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CATEGORIES OF ORTHOMODULAR POSETS

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We endow the class of orthomodular posets with three types of morphisms and obtain thus three categories. We show that in all cases the category of Boolean algebras (and Boolean mappings) becomes a reflective and a coreflective subcategory. We investigate when the reflections are embeddings. We surprisingly encounter classes of orthomodular posets important within the framework of quantum theories.

Introduction

The recent extensive study of orthomodular posets has been to a certain extent stimulated by the questions of quantum theories. The point is that the so-called logic of a quantum system (= the structure of the statments on a system) is commonly assumed to be an orthomodular poset (see [5], [7], [13], etc.). Many interesting mathematical problems then emerge as a "translation" or abstraction of real world questions.

Our present considerations may be motivated as follows. The closer the logic is to a Boolean algebra, the closer the corresponding system is to a nonquantum one. It seems therefore desirable to know if every logic has the best Boolean approximation "from inside" and "from outside", and, in the positive case, how the approximation looks and if the approximation is functorial. The purpose of this paper is to investigate the latter questions. We first equip the class of orthomodular posets with three types of morphisms. In all cases the category of Boolean algebras becomes a full subcategory of the respective category of orthomodular posets. We then prove the results stated in the abstract.

The basic notions and elementary facts, sometimes only recalled, are taken from [5] and [14]. It should be noted that Theorem 1 may be viewed as a generalization of the main results of [11] and [14].

Notions and results

Let us first review the basic notions and elementary facts (see [5]).

Definition 1. *An orthomodular poset (abbr. o. p.) is a partially ordered set* $(S, \leqslant)$ *with the least and the greatest elements* 0, 1 *and with a unary operation* $'$ *satisfying the following requirements (the symbols* $\vee$, $\wedge$ *mean the induced lattice-theoretic operations):*
(i) $(a')' = a$ *for any* $a \in S$,
(ii) *if* $a \leqslant b$ *then* $a' \geqslant b'$ *and* $b = a \vee (b \wedge a')$.

We shall write simply S instead of $(S, \leqslant, ')$ if a misunderstanding can be excluded. One should not overlook that we do not require $0 \neq 1$.

Definition 2. *Let* S *be an o. p. Two elements* $a, b \in S$ *are called orthogonal if* $a \leqslant b'$. *More generally, the elements* $a, b \in S$ *are called compatible (in symbols:* $a \leftrightarrow b$*) if there exist three mutually orthogonal elements* c, d, e *such that* $a = c \vee d$, $b = c \vee e$.

Obviously, if $a \leftrightarrow b$, then $a \vee b$, $a \wedge b$ exists in S. Recall that S is a Boolean algebra if and only if every two elements of S are compatible (see [5]).

Definition 3. *Suppose that* S *is an o. p. The set* $C(S) = \{a \in S \mid a \leftrightarrow b$ *for any* $b \in S\}$ *is called the centre of* S.

Proposition 1. *If* S *is an o. p., then* $C(S)$ *is a Boolean algebra.*

Proof. We need to show that $C(S)$ is closed under the formation of the Boolean operations. This was sketched in [13] and proved in detail in [2] and [12].

Let us consider the following properties of a mapping $f: P \rightarrow Q$ between two orthomodular posets:

(i) if $a \leqslant b$, then $f(a) \leqslant f(b)$,
(ii) $f(a)' = f(a')$ for any $a \in P$,
(iii) if $a \in C(P)$, then $f(a) \in C(Q)$,
(iv) $f(a \vee b) = f(a) \vee f(b)$ whenever $a \in C(P)$, $b \in P$, $a \leqslant b'$,
(v) $f(a \vee b) = f(a) \vee f(b)$ whenever $a \leqslant b'$,
(v)* $f(a \vee b) = f(a) \vee f(b)$ whenever $a \leftrightarrow b$,
(vi) $f(a \vee b) = f(a) \vee f(b)$ whenever $a \vee b$ exists in P.

Before stating the basic definitions, let us make a few observations.

Proposition 2.

a) *If* $f: P \rightarrow Q$ *is a mapping which fulfils the assumptions* (i), (ii), (iii), (iv), *and if* P *is a Boolean algebra, then* f *is a Boolean mapping.*
b) *The conditions* (v) *and* (v)* *are equivalent.*
c) *The identity mapping fulfils all the properties* (i)—(vi) *and the mapping fulfilling* (i)—(iv) *are closed under the formation of the compositions.*

The proof of Proposition 2 is obvious (for b) see [12]).

The message of Proposition 2, c) enables us to define the following categories.

Definition 4. *Let us denote by $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ the respective categories of orthomodular posets whose morphisms are the mappings determined as follows:*

$f \in \mathcal{O}_1 \Leftrightarrow f$ *fulfils the conditions* (i), (ii), (iii), (iv)
$f \in \mathcal{O}_2 \Leftrightarrow f$ *fulfils the conditions* (i), (ii), (iii), (v)
$f \in \mathcal{O}_3 \Leftrightarrow f$ fulfils the conditions (i), (ii), (iii), (vi).

Obviously, $\mathcal{O}_1 \supset \mathcal{O}_2 \supset \mathcal{O}_3$ and the category $\mathcal{B}$ of Boolean algebras (and Boolean mappings) is a full subcategory of each $\mathcal{O}_k$, $k=1, 2, 3$ (Proposition 2, a)). Let us now recall one category-theory notion (see e.g. [8]).

Definition 5. *Let $\mathcal{K}_2$ be a full subcategory of a category $\mathcal{K}_1$. The category $\mathcal{K}_2$ is called reflective (coreflective) in $\mathcal{K}_1$ if for any object $o_1 \in \mathcal{K}_1$ there exists an object $o_2 \in \mathcal{K}_2$ and a morphism $r\colon o_1 \to o_2$ ($c\colon o_2 \to o_1$) with the following property: For any $f\colon o_1 \to p$ ($f\colon p \to o_1$), $p \in \mathcal{K}_2$ there exists a unique $g\colon o_2 \to p$ ($g\colon p \to o_2$) such that $f = g \cdot r$ ($f = c \cdot g$). The mapping $r\colon o_1 \to o_2$ (sometimes only o_2) is called the reflection of o_1 (and dually for the coreflection).*

Theorem 1. *The category $\mathcal{B}$ of Boolean algebras is both reflective and coreflective in $\mathcal{O}_1$.*

Proof. Obviously, $\mathcal{B}$ is coreflective. The coreflector of $S \in \mathcal{O}_1$ is the inclusion $c\colon C(S) \to S$.

The proof of reflectivity will require a lemma. Let us first agree that the symbol $\{0, 1\}$ will mean the two-point Boolean algebra.

Lemma 1. *Let S be an o. p. and let $a, b \in S$, $a \not\geq b$. Then there exists such a morphism $h \in \mathcal{O}_1$, $h\colon S \to \{0, 1\}$ that $h(a) = 0$, $h(b) = 1$.*

Proof. A C-ideal I is a subset of S such that the following conditions are satisfied: 1) if $c \in I$ and $d \leqq c$, then $d \in I$, 2) if $c \in I \cap C(S)$ and $d \in I$, then $c \vee d \in I$, 3) if $c \in I$, then $c' \notin I$. Consider the collection $C_{a,b}$ of all C-ideals of S which contain the element a and do not contain b. The collection $C_{a,b}$ is clearly non-void and — by the Zorn lemma — has maximal elements when ordered by inclusion. Take a maximal element of $C_{a,b}$ and denote it by J. We shall show that $\operatorname{card}(\{c, c'\} \cap J) = 1$ for any $c \in S$.

Suppose that it is not the case. Then there exists an element $c \in S$ such that $\{c, c'\} \cap J = \emptyset$. Put $I_c = \{d \in S \mid d \leqslant c\}$ and set $K = \{x \in S \mid x \leqslant m \vee n \vee k$ for some elements $m \in J \cap C(S)$, $n \in I_c \cap C(S)$ and $k \in J \cup I_c\}$. We shall prove now that K is again a C-ideal.

If $x \in K$, $y \leqslant x$, then obviously $y \in K$. Suppose now that $x \in K$ and $y \in K \cap C(S)$. We need to show that $x \vee y \in K$. We have $x \leqslant m \vee n \vee k$ and $y \leqslant p \vee r \vee s$ for $m, p \in J \cap C(S)$, $n, r \in I_c \cap C(S)$, and $k, s \in J \cup I_c$. We may (and shall) assume that p, r, s are mutually orthogonal. (Indeed, since p, r are central, we can write $p \vee r = p \vee (r \wedge p)$. Moreover, the orthomodular law yields that $p \vee r \vee s = p \vee r \vee \bar{s}$ for an $\bar{s} \in S$, $\bar{s}$ orthogonal to $p \vee r$. Since $s \leftrightarrow p \vee r$, we obtain that $s \leqslant s$ which implies that $\bar{s} \in J \cup I_c$. It follows that we would replace p, r, s by p, $r \wedge p'$, s if necessary). Further, since $y \in C(S)$, we may write $y = (p \wedge y) \vee (r \wedge y) \vee (s \wedge y)$ and therefore $s \wedge y \in C(S)$. We see that $s \wedge y \in (J \cup I_c) \cap C(S)$ and we obtain $x \vee y \leqslant m \vee n \vee k \vee (p \wedge y) \vee (r \wedge y) \vee (s \wedge y) \in K$.

Finally, suppose that $\{x, x'\} \in K$ for an element $x \in S$. We may assume that $x \leqslant m \vee k$, $x' \leqslant n \vee s$, where $m \in J \cap C(S)$, $n \in I_c \cap C(S)$, $k \in I_c$, $s \in J$ and $m \leqslant k$, $n \leqslant s'$. As the elements m, n are central, we can write $x = (m \wedge x) \vee (m' \wedge x)$, $x' = (n \wedge x') \vee (n' \wedge x')$. Therefore $1 = x \vee x' = (m \wedge x) \vee (m' \wedge x) \vee (n \wedge x') \vee (n' \wedge x')$. Since $x < m \vee k$, we obtain that $x \wedge m' \leqslant k$ and therefore $x \wedge m' \leqslant I$. Analogically, $n' \wedge x' \in J$. Since $(x \wedge m') \vee (n \wedge x') \leqslant c$, we obtain that $c' \leqslant (m \wedge x) \vee (n' \wedge x') \leqslant m \vee (n' \wedge x') \in J$, which is absurd.

We have thus shown that K is again a C-ideal and moreover, $b \notin K$ because $b \notin J$ (otherwise $\{b, b'\} \cap J = \emptyset$ and we would use the latter method for producing a proper extension of J and $I_{b'}$). But J was supposed to be maximal in $C_{a\ b}$. Therefore $\{x, x'\} \cap J \neq \emptyset$ for any $x \in S$.

The rest is obvious. Take the mapping h: $S \to \{0, 1\}$ by putting $h(x)$ 0 if and only if $x \in J$. The required properties of h verify easily.

We shall now describe the reflection r_1: $S \to B_1$ for a given $S \in \mathcal{O}_1$. Take the set I of all two-valued $\mathcal{O}_1$-morphisms from S. We know that $I \neq \emptyset$. Put r_1: $S \to \exp I$ by setting $r_1(x) = \{h \in I \mid h(x) = 1\}$ and denote by B_1 the Boolean algebra (of subsets of I) generated by all sets $r_1(x)$, $x \in S$. Then r_1: $S \to B_1$ is the reflection of S. To show that, observe first that $r_1 \in \mathcal{O}_1$. Indeed, if $x \in C(S)$, $y \in S$ and $h(x \vee y) = 1$, then $h(x \vee y) = h(x \vee (y \wedge x')) = h(x) \vee h(y \wedge x')$ and therefore either $h(x) = 1$ or $h(y) = 1$. Secondly, suppose that we are given a morphism f: $S \to P$, $f \in \mathcal{O}_1$, $P \in \mathcal{B}$. Define a mapping g: $B_1 \to P$ by setting $g(r_1(x)) = f(x)$. Since g is thus defined on the generators of B_1, it has the properties (i), (ii). We need to show that $g(0) = 0$. The latter equality will follow by induction if we show the implication: If $r_1(x) \cap r_1(y) = \emptyset$, then $f(x) \wedge f(y) = 0$. Suppose that $f(x) \wedge f(y) \neq 0$. Since P is a Boolean algebra and $0 \neq 1$ in P, then there exists a morphism k: $P \to \{0, 1\}$ such that $k(f(x)) = k(f(y)) = 1$. Hence $r_1(x) \cap r_1(y) \neq \emptyset$ and the proof of Theorem 1 is finished.

Let us notice that we cannot use the standard category-theoretic argument for showing the reflectivity of $\mathcal{B}$ in $\mathcal{O}_1$ (the category $\mathcal{O}_1$ is not complete, in fact, it does not have the equalizers).

Theorem 2. *The category $\mathcal{B}$ is both reflective and coreflective in $\mathcal{O}_2$ and $\mathcal{O}_3$.*

Proof: Obviously, c: $C(S) \to S$ is the coreflection in both cases. Let us take up the question of reflectivity. In the case of $\mathcal{O}_2$, the reflection r_2: $S \to B_2$ is constructed as follows. Consider the reflection r_1: $S \to B_1$. Take the ideal I of B_1 generated by all elements of the type $r_1(x \vee y) \wedge r_1(x') \wedge r_1(y')$, where $x \in S$, $y \in S$, $x \leqslant y'$.

Denote by q the quotient mapping, $q: B_1 \to B_1/I = B_2$. It is easy to check that $r_2 = q \cdot r_1$ is the required reflection (the case of $\mathcal{O}_3$ argues similarly).

Let us notice that the reflectivity of $\mathcal{B}$ in $\mathcal{O}_2$, $\mathcal{O}_3$ follows also from purely category-theoretic arguments. Consider the category $\mathcal{O}_2$. It suffices to show that $\mathcal{O}_2$ is complete and $\mathcal{B}$ is closed in $\mathcal{O}_2$ under the formation of the products and the equalizers. The product in $\mathcal{O}_2$ defines "canonically" (see [4], [10]) and it is thus preserved for the Boolean algebras. Let us show that $\mathcal{O}_2$ possesses the equalizers. Suppose that we are given two $\mathcal{O}_2$-morphisms $f, g: P \to Q$. Put $E = \{x \in P \mid f(x) = g(x)\}$. We need to show that E is an o.p. Suppose that $x, y \in E$, $x \leq y$. Since $y = x \vee (y \wedge x')$ and the sum on the right-hand side is orthogonal, we see that $f(x) \vee f(y \wedge x') = g(x) \vee g(y \wedge x')$ and therefore $f(y \wedge x') = g(y \wedge x')$. It follows that $y \wedge x' \in E$ and E is thus orthomodular.

In what follows we investigate when the respective reflections are embeddings. Certain classes of orthomodular posets familiar in the quantum mechanical investigations emerge surprisingly (see Theorems 5, 6).

Definition 6. *A mapping $f \in \mathcal{O}_1$ is called an embedding if $f(x) \leq f(y)$ implies $x \leq y$.*

Theorem 3. *The mapping $r_1: S \to B_1$ is always an embedding.*

Proof: If $x \not\geq y$, then there exists an $\mathcal{O}_1$-morphism $h: S \to \{0, 1\}$ such that $h(x) = 0$, $h(y) = 1$. Therefore $r_1(x) \not\geq r_1(y)$.

As regards the case of r_2, recall that S is called representable if S is $\mathcal{O}_2$-izomorphic to an orthomodular poset of subsets of a set (see [5] — the operation $'$ means then the set-theoretic complement and $\leq$ means the inclusion). Naturally, there are many orthomodular posets which are not representable (e.g. the one of projectors on R^3, see [1], [5]).

Theorem 4. *The mapping $r_2: S \to B_2$ is an embedding if and only if S is representable.*

Proof: Suppose that $r_2: S \to B_2$ is an embedding. We need to show that S is representable. Consider the set $T = r_2(S)$ as a subset of B_2. We claim that T constitutes an o.p. with the operations inherited from B_2. Indeed, $\emptyset \in T$ and T is closed under the formation of complements. Moreover, suppose that $A, B \in T$, $A \cap B = \emptyset$. Take the elements $a, b \in S$ such that $r_2(a) = A$, $r_2(b) = B$. Since $A \subset B'$, we obtain that $a \leq b'$. Therefore $r_2(a \vee b) = A \cup B \in T$ and T is thus the representation of S.

If S is representable, $S = (R, ', \subset)$, then each two different $X, Y \in S$ can be distinguished by an $\mathcal{O}_2$-morphism to the two-point Boolean algebra $\{0, 1\}$ (a suitable two-valued measure concentrated in a point of R).

Before stating the last result, let us introduce a class of orthomodular posets (see [6]).

Definition 7. *An o.p. S is called Boolean if $x \wedge y = 0$ implies that x and y are orthogonal.*

Theorem 5. *The mapping $r_3 \colon S \to B_3$ is an embedding if and only if S is a Boolean orthomodular poset.*

Proof: Necessity. Suppose that $r_3 \colon S \to B_3$ is an embedding. If $x \wedge y = 0$, then $x' \vee y' = 1$ and therefore $r_3(x') \cup r_3(y') = r_3(1) = 1$. Since r_3 preserves the operation $'$, we have $r_3(x)' \cup r_3(y)' = r_3(0)'$. Hence $r_3(x) \cap r_3(y) = \emptyset$. This means that $r_3(x) \subset r_3(y')$ and therefore $x \leqslant y'$.

Sufficiency. If S is Boolean, then S is representable (see [9]). We need to show that the $\mathcal{O}_2$-morphisms from a Boolean o.p. coincide with the $\mathcal{O}_3$-morphisms. Suppose that S is Boolean and suppose that $x \vee y$ exists in S. According to the paper [6], Corrolary 2.5 we can write $x = (x \wedge y) \vee (x \wedge y')$, $y = (x \wedge y) \vee (y \wedge x')$ and therefore x and y are compatible. Due to Proposition 2 b., if $x \vee y$ exists in S, then every mapping $f \in \mathcal{O}_2$ fulfils $f(x \vee y) = f(x) \vee f(y)$ and the proof is finished.

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КАТЕГОРИИ ОРТОМОДУЛЯРНЫХ ПОСЕТОВ

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Резюме

В статье исследуются рефлективные и корефлективные подкатегории категории ортомодулярных частично упорядоченных множеств.