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THE EGG-BOX PROPERTY OF CONGRUENCES

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Let $\mathscr{V}$ be a variety of algebras. $\mathscr{V}$ has *directly decomposable congruences* (briefly DDC) if for every A, B of $\mathscr{V}$ and each $\Theta \in Con\, A \times B$ there exist $\Theta_A \in Con\, A$ and $\Theta_B \in Con\, B$ such that $\Theta = \Theta_A \times \Theta_B$. Varieties having DDC were characterized by a Mal'cev condition in [3]. If $\mathscr{V}$ has DDC and $A, B \in \mathscr{V}$, then for each $\Theta \in Con\, A \times B$, all congruence classes of Θ are "rectangular" and they form the so-called "egg-box", see e.g. Fig. 1.

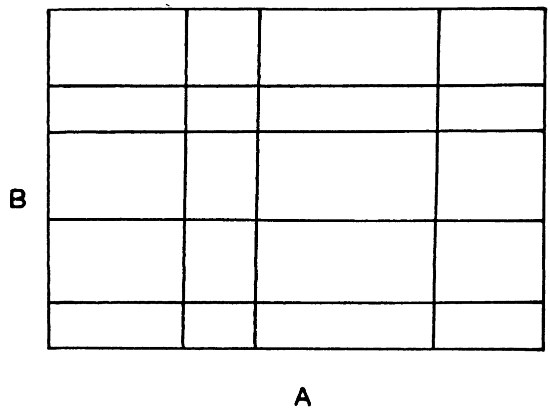

Fig. 1

We can ask: What can be said about congruences if the congruence classes of $\Theta \in Con\, A \times B$ are only "rectangular" but they need not form an "egg-box". A variety $\mathscr{V}$ is said to have *directly decomposable congruence classes* (briefly DDCC) if for every A, B of $\mathscr{V}$ and each congruence class C of $\Theta \in Con\, A \times B$, $C = pr_A C \times pr_B C$, where the symbol $pr_A C$ denotes the projection of C on A; analogously for $pr_B C$. Varieties having DDCC were characterized by a Mal'cev condition in [2]. If $\mathscr{V}$ has DDCC and $A, B \in \mathscr{V}$, then for each $\Theta \in Con\, A \times B$ the congruence classes of Θ from "bricks", see, e.g., Fig. 2.

Clearly, if $\mathscr{V}$ has DDC, then $\mathscr{V}$ has also DDCC but not vice versa in a general case. The aim of this paper is to characterize (by a Mal'cev condition) varieties of algebras in which DDC and DDCC are equivalent conditions.

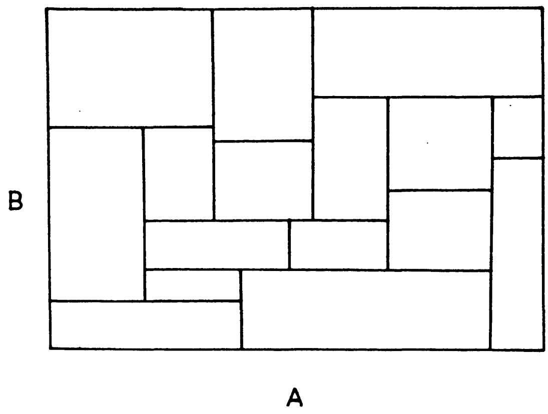

Fig. 2

Definition. A variety $\mathscr{V}$ has the *egg-box property* if for every A, B of $\mathscr{V}$ and each $\Theta \in Con\, A \times B$ and each $x, y \in A$, $z, v \in B$

$$\text{if} \quad \langle (x, z), (x, v) \rangle \in \Theta, \quad \text{then} \quad \langle (y, z), (y, v) \rangle \in \Theta \quad \text{and}$$

$$\text{if} \quad \langle (x, z), (y, z) \rangle \in \Theta, \quad \text{then} \quad \langle (x, v), (y, v) \rangle \in \Theta.$$

The egg-box property can be visualized by the following diagrams, see Fig. 3.

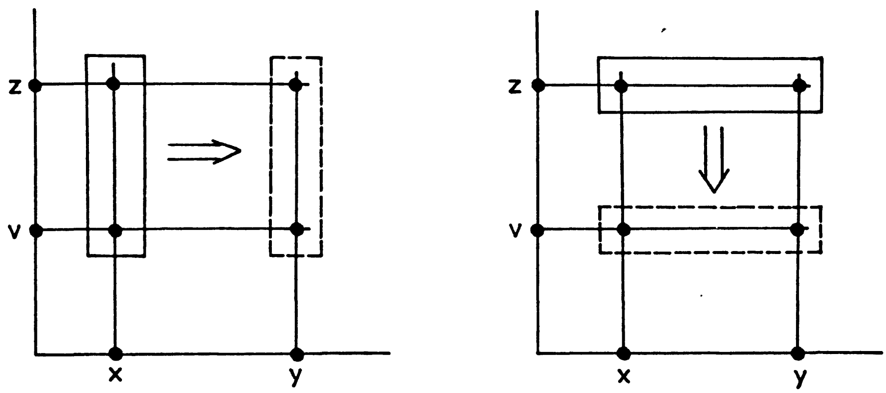

Fig. 3

Clearly, it means that varieties having the egg-box property are the best candidates fof DDC, since if each congruence class of $\Theta \in Con\, A \times B$ has at least one horizontal segment (i.e. $\langle (x, z), (y, z) \rangle \in \Theta$) or at least one vertical segment (i.e. $\langle (x, z), (x, v) \rangle \in \Theta$), then each congruence class is rectangular and all congruence classes form the egg-box, i.e. $\mathscr{V}$ has DDC.

Lemma. *Let $\mathscr{V}$ be a variety. Then $\mathscr{V}$ has DDC if and only if it has DDCC and the egg-box property.*

The proof follows directly from the previous remark.

Theorem 1. *Let $\mathscr{V}$ be a variety, the following two conditions are equivalent:*

(1) *$\mathscr{V}$ has the egg-box property;*

(2) *there exist 5-ary polynomials $p_0, \ldots, p_n$ such that*

$p_i(x, x, x, y, y) = y$ *for* $i = 0, \ldots, n$,

$p_0(x, x, y, x, y) = x$, $p_n(y, x, y, x, y) = y$ *and*

$p_i(y, x, y, x, y) = p_{i+1}(y, x, y, x, y)$ *for i even*

$p_i(x, x, y, x, y) = p_{i+1}(x, x, y, x, y)$ *for i odd.*

Proof. (1)$\Rightarrow$(2): Let $F_V(x, y)$ be a free algebra of $\mathscr{V}$ generated by two free generators x, y and put $A = F_V(x, y) = B$. Let $\Theta \in Con\, A \times B$ be a principal congruence generated by the pair $\langle (x, x), (x, y) \rangle$, i.e. $\Theta = \Theta((x, x), (x, y))$. By the Definition, we have

$$\langle (y, x), (y, y) \rangle \in \Theta((x, x), (x, y)).$$

By the Mal'cev Lemma, There exist elements $z_0, \ldots, z_n$ of $A \times B$ and unary algebraic functions $\tau_0, \ldots, \tau_n$ such that $z_0 = (y, x)$, $z_n = (y, y)$ and $\{z_i, z_{i+1}\} = \{\tau_i((x, x)), \tau_{i+1}((x, y))\}$. Since $A \times B$ is generated by the elements (x, x), (x, y), (y, x), (y, y), there exist 5-ary polynomials $p_0, \ldots, p_n$ such that

$$\tau_i(z) = p_i(z, (x, x), (x, y), (y, x), (y, y)) \quad \text{for all} \quad i = 0, \ldots, n.$$

Hence, $p_0((x, x), (x, x), (x, y), (y, x), (y, y)) = (y, x)$,

$$p_i((x, y), (x, x), (x, y), (y, x), (y, y)) = \\ = p_{i+1}((x, y), (x, x), (x, y), (y, x), (y, y)) \quad \textit{for i even},$$

$$p_i((x, x), (x, x), (x, y), (y, x), (y, y)) = \\ = p_{i+1}((x, x), (x, x), (x, y), (y, x), (y, y)) \quad \textit{for i odd, and}$$

$$p_n((x, y), (x, x), (x, y), (y, x), (y, y)) = (y, y).$$

If we write it componentwise, we obtain (2).

(2)$\Rightarrow$(1): Let $\mathscr{V}$ satisfy (2) and $A, B \in \mathscr{V}$, $\Theta \in Con\, A \times B$, $x, y \in A$ and $z, v \in B$. Suppose $\langle (x, z), (x, v) \rangle \in \Theta$. Then, by (2),

$$(y, z) = p_0((x, z), (x, z), (x, v), (y, z), (y, v))$$

$$(y, v) = p_n((x, v), (x, z), (x, v), (y, z), (y, v)),$$

thus (2) implies $\langle (y, z), (y, v)\rangle \in \Theta((x, z), (x, v)) \subseteq \Theta$. Analogously, we can show that $\langle (x, z), (y, z)\rangle \in \Theta$ implies $\langle (x, v), (y, v)\rangle \in \Theta$, thus (2) $\Rightarrow$ (1) is proved.

Example 1. *Every congruence permutable variety has the egg-box property.*

If $\mathscr{V}$ is congruence permutable, then there exists a Mal'cev polynomial, i.e. a ternary polynomial $t(x, y, z)$ satisfying $t(x, y, y) = x$ and $t(x, x, y) = y$. We can put $n = 0$ and $p_0(a, x, b, c, d) = t(c, t(b, a, c), d)$. Then

$$p_0(x, x, x, y, y) = t(y, t(x, x, y), y) = t(y, y, y) = y$$

and

$$p_0(x, x, y, x, y) = t(x, t(y, x, x), y) = t(x, y, y) = x$$

$$p_0(y, x, y, x, y) = t(x, t(y, y, x), y) = t(x, x, y) = y.$$

Example 2. *Every 3-permutable variety has the egg-box property.*

By [4], $\mathscr{V}$ is 3-permutable if there exist ternary polynomials $t_1(x, y, z)$ and $t_2(x, y, z)$ such that

$$x = t_1(x, y, y), \quad t_1(x, x, y) = t_2(x, y, y), \quad t_2(x, x, y) = y.$$

Put $n = 1$ and

$$p_0(a, b, c, d, y) = t_1(d, t_1(b, c, c), t_2(d, d, a))$$

$$p_1(a, x, b, c, d) = t_2(a, b, t_2(c, c, d)).$$

Then

$$p_0(x, x, x, y, y) = t_1(y, t_1(x, x, x), t_2(y, y, x)) = t_1(y, x, x) = y$$

$$p_1(x, x, x, y, y) = t_2(x, x, t_2(y, y, y)) = t_2(x, x, y) = y$$

and

$$p_0(x, x, y, x, y) = t_1(x, t_1(x, y, y), t_2(x, x, x)) = t_1(x, x, x) = x$$

$$p_0(y, x, y, x, y) = t_1(x, t_1(x, y, y), t_2(x, x, y)) = t_1(x, x, y) =$$
$$= t_2(x, y, y) = t_2(x, y, t_2(x, x, y)) = p_1(x, x, y, x, y)$$

$$p_1(y, x, y, x, y) = t_2(y, y, t(x, x, y)) = t_2(y, y, y) = y.$$

Remark 1. Both previous examples are special cases of congruence modular varieties. It is a question if also congruence modular varieties have the egg-box property in a general case. The following theorem gives the answer:

Theorem 2. *Every congruence modular variety has the egg-box property.*

Proof. By [1], $\mathscr{V}$ is congruence modular if and only if there exist an integer $k \geqslant 0$ and 4-ary polynomials $t_0, \ldots, t_k$ such that

$$t_0(x, y, z, u) = x, \; t_k(x, y, z, u) = y \quad \text{and}$$

$$t_i(x, y, y, x) = x \quad \text{for } i = 0, \ldots, k$$

$$t_i(x, y, y, z) = t_{i+1}(x, y, y, z) \quad \text{for } i \text{ odd}$$

$$t_i(x, x, y, y) = t_{i+1}(x, x, y, y) \quad \text{for } i \text{ even.}$$

Put $n = k - 1$ and $p_i(a, x, b, c, d) = t_{i+1}(c, a, b, d)$. Then

$$p_0(x, x, y, x, y) = t_1(x, x, y, y) = t_0(x, x, y, y) = x$$

$$p_n(y, x, y, x, y) = t_k(x, y, y, y) = y.$$

Moreover,

$$p_i(x, x, x, y, y) = t_{i+1}(y, x, x, y) = y \quad \text{for } i = 0, \ldots, n$$

$$p_i(y, x, y, x, y) = t_{i+1}(x, y, y, y) = t_{i+2}(x, y, y, y) = p_{i+1}(y, x, y, x, y)$$

for i even, and

$$p_i(x, x, y, x, y) = t_{i+1}(x, x, y, y) = t_{i+2}(x, x, y, y) = p_{i+1}(x, x, y, x, y)$$

for i odd.

Remark 2. By [3], every congruence distributive variety has DDC. As it was mentioned above, each variety having egg-box property is the best candidate for DDC. Therefore, Theorem 2 implies that congruence modular varieties have "almost" DDC.

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ПЕРПЕНДИКУЛЯРНЫЕ КЛАССЫ КОНГРУЭНЦИЙ

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Резюме

Многообразные $\mathscr{V}$ иммет перпендикулярные классы конгруэнцый, если для любых алгебр A, $B \in \mathscr{V}$ и любой конгруенции $\Theta \in Con\, A \times B$ справедливо: если $\langle (x, z), (x, v) \rangle \in \Theta$ для некоторых $x \in A$, z, $v \in B$, то $\langle (y, z), (y, v) \rangle \in \Theta$ для каждого $y \in A$ и если $\langle (x, z), (y, z) \rangle \in \Theta$ для некоторых x, $y \in A$, $z \in B$, то $\langle (x, v), (y, v) \rangle \in \Theta$ для каждого $v \in B$. Мы даем условия Мальцева, характеризующие многообразия с этим свойством.