element in $\mathrm{P\Gamma L}_2(q) \backslash \mathrm{P\Sigma L}_2(q)$ acts as a non-trivial involutory automorphism on $Z_2 \times Z_2$.

Proof. Indeed, using Lemma 1.4, part (i), one concludes easily that for a fixed non-square t, the mapping $\langle c_t, g_T \mid T \in \mathrm{PSL}_2(q) \rangle \to \mathrm{PGL}_2(q)$ given by

$$g_T \mapsto T, \quad c_t g_T \mapsto \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix} T, \qquad T \in \mathrm{PSL}_2(q),$$

defines an isomorphism. The rest of the proof follows from above Lemma 1.4, part (ii). □

2. Graphs related to $\mathrm{PSL}_2(q)$

In [7; Sec. 4], I gave another infinite family of unstable graphs $\Gamma = \Gamma(q)$, where q is a power of the prime p and $q \equiv 5, 7 \pmod{12}$. Recall that the group

$\mathrm{PSL}_2(q)$ has two conjugacy classes $\mathcal{C}$, $\mathcal{C}'$ of elements of order p, and a unique class $\mathcal{T}$ of elements of order 3. Assume that the class $\mathcal{C}$ contains the (residue of the) matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. The graph Γ has vertex set $\mathcal{C}$ and edge relation $x \sim y$ if and only if $xy \in \mathcal{T}$. We have already shown in [7] that this graph is acted on edge transitively by $G = \mathrm{PSL}_2(q)$.

We shall investigate the automorphism group of Γ when p is prime. Indeed, from the point of view of regular map embeddings, this is the only relevant case as the case of non-trivial prime powers does not admit dihedral vertex stabilizers. In what follows, we denote $\mathbb{F} = \mathbb{F}_p$, the field of p elements.

THEOREM. *If $q = p$ is prime, we have*

$$\mathrm{Aut}(\Gamma) \cong \begin{cases} Z_2 \times \mathrm{PSL}_2(p) & \text{if } p \equiv 5 \pmod{12}, \\ \mathrm{PGL}_2(p) & \text{if } p \equiv 7 \pmod{12}. \end{cases}$$

Proof. Define the elements

$$a = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \quad \tau = \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix},$$

where t generates $\mathbb{F}^\times$. Note that $\mathrm{N}_G(\langle a \rangle) = \langle a \rangle \rtimes \langle \tau \rangle$. By Burnside fusion (see [3; p. 240, Theorem 1.1]) it follows that there are $(p-1)/2$ conjugates of a contained in $\langle a \rangle$, and are of the form $\tau^j a \tau^{-j}$, $j = 0, 1, \dots, (p-3)/2$. In [7] it is proved that $\Gamma(p)$ has valency p and that the vertices adjacent to a are the p conjugates of b by powers of a. For any conjugate x of a, let $\Gamma(x)$ denote the vertices of Γ adjacent to x.

The following simple lemma shall prove useful.

LEMMA 2.1. *For any odd prime power q not divisible by 3, the matrix $A \in \mathrm{SL}_2(q)$ satisfies $A^3 = \pm I$ if and only if* $\operatorname{trace} A = \mp 1$.

Proof. If $\operatorname{trace} A = \varepsilon = \pm 1$, and if λ_1, λ_2 are the eigenvalues of A (which might be in a quadratic extension field of $\mathbb{F}_q$), then $\lambda_1 + \lambda_2 = \varepsilon$, $\lambda_1 \lambda_2 = 1$ jointly imply that $\lambda_1^3 = \lambda_2^3 = -\varepsilon$. Since q is not divisible by 3, A must be diagonalizable (over a quadratic extension of $\mathbb{F}_q$) and so it follows that $A^3 = -\varepsilon I$, $\varepsilon = \pm 1$. The converse is equally simple. □

We resume the proof of the above theorem; a few useful facts are singled out below.

(1) *For any pair of distinct elements $x, y \in \langle a \rangle$, $\Gamma(x) \cap \Gamma(y) = \emptyset$. As a result, the vertex set $\mathcal{C}$ can be represented as*

$$\mathcal{C} = (\mathcal{C} \cap \langle a \rangle) \cup \bigcup_{x \in \mathcal{C} \cap \langle a \rangle} \Gamma(x).$$

Indeed, it is sufficient to prove that $\Gamma(a)\cap\Gamma(\tau^k a\tau^{-k})=\emptyset$, where $1\le k\le (p-3)/2$. In turn, since a typical element of $\Gamma(a)$ has the form $a^jba^{-j}=\left[\begin{smallmatrix}1-j & j^2\\ -1 & 1+j\end{smallmatrix}\right]$, and since $\tau^k a\tau^{-k}=\left[\begin{smallmatrix}1 & \nu\\ 0 & 1\end{smallmatrix}\right]$ for some $\nu\in\mathbb{F}^\times$, one shows, using the above lemma, that the product

$$\begin{bmatrix}1-j & j^2\\ -1 & 1+j\end{bmatrix}\begin{bmatrix}1 & \nu\\ 0 & 1\end{bmatrix}$$

has order 3 in G if and only if $\nu=1,3\in\mathbb{F}$. However, since $q\equiv 5,7 \pmod{12}$, an easy application of quadratic reciprocity show that $3\notin\mathbb{F}^{\times 2}$, which precludes the possibility that

$$\tau^k a\tau^{-k}=\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix}$$

for any power k. Therefore, we must have $\nu=1$, i.e., that $\tau^k a\tau^{-k}=a$, which proves (1).

(2) *If $x,y\in\langle a\rangle\cap C$, there are exactly two edges from a given vertex in $\Gamma(x)$ to vertices in $\Gamma(y)$.*

Here, it suffices to assume that $x=a$, that

$$y=\begin{bmatrix}\nu & 0\\ 0 & \nu^{-1}\end{bmatrix}a\begin{bmatrix}\nu^{-1} & 0\\ 0 & \nu\end{bmatrix}=\begin{bmatrix}1 & \nu^2\\ 0 & 1\end{bmatrix}$$

for some $\nu\in\mathbb{F}^\times$, and to prove that there are exactly two vertices in $\Gamma(y)$ adjacent to b ($\in\Gamma(a)$). As a result, the vertices in $\Gamma(y)$ are of the form

$$\begin{bmatrix}\nu & 0\\ 0 & \nu^{-1}\end{bmatrix}a^jba^{-j}\begin{bmatrix}\nu^{-1} & 0\\ 0 & \nu\end{bmatrix}=\begin{bmatrix}1-j & \nu^2j^2\\ -\nu^{-2} & 1+j\end{bmatrix}.$$

Such an element can be checked to be adjacent to $x=b$ if and only if $(\nu j)^2=1,3$. As already observed, $3\notin\mathbb{F}^{\times 2}$ and so $\nu j=\pm 1$, affording the required two vertices.

Note that as a result of (1) and (2) we conclude that Γ is connected, has diameter 3, and that vertices x, y have distance 3 if and only if x and y commute. Therefore the $p+1$ conjugates of $C\cap\langle a\rangle$ form a system of imprimitivity in C for the group $\mathrm{Aut}(\Gamma)$.

(3) *The subgraph of Γ induced by $\Gamma(x)$, where x is a conjugate of a, is an ordinary p-gon.*

Note that by (2) above, together with the fact that Γ has valency p, we see that the subgraph induced by $\Gamma(a)$ must have valency 2. Next, an easy calculation shows that aba^{-1} and b are adjacent in Γ. This is clearly enough.

Next a routine calculation reveals the following:

(4) *The distance between the vertices* $c_i = a^iba^{-i}$, $c_{-i} = a^{-i}ba^i$ *as measured within the subgraph induced by* $\Gamma(a)$ *is*

$$\mathrm{dist}_{\Gamma(a)}(c_i, c_{-i}) = \begin{cases} 2i & \text{if } i \le \left[\frac{p-1}{4}\right], \\ p-2i & \text{if } \left[\frac{p-1}{4}\right] < i \le \frac{p-1}{2}. \end{cases}$$

Now assume that $\nu, \nu' \neq 1$ and assume that $\nu \neq \pm\nu'$. Set

$$x = \begin{bmatrix} 1 & \nu \\ 0 & 1 \end{bmatrix}, \qquad x' = \begin{bmatrix} 1 & \nu' \\ 0 & 1 \end{bmatrix}.$$

As a result of (4) and the calculations in (2) we infer that if $b_1, b_2 \in \Gamma(x)$, $b_1', b_2' \in \Gamma(x')$ are the vertices adjacent to b, then the distance between b_1 and b_2 *as measured from within* $\Gamma(x)$ is different from the distance between b_1' and b_2' *as measured from within* $\Gamma(x')$. Put slightly differently:

(5) *Let* $x \neq x' \in \mathcal{C} \cap \langle a\rangle$ *and assume that* $x, x' \neq a$. *Let* $b_1, b_2 \in \Gamma(x)$, $b_1', b_2' \in \Gamma(x')$ *be the unique vertices in* $\Gamma(x)$, $\Gamma(x')$, *respectively, adjacent to* b. *Then the distance between* b_1, b_2 *as measured within* $\Gamma(x)$ *is different from the distance between* b_1', b_2' *as measured within* $\Gamma(x')$.

Note that (5) implies the following:

(6) *Let* $a \neq x \in \mathcal{C} \cap \langle a\rangle$, *let* $b \in \Gamma(x)$, *and let* $b_1, b_2 \in \Gamma(a)$ *the two vertices in* $\Gamma(a)$ *adjacent to* b. *Then* $\Gamma(b_1) \cap \Gamma(b_2) = \{a, b\}$, *i.e., the only vertices in* Γ *adjacent to both* b_1 *and* b_2 *are* a *and* b.

(7) $|\mathrm{Aut}(\Gamma)| \le p(p^2-1)$.

Since G (and hence $\mathrm{Aut}(\Gamma)$) acts transitively by conjugation on the directed edges of Γ, and since there are $\frac{1}{2}p(p^2-1)$ directed edges in Γ, it suffices to prove that the stabilizer of a directed edge is cyclic of order at most two. Thus, let $g \in \mathrm{Aut}(\Gamma)$ fix the directed edge (a, b). Since g fixes a, and since the conjugates of $\mathcal{C} \cap \langle a\rangle$ is a system of imprimitivity for $\mathrm{Aut}(\Gamma)$, we conclude that g must fix (setwise) the set $\mathcal{C} \cap \langle a\rangle$. Since $\Gamma(a)$ induces a p-gon, we conclude immediately that either g induces the identity on $\Gamma(a)$, or g induces an involutory involution on $\Gamma(a)$, fixing only the vertex b. Assume that g induces the identity on $\Gamma(a)$. We shall argue that, in fact, $g = 1 \in \mathrm{Aut}(\Gamma)$. To accomplish this, it suffices to show that

(i) g fixes $\mathcal{C} \cap \langle a\rangle$ *elementwise*,
(ii) g fixes $\Gamma(x)$ *elementwise* for each $x \in \mathcal{C} \cap \langle a\rangle$.

For each $a \neq x \in \mathcal{C} \cap \langle a\rangle$, there are, by (2) above, exactly two elements, $b_1, b_2 \in \Gamma(x)$ adjacent to b. As already noticed above, g must map the vertex x to another vertex $x' \in \mathcal{C} \cap \langle a\rangle$, $x' \neq a$. As a result, g maps the vertices

b_1, b_2 to the two unique vertices $b_1', b_2' \in \Gamma(x')$ adjacent to b. By (5), if $x' \neq x$, this is impossible; therefore we infer already that g fixes $\mathcal{C} \cap \langle a \rangle$ elementwise.

Next, as g fixes the vertices b, x, it is clear that g stabilizes the set $\{b_1, b_2\}$. Using (2) again, we infer that there is a second vertex $b' \in \Gamma(a)$ adjacent to b_1. Thus, b' is adjacent to vertices $b_1, b_1' \in \Gamma(x)$; we shall show that $b_1' \neq b_2$. Note that b' is a conjugate of b by some power a^j of a; if it were the case that $b_1' = b_2$, then we would infer that $\{a^j b_1 a^{-j}, a^j b_2 a^{-j}\} = \{b_1, b_2\}$. However, since a^j has order p and centralizes neither of b_1, b_2, we see that this is impossible. Therefore, using the fact that g fixes b, b', we may infer that g also fixes b_1 (and hence also b_2).

Finally, since $\Gamma(x)$ induces a p-gon, it follows that g fixes $\Gamma(x)$ elementwise, and so $g = 1$. It follows, therefore, that the action of an arbitrary element in the stabilizer of the directed edge (a, b) is uniquely determined by its effect on $\Gamma(a)$. As already noted, this stabilizer has order at most 2, and the result follows.

The proof of the above theorem is concluded by separately considering the two relevant cases.

Case 1: $p \equiv 5 \pmod{12}$.
In this case, if $x \in \mathcal{C}$, so is x^{-1}, and the mapping $x \mapsto x^{-1}$ defines an involutory automorphism of $\Gamma(p)$. Since this mapping clearly commutes with conjugation by elements of $\mathrm{PSL}_2(p)$, it follows that we have a group isomorphic with $Z_2 \times \mathrm{PSL}_2(p)$ acting on $\Gamma(p)$. Since this has the maximum allowable order, it is the full automorphism group.

Case 2: $p \equiv 7 \pmod{12}$.
In this case, if $x \in \mathcal{C}$, then $x^{-1} \notin \mathcal{C}$. We have a surjective homomorphism $\varepsilon\colon \mathrm{PGL}_2(p) \to \{\pm 1\}$ whose kernel is $\mathrm{PSL}_2(p)$. Using this define, for each element $g \in \mathrm{PGL}_2(p)$, the mapping $\gamma_g\colon \mathcal{C} \to \mathcal{C}$ by $\gamma_g(x) = g x^{\varepsilon(g)} g^{-1}$. It is routine to verify that $g \mapsto \gamma_g$ defines an injective homomorphism of $\mathrm{PGL}_2(p)$ into $\mathrm{Aut}(\Gamma)$, and hence $\mathrm{PGL}_2(p)$ must be the full automorphism group of $\Gamma(p)$.

The proof is concluded. □

The next task is to determine the full automorphism group of the canonical double cover $\widetilde{\Gamma}$. As in [7], identify $\widetilde{\Gamma}$ with the bipartite graph $\Gamma^\dagger$ having partite classes $\mathcal{C}$, $\mathcal{C}'$ with incidence $x \sim x'$ if and only if $xx' \in \mathcal{T}$, $x \in \mathcal{C}$, $x' \in \mathcal{C}'$. The isomorphism $\widetilde{\Gamma} \to \Gamma^\dagger$ is given by $(x, \zeta) \mapsto x^{2-\zeta}$; as a result, the deck transformation $(x, 1) \leftrightarrow (x, -1)$ corresponds to the "polarity" $\Gamma^\dagger \to \Gamma^\dagger$ where

$$x \mapsto x^3, \quad x' \mapsto x'^k, \qquad x \in \mathcal{C}, \ x' \in \mathcal{C}' \ \text{ where } 3k \equiv 1 \pmod{p}.$$

Furthermore, $\mathrm{Aut}(\Gamma)$ acts on $\Gamma^\dagger$, stabilizing the partite classes and commuting with the above polarity.

We proceed to define two more involutory automorphisms on $\Gamma^\dagger$, as follows. Let r be a generator of $\mathbb{F}^\times$. If $rs \equiv 1 \pmod{p}$, define the involutory mapping

$\Delta\colon \Gamma^\dagger \to \Gamma^\dagger$ by setting

$$(x)\Delta = x^r, \quad (x')\Delta = x'^s, \qquad x \in \mathcal{C},\ x' \in \mathcal{C}'.$$

We show that Δ is an automorphism $\Gamma^\dagger$. Indeed, if $x \in \mathcal{C}$, $x' \in \mathcal{C}'$ with $xx' \in \mathcal{T}$, then since $\mathrm{PSL}_2(p)$ acts transitively on directed edges of the form (x, x'), $x \in \mathcal{C}$, $x' \in \mathcal{C}'$, we may assume that

$$x = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad x' = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}.$$

Therefore,

$$x^r = \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix}, \qquad x'^s = \begin{bmatrix} 1 & 0 \\ -3s & 1 \end{bmatrix};$$

an easy calculation shows that $x^r x'^s$ has trace -1 and so represents an element of $\mathcal{T}$. Therefore, $(x)\Delta$ and $(x')\Delta$ are adjacent in $\Gamma^\dagger$.

It is clear that $\mathrm{PGL}_2(p)$ acts on $\Gamma^\dagger$ by conjugation. If we let c_g denote conjugation by $g \in \mathrm{PGL}_2(p)$, then the element $\Delta^* := c_g^{-1}\Delta c_g$, $g \in \mathrm{PGL}_2(p) \setminus \mathrm{PSL}_2(p)$, does not depend on the particular choice of element $g \in \mathrm{PGL}_2(p) \setminus \mathrm{PSL}_2(p)$. Note that

$$(z)\Delta^* = \begin{cases} z^s & \text{if } z \in \mathcal{C}, \\ z^r & \text{if } z \in \mathcal{C}'. \end{cases}$$

Therefore, Δ^* is a second involutory automorphism of $\Gamma^\dagger$, and

$$(z)\Delta\Delta^* = \begin{cases} z^{r^2} & \text{if } z \in \mathcal{C}, \\ z^{s^2} & \text{if } z \in \mathcal{C}'. \end{cases}$$

It follows, therefore, that the dihedral group $\langle \Delta, \Delta^* \rangle$ has order equal to the order of the element r in the cyclic group $\mathbb{F}^\times$. Since r was chosen to be a generator, this order is $p - 1$.

From this we see that the automorphism group of $\Gamma^\dagger$ has a subgroup of the form $\langle \Delta, \Delta^* \rangle \rtimes \mathrm{PGL}_2(p)$, where the action of $\mathrm{PGL}_2(p)$ on the dihedral group $\langle \Delta, \Delta^* \rangle$ has $\mathrm{PSL}_2(p)$ in the kernel. As a result of the above, (since $|\mathrm{Aut}(\Gamma)| = |\mathrm{PGL}_2(p)|$), we have

$$|\mathrm{Aut}(\Gamma^\dagger)| \geq (p - 1) \times |\mathrm{Aut}(\Gamma)|,$$

We contend that the above inequality is an equality:

THEOREM 2.2. *Let p be a prime congruent to 5 or 7 modulo 12, and let $\Gamma = \Gamma(p)$. Let $\widetilde{\Gamma}$ be the canonical double cover of Γ. Then $|\mathrm{Aut}(\widetilde{\Gamma})| = (p-1) \times |\mathrm{Aut}(\Gamma)|$. Furthermore, the structure of $\mathrm{Aut}(\widetilde{\Gamma})$ is that of $\mathrm{PGL}_2(p)$ acting on a dihedral group of order $p-1$.*

P r o o f. For any element $\mu \in \mathbb{F}^{\times 2}$, let

$$a_\mu = \begin{bmatrix} 1 & \mu \\ 0 & 1 \end{bmatrix};$$

as above, we set $a = a_1$. As usual, let $\Gamma(a_\mu)$ be the vertices in the graph Γ adjacent to a_μ. If $\widetilde{\Gamma}_i(a,1)$ is the set of vertices in $\widetilde{\Gamma}$ at distance i from the vertex $(a,1)$ of $\widetilde{\Gamma}$, then it follows easily that

$$\widetilde{\Gamma}_1(a,1) = \Gamma(a) \times \{-1\},$$
$$\widetilde{\Gamma}_2(a,1) = \bigcup_{\mu \in \mathbb{F}^{\times 2}} \Gamma(a_\mu) \times \{1\},$$
$$\widetilde{\Gamma}_3(a,1) = \{(a_\mu, -1):\ \mu \in \mathbb{F}^{\times 2}\} \cup \bigcup_{1 \neq \mu \in \mathbb{F}^{\times 2}} \Gamma(a_\mu) \times \{-1\},$$
$$\widetilde{\Gamma}_4(a,1) = \{(a_\mu, 1):\ 1 \neq \mu \in \mathbb{F}^{\times 2}\}.$$

Since each element $(a_\mu, -1)$ is adjacent to every vertex in $\Gamma(a_\mu) \times \{1\}$, we infer easily that the sets $\Gamma(a_\mu) \times \{1\}$, $\mu \in \mathbb{F}^{\times 2}$, form a system of imprimitivity for the stabilizer $\widetilde{G}_{(a,1)}$ of $(a,1)$ in $\widetilde{G} = \mathrm{Aut}(\widetilde{\Gamma})$. Let $H \leq \widetilde{G}_{(a,1)}$ be the stabilizer in $\widetilde{G}_{(a,1)}$ of the set $\Gamma(a) \times \{1\}$. Therefore, H acts on the subgraph induced by the vertices in $\Gamma(a) \times \{\pm 1\}$, which clearly induces the canonical double cover of the ordinary p-gon. Since p is odd, this p-gon is not bipartite and it follows that its canonical double cover is the ordinary $2p$-gon. It is clear that H must fix the partite classes of vertices of this ordinary $2p$-gon, from which we infer that H must map into the dihedral group D_{2p} of order $2p$. We claim that the homomorphism $H \to D_{2p}$ is injective. Thus, let h fix pointwise the set $\Gamma(a) \times \{\pm 1\}$; we shall show that, in fact, h must induce the identity on all of $\widetilde{\Gamma}$.

We first show that h must fix pointwise each of the sets $\Gamma(a_\mu) \times \{1\}$, where $1 \neq \mu \in \mathbb{F}^{\times 2}$. If $(b_\mu, 1) \in \Gamma(a_\mu) \times \{1\}$, then by (2) above there are exactly two vertices $(b_1, -1), (b_2, -1) \in \Gamma(a) \times \{-1\}$ adjacent to $(b_\mu, 1)$. By (6), the pair b_1, b_2 are adjacent to exactly a and b_μ in Γ; therefore, $(b_1, -1), (b_2, -1)$ are adjacent to exactly $(a,1)$ and $(b_\mu, 1)$. From this, it follows that h fixes $(b_\mu, 1)$, proving that h fixes pointwise each of the sets $\Gamma(a_\mu) \times \{1\}$.

From the above, it is clear that h fixes each of the points $(a_\mu, -1)$, $\mu \in \mathbb{F}^{\times 2}$. Finally, arguments similar to those above show also that h must fix each of the sets $\Gamma(a_\mu) \times \{-1\}$, $1 \neq \mu \in \mathbb{F}^{\times 2}$.

Therefore, we conclude that $|H| \le 2p$, from which it follows that

$$|\widetilde{G}_{(a,1)}| \le |H| \times \frac{1}{2}(p-1) \le p(p-1).$$

Since the graph $\widetilde{\Gamma}$ has $2 \times \frac{1}{2}(p^2-1) = p^2-1$ vertices, and since $\widetilde{G}$ acts vertex-transitively on $\widetilde{\Gamma}$, we infer that

$$\begin{aligned}|\mathrm{Aut}(\widetilde{\Gamma})| &= |\widetilde{G}_{(a,1)}| \times (p^2-1) \le p(p-1)^2(p+1)\\ &= (p-1) \times |\mathrm{PGL}_2(p)| \;(= |\mathrm{GL}_2(p)|).\end{aligned}$$

Since the reverse inequality has already been established, we are finished. □

Concluding remarks. Since the graphs $\Gamma(p)$, $p \equiv 5 \pmod{12}$, considered above admit a central involutory automorphism ($\iota\colon x \mapsto x^{-1}$) one might wonder (as did the referee!) whether the corresponding halved quotients $\overline{\Gamma}(p) := \Gamma(p)/\langle\iota\rangle$ are also unstable. In fact, excluding only the case $p = 5$ ($\overline{\Gamma}(5)$ is complete on 6 vertices, and is hence stable), these graphs are, in fact, unstable. This is easily demonstrated, as follows. Set $\Gamma = \Gamma(p)$, $p \equiv 5 \pmod{12}$, and set $\overline{\Gamma} = \Gamma/\langle\iota\rangle$, the halved quotient of Γ by the involution ι. Then ι also defines an automorphism of the graph $\Gamma^{\dagger}$ via $x \mapsto x^{-1}$; from this one can identify the canonical double cover of the halved quotient $\overline{\Gamma}$ with $\overline{\Gamma^{\dagger}} := \Gamma^{\dagger}/\langle\iota\rangle$. Next, if Δ, Δ^* are the involutory automorphisms of $\Gamma^{\dagger}$ given above, then (since $p \ge 13$) we have $\langle\iota\rangle = Z(\langle\Delta, \Delta^*\rangle)$, the center of $\langle\Delta, \Delta^*\rangle$, (one checks that $\iota = (\Delta\Delta^*)^{\frac{1}{4}(p-1)}$), and so the dihedral group $\langle\Delta, \Delta^*\rangle$ of order $p-1$ determines a dihedral group of automorphisms (of order $(p-1)/2$) of $\overline{\Gamma^{\dagger}}$. Finally, let $r \in \mathbb{F}^{\times}$ be the generator in the definition of Δ; then as (3) is not a quadratic residue modulo p, we may write $3 = r^{2l+1}$ for some integer l. One then computes that the deck transformation δ on $\Gamma^{\dagger}$ ($x \mapsto x^3$, $x \in \mathcal{C}$, $x' \mapsto x'^k$, $x' \in \mathcal{C}'$, $3k \equiv 1 \pmod{p}$) is given by $\delta = (\Delta\Delta^*)^l\Delta^*$. This implies that the image of δ in $\langle\Delta, \Delta^*\rangle / Z(\langle\Delta, \Delta^*\rangle)$ is not central, and so $\mathrm{Aut}(\overline{\Gamma^{\dagger}})$ contains automorphisms not commuting with the deck transformation of $\overline{\Gamma^{\dagger}}$. Thus, $\overline{\Gamma}$, $p \equiv 1 \pmod{p}$, $p \ge 13$ is not stable. It is conceivable, if not likely, that the full automorphism group of $\overline{\Gamma^{\dagger}}$ is $\langle\Delta, \Delta^*\rangle/\langle\iota\rangle \rtimes \mathrm{PGL}_2(p)$; we shall leave the details for a separate investigation.

In addition to the automorphism groups of the halved quotients $\overline{\Gamma^{\dagger}}$ considered in the above paragraph, it would be of interest also to compute the automorphism groups of the canonical double covers of the graphs in [7] not considered herein. These are the unstable graphs based on the space $\Omega^{+}(2n, 2)$ and the graphs based on $\mathrm{PSL}_2(q)$, where $q = p^n$, p prime and $n > 1$. The automorphism groups of the canonical double covers of these graphs will be considered elsewhere.

Finally, with the automorphism groups of the graphs in this investigation having been determined, it would be of considerable interest to determine the regular embeddings of these graphs (and their canonical double covers; keeping in mind that the results of [5] no longer apply) into orientable surfaces (or for the above halved quotients, into nonorientable surfaces); again, this is best left to a separate project.

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Received February 7, 2002
Revised April 6, 2003

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