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# BINOMIAL MATRICES

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**Dedicated to Academician Štefan Schwarz on the occasion of his 70th birthday**

## Introduction

In [1], a class of matrices has been introduced defined as follows:

If $\mathbf{A}=\begin{pmatrix} a_{11} a_{12} \\ a_{21} a_{22}\end{pmatrix}$ is a $2\times 2$ matrix and $k$ is a positive integer then $\mathbf{A}_{[k]}$ is the $(k+1)\times(k+1)$ matrix defined by the identity

$$\begin{aligned}&\mathbf{A}_{[k]}(x_1^k, x_1^{k-1}x_2, x_1^{k-2}x_2^2, \dots, x_2^k)^{\mathrm{T}}=\\ &=((a_{11}x_1+a_{12}x_2)^k, (a_{11}x_1+a_{12}x_2)^{k-1}(a_{21}x_1+a_{22}x_2), \dots, (a_{21}x_1+a_{22}x_2)^k)^{\mathrm{T}}.\end{aligned} \quad (1)$$

The matrix $\mathbf{A}_{[k]}$ has been called Kronecker power of the matrix $\mathbf{A}$.

The purpose of the present note is to show a relation of this class to the class of Hankel matrices, to introduce a closely related class of binomial matrices and to find some of its properties including its additive version.

We shall denote here by $\mathcal{B}_{[k]}$ the class of all Kronecker $k$-th powers of complex $2\times 2$ matrices.

## 1. Hankel matrices and Kronecker powers

As is well known [3], Hankel matrices of order $n$ are square matrices of the form $(p_{i+k})$, $i, k=0, \dots, n-1$ where $p_0, p_1, \dots, p_{2n-2}$ are (in general complex) numbers. The following lemma is obvious:

**(1.1) Lemma.** *The matrix*

$$\mathbf{H}(t)=(t^{i+k}), \quad i, k=0, \dots, n-1 \quad (2)$$

*as well as*

$$\mathbf{H}_\infty=\begin{bmatrix} 0\dots 0\ 0 \\ \dots\dots\dots \\ 0\dots 0\ 0 \\ 0\dots 0\ 1\end{bmatrix} \quad (3)$$

*are Hankel matrices with rank one. Conversely, any Hankel matrix with rank one is a non-zero multiple of* $\mathbf{H}_\infty$ *or of* $\mathbf{H}(t)$ *for some* $t$.

**(1.2) Theorem.** *The set* $\mathcal{H}_n$ *of all complex Hankel matrices of order* $n$ *forms a linear subspace in the* $n^2$*-dimensional space of all complex square matrices of order* $n$. *The dimension of* $\mathcal{H}_n$ *is* $2n-1$ *and one of its bases is* $\mathbf{H}(\varepsilon^k)$, $k=1, \ldots, 2n-1$ *where* $\varepsilon = \exp(n^{-1}\pi\mathrm{i})$. *Moreover, the* $2n$ *matrices* $\mathbf{H}(\varepsilon^s)$, $s=0, 1, \ldots, 2n-1$ *satisfy the relation*

$$\sum_{k=0}^{2n-1} \varepsilon^k \mathbf{H}(\varepsilon^k) = 0 \tag{4}$$

*and any Hankel matrix* $\mathbf{H}=(p_{i+k})$ *can be expressed as*

$$\mathbf{H} = \frac{1}{2n} \sum_{l,k=0}^{2n-1} p_l \varepsilon^{-lk} \mathbf{H}(\varepsilon^k). \tag{5}$$

Proof. The first assertion is obvious. The second follows from (4) and (5) which are easy consequences of (2).

(1.3) Remark. The matrices $\mathbf{H}(\varepsilon^s)\mathbf{P}$, $s=0, 2, \ldots, 2n-2$ form a basis for the linear space of the so called circulant matrices [4]. Here, $\mathbf{P}$ is the permutation matrix $(\delta_{i,n-1-k})$, $i, k=0, \ldots, n-1$, $\delta_{ij}$ being the Kronecker symbol.

In the following main theorem of this section, the superscript T means transposition.

**(1.4) Theorem.** *Let* $n \geqq 2$ *be an integer, let* $\mathbf{B}$ *be a complex* $n \times n$ *matrix. Then the following are equivalent:*

(i) $\mathbf{B} \in \mathcal{B}_{[n-1]}$;

(ii) $\mathbf{BHB}^\mathrm{T} \in \mathcal{H}_n$ *for any matrix* $\mathbf{H} \in \mathcal{H}_n$.

Proof. We can assume that $n>2$. (i) $\Rightarrow$ (ii). Let $\mathbf{B} \in \mathcal{B}_{[n-1]}$. For any $x$,

$$\mathbf{BH}(x)\mathbf{B}^\mathrm{T} = \mathbf{BXX}^\mathrm{T}\mathbf{B}^\mathrm{T} = (\mathbf{BX})(\mathbf{BX})^\mathrm{T}$$

with

$$\mathbf{X} = (1, x, x^2, \ldots, x^{n-1})^\mathrm{T}; \tag{6}$$

however, $\mathbf{BX} = c\mathbf{Y}$ where $\mathbf{Y} = 1, y, \ldots, y^{n-1})^\mathrm{T}$ for some $y$ of the form $(a_{21}+a_{22}x)$. $.(a_{11}+a_{12}x)^{-1}$, or $\mathbf{BX} = c'e_n$, $e_n = (0, \ldots, 0, 1)^\mathrm{T}$.

Consequently,

$$\mathbf{BH}(x)\mathbf{B}^\mathrm{T} = c^2\mathbf{H}(y)$$

for some $y$, or $\mathbf{BH}(x)\mathbf{B}^\mathrm{T} = c'^2\mathbf{H}_\infty$. The assertion follows since, by (5), each matrix $\mathbf{H} \in \mathcal{H}_n$ is a linear combination of matrices of the form (2) and $\mathcal{H}_n$ is a linear space by Lemma (1.1).

(ii) $\Rightarrow$ (i): Let $\mathbf{B} = (b_{ik})$, $i, k=0, \ldots, n-1$ and let $\mathbf{BHB}^\mathrm{T} \in \mathcal{H}_n$ for each $\mathbf{H} \in \mathcal{H}_n$. In particular, $\mathbf{BH}(x)\mathbf{B}^\mathrm{T} \in \mathcal{H}_n$ for any $x$. Since this matrix has rank one, we have by Theorem (1.2) either

$$\mathbf{B}\mathbf{H}(x)\mathbf{B}^{\mathrm{T}} = \gamma \mathbf{H}(y), \tag{7}$$

or

$$\mathbf{B}\mathbf{H}(x)\mathbf{B}^{\mathrm{T}} = \gamma_0 \mathbf{H}_{\infty}. \tag{8}$$

Define the polynomials $f_j$, $j = 0, \ldots, n-1$ by

$$f_j(z) = \sum_{k=0}^{n-1} b_{jk} z^k.$$

In terms of these polynomials,

$$\mathbf{B}\mathbf{H}(x)\mathbf{B}^{\mathrm{T}} = (\mathbf{B}\mathbf{X})(\mathbf{B}\mathbf{X})^{\mathrm{T}} = \mathbf{U}\mathbf{U}^{\mathrm{T}},$$
$$\mathbf{U} = (f_0(x), f_1(x), \ldots, f_{n-1}(x))^{\mathrm{T}}.$$

Therefore, both (7) and (8) imply that for any $x$,

$$f_{i-1}(x) f_{i+1}(x) = f_i^2(x), \quad i = 1, \ldots, n-2. \tag{9}$$

It is easy to prove by induction with respect to $n$ the following:

**(1.5) Lemma.** *Let $n \geqq 2$ and let $f_0, \ldots, f_{n-1}$ be non-zero polynomials such that* (9) *is identically satisfied. Then there exist relatively prime polynomials $g_0$, $g_1$ and a non-zero polynomial $d$ such that*

$$f_k = d\, g_0^{n-1-k} g_1^k, \quad k = 0, \ldots, n-1. \tag{10}$$

Applying this lemma to our case, we obtain that $d$ is a constant, $g_0$, $g_1$ are polynomials of degree at most one (and at least one of them has degree exactly one). Consequently, $\mathbf{B} \in \mathscr{B}_{[n-1]}$.

## 2. Binomial matrices and their properties

In the sequel, we shall denote by $R^n$, $C^n$ respectively the linear space of real (complex) column vectors with $n$ coordinates. In such spaces, we denote by $((\mathbf{x}, \mathbf{y}))$ the inner product of the vectors

$$\mathbf{x} = (x_1, \ldots, x_n)^{\mathrm{T}}, \quad \mathbf{y} = (y_1, \ldots, y_n)^{\mathrm{T}}, \quad \text{i.e.} \quad ((\mathbf{x}, \mathbf{y})) = \sum_{i=1}^{n} x_i \bar{y}_i$$

($\bar{y}$ is the complex conjugate number to $y$, the superscript T means transposition, the superscript $*$ transposition and complex conjugation).

We denote by $R^{m,n}$, $C^{m,n}$ respectively the set of all $m \times n$ real (complex) matrices.

**(2.1) Definition.** *For a positive integer $m$ and $x = (x_1, x_2)^{\mathrm{T}} \in C^2$, we denote by $\mathbf{x}^{[m]}$ the vector*

$$\mathbf{x}^{[m]} = \left(x_1^m, \binom{m}{1}^{1/2} x_1^{m-1}x_2, \binom{m}{2}^{1/2} x_1^{m-2}x_2^2, \ldots, x_2^m\right)^{\mathrm{T}} \in C^{m+1}$$

*and call it the m-binomial vector to* $\mathbf{x}$.

(2.2) Remark. The including of the binomial coefficients in the definition of $\mathbf{x}^{[m]}$ is justified by the following

$$((\mathbf{x}^{[m]}, \mathbf{y}^{[m]})) = ((\mathbf{x}, \mathbf{y}))^m. \tag{11}$$

**(2.3) Definition.** *For* $\mathbf{A} \in C^{2,2}$ *and* $m$ *positive integer,* $\mathbf{A}^{[m]}$ *is the matrix from* $C^{m+1,m+1}$ *for which, whenever* $\mathbf{x} \in R^2$,

$$(\mathbf{Ax})^{[m]} = \mathbf{A}^{[m]}\mathbf{x}^{[m]}. \tag{12}$$

We shall denote by $\mathscr{B}_R^{[m]}$, $\mathscr{B}_C^{[m]}$ respectively the set of all real (complex) matrices obtained as $\mathbf{A}^{[m]}$ for $\mathbf{A} \in R^{2,2}$ ($\mathbf{A} \in C^{2,2}$); we shall call $\mathbf{A}^{[m]}$ the $m$-binomial matrix corresponding to $\mathbf{A}$.

(2.4) Remark. The classes $\mathscr{B}^{[m]}$ are closely related to the class $\mathscr{B}_{[m]}$ of $m$-th Kronecker powers of $2\times2$ matrices mentioned above. Indeed, if $\mathbf{D} = \operatorname{diag}\left(\binom{m}{k}^{1/2}\right)$, $k = 0, \ldots, m$, then $\mathbf{P} \in \mathscr{B}^{[m]}$ if and only if $\mathbf{D}^{-1}\mathbf{PD} \in \mathscr{B}_{[m]}$.

(2.5) Example. Clearly $\mathbf{A}^{[1]} = \mathbf{A}$. If

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

then

$$\mathbf{A}^{[2]} = \begin{pmatrix} a_{11}^2, & \sqrt{2}\, a_{11}a_{12}, & a_{12}^2 \\ \sqrt{2}\, a_{11}a_{21}, & a_{11}a_{22} + a_{12}a_{21}, & \sqrt{2}\, a_{12}a_{22} \\ a_{21}^2, & \sqrt{2}\, a_{21}a_{22}, & a_{22}^2 \end{pmatrix}. \tag{13}$$

The following theorem comprises several elementary properties of $m$-binomial matrices. We denote by $\mathbf{I}$ the identity matrix; if its size should be emphasized, $\mathbf{I}_n$ is the $n\times n$ identity matrix.

**(2.6) Theorem.** *We have*

(a) $\mathbf{I}_2^{[m]} = \mathbf{I}_{m+1}$;

(b) *for* $\mathbf{A}, \mathbf{B} \in C^{2,2}$, $(\mathbf{AB})^{[m]} = \mathbf{A}^{[m]}\mathbf{B}^{[m]}$;

(c) *If* $\mathbf{A}, \mathbf{B} \in C^{2,2}$ *commute then* $\mathbf{A}^{[m]}$, $\mathbf{B}^{[m]}$ *commute as well;*

(d) *if* $\mathbf{A} \in C^{2,2}$ *is nonsingular then* $\mathbf{A}^{[m]}$ *is nonsingular and* $(\mathbf{A}^{[m]})^{-1} = (\mathbf{A}^{-1})^{[m]}$;

(e) $(\mathbf{A}^{[m]})^{\mathrm{T}} = (\mathbf{A}^{\mathrm{T}})^{[m]}$ *for* $\mathbf{A} \in C^{2,2}$;

(f) $(\mathbf{A}^{[m]})^* = (\mathbf{A}^*)^{[m]}$ *for* $\mathbf{A} \in C^{2,2}$;

(g) *if* $\mathbf{A} \in C^{2,2}$ *is lower triangular (upper triangular, diagonal) then so is* $\mathbf{A}^{[m]}$; *moreover, if* $a_{11}$, $a_{22}$ *are diagonal entries of* $\mathbf{A}$ *then*
$a_{11}^m, a_{11}^{m-1}a_{22}, a_{11}^{m-2}a_{22}^2, \ldots, a_{22}^m$

*are, in this order, the diagonal entries of* $\mathbf{A}^{[m]}$ *in each case;*

(h) *if* $\mathbf{A} \in C^{2,2}$ *is symmetric (Hermitian, orthogonal, unitary, normal) then* $\mathbf{A}^{[m]}$ *is symmetric (Hermitian, orthogonal, unitary, normal).*

Proof. All these properties follow in a standard way [2] from (2) and (1). We shall prove (b), (g) and a part of (h) only:

(b): Let $\mathbf{A}, \mathbf{B} \in C^{2,2}$, $\mathbf{x} \in C^2$, let $\mathbf{y} = \mathbf{B}\mathbf{x}$, $\mathbf{z} = \mathbf{A}\mathbf{B}\mathbf{x}$.

Then

$$\mathbf{z}^{[m]} = (\mathbf{A}\mathbf{y})^{[m]} = \mathbf{A}^{[m]}\mathbf{y}^{[m]} = \mathbf{A}^{[m]}\mathbf{B}^{[m]}\mathbf{x}^{[m]}.$$

On the other hand,

$$\mathbf{z}^{[m]} = (\mathbf{A}\mathbf{B})^{[m]}\mathbf{x}^{[m]}$$

so that

$$(\mathbf{A}\mathbf{B})^{[m]}\mathbf{x}^{[m]} = \mathbf{A}^{[m]}\mathbf{B}^{[m]}\mathbf{x}^{[m]}. \tag{14}$$

It is easily seen that $R^{m+1}$ possesses a basis of the form

$$\begin{pmatrix} 1 \\ t_1 \end{pmatrix}^{[m]}, \begin{pmatrix} 1 \\ t_2 \end{pmatrix}^{[m]}, \ldots, \begin{pmatrix} 1 \\ t_{m+1} \end{pmatrix}^{[m]}$$

(if $t_1, \ldots, t_{m+1}$ are mutually distinct since the determinant of the coordinates of these vectors is essentially the Vandermonde determinant). Consequently, (14) implies (b).

To prove (g), observe that for $\mathbf{A}$ lower triangular, the $k$-the coordinate of $(\mathbf{A}\mathbf{x})^{[m]}$ contains $x_2$ in the power at most $k-1$ and the coefficient at

$$\binom{m}{k-1}^{1/2} x_1^{m-k+1} x_2^{k-1} \quad \text{is} \quad a_{11}^{m-k+1} a_{22}^{k-1}.$$

To prove the first assertion of (h), observe that $\mathbf{A} = \mathbf{A}^{\mathrm{T}}$ is equivalent to $((\mathbf{A}\mathbf{x}, \mathbf{y})) = ((\mathbf{x}, \mathbf{A}\mathbf{y}))$ for all $\mathbf{x}, \mathbf{y} \in R^2$ so that by (11),

$$((\mathbf{A}^{[m]}\mathbf{x}^{[m]}, \mathbf{y}^{[m]})) = ((\mathbf{x}^{[m]}, \mathbf{A}^{[m]}\mathbf{y}^{[m]})).$$

The same reasoning as above yields that then

$$((\mathbf{A}^{[m]}\mathbf{X}, \mathbf{Y})) = ((\mathbf{X}, \mathbf{A}^{[m]}\mathbf{Y})) \quad \text{for all} \quad \mathbf{X}, \mathbf{Y} \in R^{m+1}$$

so that $\mathbf{A}^{[m]} = (\mathbf{A}^{[m]})^{\mathrm{T}}$. A similar argument proves (e) and (f).

(2.6) Remark. In the class $\mathscr{B}_{[k]}$, the properties (e), (f) are not satisfied in general.

**(2.7) Theorem.** *The classes* $\mathscr{B}_R^{[m]}$, $\mathscr{B}_C^{[m]}$ *are closed under multiplication, the nonsingular matrices of both classes forming a group (with respect to multiplication). If the upper-left-corner entry of a matrix* $\mathbf{P} \in \mathscr{B}_R^{[m]}$ *or* $\mathscr{B}_C^{[m]}$ *is different* *from zero then* $\mathbf{P} = \mathbf{A}^{[m]}\mathbf{B}^{[m]}$ *for some lower triangular matrix* $\mathbf{A} \in R^{2\times 2}$ (*or* $C^{2\times 2}$) *and some upper triangular matrix* $B \in B^{2\times 2}$. *Any matrix* $\mathbf{Q} \in \mathcal{B}_C^{[m]}$ *is equal to*

$$\mathbf{Q} = \mathbf{U}^{[m]}\mathbf{T}^{[m]}(\mathbf{U}^*)^{[m]} \tag{15}$$

*where* $\mathbf{U}$ *is a unitary and* $\mathbf{T}$ *an upper triangular matrix from* $C^{2\times 2}$.

Proof. The first two assertions are corollaries of Theorem (2.4). The remaining assertions follow from similar assertions for 2×2 matrices.

**(2.8) Theorem.** *If* $\alpha_1, \alpha_2$ *are eigenvalues of* $\mathbf{A} \in C^{2\times 2}$ *and* $m$ *is a positive integer then* $\alpha_1^m, \alpha_1^{m-1}\alpha_2, \alpha_1^{m-2}\alpha_2^2, \ldots, \alpha_2^m$ *are all eigenvalues of* $\mathbf{A}^{[m]}$. *In the case that* $\mathbf{A}$ *has linear elementary divisors, all elementary divisors of* $\mathbf{A}^{[m]}$ *are linear as well. In the case that* $\mathbf{A}$ *has one quadratic elementary divisor then for* $\mathbf{A}$ *nonsingular,* $\mathbf{A}^{[m]}$ *has a single elementary divisor of degree* $m+1$, *for* $\mathbf{A}$ *singular,* $\mathbf{A}^{[m]}$ *has one quadratic elementary divisor, all* $m-1$ *remaining ones being linear.*

*In the first case, eigenvectors of* $\mathbf{A}^{[m]}$ *corresponding to* $\alpha_1^m, \alpha_1^{m-1}\alpha_2, \ldots, \alpha_2^m$ *can be chosen as columns of the matrix* $\mathbf{X}^{[m]}$ *where* $\mathbf{X}$ *is a matrix whose columns are some two linearly independent eigenvectors of* $\mathbf{A}$.

Proof. Follows easily from the Jordan theorem since $\mathbf{A} = \mathbf{T}\mathbf{J}_\mathbf{A}\mathbf{T}^{-1}$, $\mathbf{J}_\mathbf{A}$ being either diagonal or of the form $\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$, implies

$$\mathbf{A}^{[m]} = \mathbf{T}^{[m]}\mathbf{J}_\mathbf{A}^{[m]}(\mathbf{T}^{[m]})^{-1};$$

$\mathbf{J}_\mathbf{A}$ being always upper triangular, (g) of Theorem (2.5) applies. The asserted properties of elementary divisors of $\mathbf{A}^{[m]}$ are easily checked.

For $\mathbf{J}_\mathbf{A}$ diagonal and $\mathbf{X}$ a matrix described above, $\mathbf{AX} = \mathbf{XJ}_\mathbf{A}$ implies

$$\mathbf{A}^{[m]}\mathbf{X}^{[m]} = \mathbf{X}^{[m]}\mathbf{J}_\mathbf{A}^{[m]}$$

and $\mathbf{J}_\mathbf{A}^{[m]}$ being again diagonal, the assertion follows.

Since the determinant is the product of all eigenvalues, we have:

**(2.9) Corollary.** *For* $\mathbf{A} \in C^{2\times 2}$

$$\det \mathbf{A}^{[m]} = (\det \mathbf{A})^{\binom{m}{2}}.$$

**(2.10) Corollary.** *The rank of a matrix in* $\mathcal{B}^{[m]}$, $m \geqq 1$, *is either* $m+1$, *or* 1, *or* 0.

**(2.11) Theorem.** *If* $\mathbf{A}$ *is positive semidefinite (positive definite) then so is* $\mathbf{A}^{[m]}$.

Proof. In such case there exists a unitary matrix $\mathbf{U}$ and a diagonal matrix $\mathbf{D}$ with nonnegative (positive) diagonal entries such that

$$\mathbf{A} = \mathbf{UDU}^*. \tag{16}$$

Therefore,

$$\mathbf{A}^{[m]} = \mathbf{U}^{[m]}\mathbf{D}^{[m]}(\mathbf{U}^{[m]})^* \tag{17}$$

where $\mathbf{U}^{[m]}$ is unitary and $\mathbf{D}^{[m]}$ diagonal with nonnegative (positive) diagonal entries. The assertion follows.

**(2.12) Theorem.** *For any positive definite* $\mathbf{P} \in \mathscr{B}_C^{[m]}$, *its positive definite square root, commuting with* $\mathbf{P}$, *is in* $\mathscr{B}_C^{[m]}$ *as well.*

Proof. Let $\mathbf{P} \in \mathscr{B}_C^{[m]}$ satisfy $\mathbf{P} = \mathbf{A}^{[m]}$ for $\mathbf{A} \in C^{2,2}$. Since $\mathbf{P} = \mathbf{P}^*$, $\mathbf{A} = \mathbf{A}^*$ as well and (16) holds with $\mathbf{D}$ having positive diagonal entries. Define $\mathbf{B} = \mathbf{U}\mathbf{D}^{1/2}\mathbf{U}^*$ where the diagonal entries of $\mathbf{D}^{1/2}$ are positive square roots of the diagonal entries of $\mathbf{D}$. Since

$$\mathbf{B}^2 = \mathbf{A}, \quad \mathbf{AB} = \mathbf{BA},$$

the matrix $\mathbf{Q} = \mathbf{B}^{[m]}$ satisfies $\mathbf{Q}^2 = \mathbf{P}$, $\mathbf{PQ} = \mathbf{QP}$ and is positive definite by Theorem (2.11).

## 3. Additive binomial matrices

**(3.1) Definition.** *Let* $\mathbf{A} \in C^{2,2}$, *m positive integer and k integer,* $0 \leqq k \leqq m$. *The generalized m-binomial matrices* $\mathbf{A}^{[m,k]}$ *are defined as coefficient matrices in* $(\mathbf{I} + t\mathbf{A})^{[m]}$:

$$(\mathbf{I} + t\mathbf{A})^{[m]} = \sum_{k=0}^{m} t^k \mathbf{A}^{[m,k]}. \tag{18}$$

In particular, the matrix $\mathbf{A}^{[m,1]}$ will be called additive $m$-binomial matrix of $\mathbf{A}$.

**(3.2) Theorem.** *For a fixed* $\mathbf{A}$ *and fixed m, all the matrices* $\mathbf{A}^{[m,k]}$, $k = 0, \ldots, m$, *commute with each other;* $\mathbf{A}^{[m,0]} = \mathbf{I}$, $\mathbf{A}^{[m,m]} = \mathbf{A}^{[m]}$. *If* $\mathbf{A}$ *has eigenvalues* $\alpha_1$, $\alpha_2$ *then all eigenvalues of* $\mathbf{A}^{[m,k]}$ *are* $f_{k0}, f_{k1}, \ldots, f_{km}$ *where the numbers* $f_{ks}$ *are coefficients of the polynomials*

$$(1 + t\alpha_1)^{m-s}(1 + t\alpha_2)^s = f_{0s} + f_{1s}t + \ldots + f_{ms}t^m (= f_s(t)), \quad s = 0, \ldots, m.$$

Proof. By (c) of Theorem (2.5), the matrices $\sum_{k=0}^{m} t^k \mathbf{A}^{[m,k]}$ (for varying $t$) commute with each other. Therefore, any two matrices of the form $\sum_{k=0}^{m} \gamma_k \mathbf{A}^{[m,k]}$ commute.

If $\mathbf{A}$ is diagonizable, $\mathbf{T}^{-1}\mathbf{AT} = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$ for some nonsingular $\mathbf{T}$. Consequently,

$$(\mathbf{T}^{[m]})^{-1}(\mathbf{I} + t\mathbf{A})^{[m]}\mathbf{T}^{[m]} = \begin{pmatrix} 1 + t\alpha_1 & 0 \\ 0 & 1 + t\alpha_2 \end{pmatrix}^{[m]} =$$

$$= \operatorname{diag}((1 + t\alpha_1)^m, (1 + t\alpha_1)^{m-1}(1 + t\alpha_2), \ldots, (1 + t\alpha\text{-}2)^m) =$$

$$= \operatorname{diag}(f_0(t), f_1(T), \ldots, f_m(t)).$$

It follows easily that the eigenvalues of $\sum_{k=0}^{m} \gamma_k \mathbf{A}^{[m,k]}$ are equal to $f_0(\gamma), f_1(\gamma), \dots, f_m(\gamma)$ where symbolically

$$f_s(\gamma) = f_{0s}\gamma_0 + f_{1s}\gamma_1 + \dots + f_{ms}\gamma_m .$$

The same is true if $\mathbf{A}$ is not diagonalizable.

In the following theorem we shall summarize properties of the additive binomial matrices.

**(3.3) Theorem.** *For* $\mathbf{A}, \mathbf{B} \in C^{2,2}$,

$$(\mathbf{A}+\mathbf{B})^{[m,1]} = \mathbf{A}^{[m,1]} + \mathbf{B}^{[m,1]}. \tag{19}$$

*If* $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ *has eigenvalues* $\alpha_1, \alpha_2$ *then:*

(a) $\mathbf{A}^{[m,1]}$ *is tridiagonal with the entries*

$\mathbf{A}^{[m,1]}_{kk} = (m-k)a_{11} + ka_{22}, \quad k = 0, \dots, m,$

$\mathbf{A}^{[m,1]}_{k,k+1} = a_{12}\sqrt{(k+1)(m-k)}, \quad k = 0, \dots, m-1,$

$\mathbf{A}^{[m,1]}_{k+1,k} = a_{21}\sqrt{(k+1)(m-k)}, \quad k = 0, \dots, m-1,$

$\mathbf{A}^{[m,1]}_{ij} = 0$ *in all other cases;*

(b) *the eigenvalues of* $\mathbf{A}^{[m,1]}$ *are* $(m-s)\alpha_1 + s\alpha_2$, $s = 0, \dots, m$;

(c) *if* $\mathbf{A}$ *is positive semidefinite (positive definite), the same is true of* $\mathbf{A}^{[m,1]}$.

Proof. (19) follows from the definition, (a) by direct computation, (b) is a corollary of Theorem (3.2) and (c) follows from the commutativity property and (17).

(3.4) Remark. In Theorem (3.3), (b) means, of course, that the eigenvalues of $\mathbf{A}^{[m,1]}$ correspond in the complex plane to $m+1$ equidistant points on the segment joining the points $m\alpha_1$ and $m\alpha_2$.

(3.5) Remark. The matrix $\mathbf{A}^{[m,1]}$ being nonderogatory [4], it follows from Theorem (3.2) that the matrices $\mathbf{A}^{[m,k]}$, $k = 2, \dots, m$ are polynomials in $\mathbf{A}^{[m,1]}$. For instance, the matrix $\mathbf{A}^{[2]}$ from (13) can be expressed as

$$(\det \mathbf{A})\mathbf{I} - \tfrac{1}{2}(a_{11}+a_{22})\,\mathbf{A}^{[2,1]} + \tfrac{1}{2}(\mathbf{A}^{[2,1]})^2.$$

Several other properties of binomial matrices follow from analogous properties of matrices in $C^{2,2}$. An example is the following:

**(3.6) Theorem.** *If* $\mathbf{A} \in R^{2,2}$ *is (elementwise) nonnegative then all matrices* $\mathbf{A}^{[m,k]}$, $k = 0, \dots, m$ *(and thus* $\mathbf{A}^{[m]}$*) are nonnegative as well. If* $\mathbf{A}$ *is positive,* $\mathbf{A}^{[m]}$ *is positive.*

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## БИНОМИАЛЬНЫЕ МАТРИЦЫ

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### Резюме

В связи с классом $\mathcal{B}_{[k]}$ кронекеровских степеней [1] матриц порядка 2 доказывается, что невырожденная матрица порядка $n$ принадлежит $\mathcal{B}_{[n-1]}$ тогда и только тогда, когда $\mathbf{BHB}'$ является матрицей Ганкеля для всех матриц Ганкеля $\mathbf{H}$. Во второй части модифицируется определение класса $\mathcal{B}_{[k]}$ и изучается полученный класс $\mathcal{B}^{[k]}$ т. наз. биномиальных матриц. Также изучается аддитивная версия биномиальных матриц.