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DARBOUX PROPERTY OF FINITELY ADDITIVE MEASURE ON δ-RING

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1. Introduction. Let $\mathscr{S}$ be a δ-ring (i.e. a ring closed over countable intersections) and let φ be a positive finitely additive measure on $\mathscr{S}$, i.e. φ is a real function defined on $\mathscr{S}$, where

(i) $E\in\mathscr{S}\Rightarrow 0\leq\varphi(E)<\infty$,
(ii) $E\in\mathscr{S},\ F\in\mathscr{S},\ E\cap F=\emptyset\Rightarrow\varphi(E\cup F)=\varphi(E)+\varphi(F)$.

Definition. *We say that a set $E\in\mathscr{S}$ is an atom (with respect to φ) if $\varphi(E)>0$ and if for every set $A\in\mathscr{S}$, $A\subset E$ we have either $\varphi(A)=0$ or $\varphi(A)=\varphi(E)$. We say that φ is nonatomic on E if $\mathscr{S}$ contains no atom $A\subset E$. We say that φ has the Darboux property on E if for every number α such that $0<\alpha<\varphi(E)$ there exists a set $A\in\mathscr{S}$, $A\subset E$ and such that $\varphi(A)=\alpha$. We say that φ is nonatomic (φ has the Darboux property) if φ is nonatomic (φ has the Darboux property) on E for every set $E\in\mathscr{S}$.*

This is an extended definition of that in [2, §2, 9, def. 4] and moreover we use the formulation "*φ has the Darboux property on E*" instead of "*E has the Darboux property with respect to φ*".

It is well known that if φ is countably additive on $\mathscr{S}$ and σ-finite on E, then a sufficient condition for φ having the Darboux property on E is the fact that φ is nonatomic on E ([2, §2, 9, prop. 7]).

In this paper it will be shown in section 2 that the preceding proposition for a finitely additive measure is false in general. We shall prove the existence of a nonatomic positive finitely additive measure without the Darboux property.

In section 3 there are given some sufficient conditions for a finitely additive measure φ having the Darboux property, namely by means of the decomposition of φ on the σ-additive part and the purely additive part by [4].

2. Nonatomic measure and the Darboux property.

Lemma 1 ([3, lemme 1]). *Let E be an arbitrary set and let $\mathscr{K}$ be a nonempty class of subsets of the set E. There exists a set function m, where the following conditions hold;*

(α) *m attains on every subset of the set E one of the two values* 0 *and* 1,
(β) *if* $A \subset E$, $B \subset E$ *and* $A \cap B = \emptyset$, *then* $m(A \cup B) = m(A) + m(B)$,
(γ) *if* $A \in \mathcal{K}$, *then* $m(A) = 0$,
(δ) $m(E) = 1$,

if and only if there exists a class of sets $\mathcal{M}$, *satisfying the next conditions:*
(a) *every element of the class* $\mathcal{M}$ *is a subset of the set E,*
(b) *if* $A \in \mathcal{M}$ *and* $B \in \mathcal{M}$, *then* $A \cup B \in \mathcal{M}$,
(c) *if* $A \subset \mathcal{M}$ *and* $B \subset A$, *then* $B \in \mathcal{M}$,
(d) $E \notin \mathcal{M}$,
(e) *if* $A \subset E$, *then* $A \in \mathcal{M}$ *or* $E - A \in \mathcal{M}$,
(f) $\mathcal{K} \subset \mathcal{M}$.

Note 1. In lemma 1 we can write an arbitrary positive real number a instead of 1.

Note 2. The conditions (a)—(d) ((a)—(e)) are equivalent to the fact that $\mathcal{M}$ is an ideal (maximal ideal) in E.

Lemma 2. *Let* μ *be an arbitrary nontrivial positive finitely additive measure defined on the* σ*-algebra of all subsets of the set of positive integers (i.e. on* 2^N*). Then for an arbitrary positive real number a there exists a nontrivial positive finitely additive measure* ν *defined on* 2^N *such that either* $\nu(A) = 0$ *or* $\nu(A) = a$ *for every* $A \in 2^N$ *and*

$$\mu(A) = 0 \Rightarrow \nu(A) = 0.$$

Proof. Put $\mathcal{K} = \{A \subset N: \mu(A) = 0\}$. Evidently in $\mathcal{K}$ holds conditions (a)—(d), i.e. $\mathcal{K}$ is an ideal in N and it can be extended to a maximal ideal $\mathcal{M}$, for which the conditions (a)—(f) are fulfilled. From lemma 1 it follows that there exists a nontrivial two-valued positive finitely additive measure ν defined on 2^N and vanishing on $\mathcal{K}$, i.e. such that $\mu(A) = 0 \Rightarrow \nu(A) = 0$.

Let $A \subset N$. Denote $A(n)$ the number of elements of A, which are smaller or equal to n, i.e.

$$A(n) = \sum_{a \in A,\, a \leq n} 1.$$

Put $\alpha_n = n^{-1} A(n)$. The numbers $h_*(A) = \liminf_n \alpha_n$ and $h^*(A) = \limsup_n \alpha_n$ are called lower and upper asymptotic density of the set A, respectively. If $h_*(A) = h^*(A)$, i.e. if there exists $\lim_n \alpha_n$, it is called the asymptotic density of the set A and is denoted $h(A)$. The set function h is positive finitely additive on the family $\mathcal{H} \subset 2^N$ of all subsets of the set N for which there exists the asymptotic density and it can be extended to a positive finitely additive measure μ, defined on 2^N such that

$$\text{(I)} \qquad h_*(A) \leq \mu(A) \leq h^*(A) \qquad ([1, \text{p. } 231]).$$

For example we can put $\mu(A)=\text{Lim}\ \alpha_n$, where Lim means the Banach limit ([1, p. 34]).

Lemma 3. *The measure μ is nonatomic on N.*

Proof. Let $A\subset N$ and $\mu(A)>0$. Let k be an integer, for which $k^{-1}h^*(A)<\mu(A)$. Decompose the set $A=\{a_1, a_2, a_3, \dots\}$ into k sets

$$B_1=\{a_1, a_{k+1}, \dots, a_{mk+1}, a_{(m+1)k+1}, \dots\}$$
$$B_2=\{a_2, a_{k+2}, \dots, a_{mk+2}, a_{(m+1)k+2}, \dots\}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$B_k=\{a_k, a_{2k}, \dots, a_{(m+1)k}, a_{(m+2)k}, \dots\}.$$

For an arbitrary $i\leqslant k$ put

$$B_i(n)=\sum_{a\in B_i,\, a\leqslant n} 1.$$

Then $B_i(n)=m+1$ if and only if

$$a_{mk+i}\leqslant n<a_{(m+1)k+i}.$$

It follows that

$$mk+i=(m+1)k+i-k\leqslant A(n)<(m+1)k+i$$

and therefore

$$B_i(n)+\frac{i-k}{k}\leqslant\frac{A(n)}{k}<B_i(n)+\frac{i}{k}.$$

Hence multiplying n^{-1} and passing $n\to\infty$ we obtain

$$h^*(B_i)=\limsup_n\frac{B_i(n)}{n}=\limsup_n\frac{A(n)}{kn}=\frac{1}{k}h^*(A).$$

The measure μ is additive and $B_i\cap B_j=\emptyset$ for $i\neq j$. It follows that there exists an integer $i_0\leqslant k$ such that $\mu(B_{i_0})>0$. Summarizing we have

$$0<\mu(B_{i_0})\leqslant h^*(B_{i_0})=\frac{1}{k}h^*(A)<\mu(A).$$

Consequently A is not an atom.

Theorem 1. *There exists a positive finitely additive measure, which is nonatomic and which has not the Darboux property.*

Proof. Let $\mathcal{S}$ be the σ-algebra of all subsets of the set N of positive integers. Let μ be the measure defined above for which (I) holds. According to lemma 2 there exists a positive finitely additive measure ν defined on $\mathcal{S}$ such that $\nu(A)=0$ or $\nu(A)=2$ and $\mu(A)=0\Rightarrow\nu(A)=0$ for every set $A\in\mathcal{S}$. Put $\varphi=\mu+\nu$. Then φ is a positive finitely additive measure on $\mathcal{S}$. Let $\varphi(A)>0$. If there were $\mu(A)=0$, then $\nu(A)=0$ as well and hence $\varphi(A)=\mu(A)+\nu(A)=0$. Consequently $\mu(A)>0$. Since μ is a nonatomic measure, there exists a set $B\in\mathcal{S}$ such that $B\subset A$ and $0<\mu(B)<\mu(A)$. But then

$$0<\varphi(B)=\mu(B)+\nu(B)<\mu(A)+\nu(A)=\varphi(A).$$

Consequently A is not an atom with respect to φ and therefore φ is nonatomic on N. Since φ attains no value in the open interval $(1, 2)$, φ has not the Darboux property on N.

3. Decomposition and the Darboux property.

Let φ be a positive finitely additive measure defined on a δ-ring φ of subsets of a set X. Let for a set $E\in\mathcal{S}$ be $0<\varphi(E)<\infty$. According to [4, 1.24] there exists one and only one pair of positive measures ψ, π defined on the σ-algebra $\mathcal{S}_E=\{A\in\mathcal{S}: A\subset E\}$ such that ψ is σ-additive, π is purely additive and $\varphi(A)=\psi(A)+\pi(A)$ for every $A\in\mathcal{S}_E$.

Theorem 2. *The measure φ has the Darboux property on the set $E\in\mathcal{S}$ if at least one of the following two conditions is fulfilled:*

(i) $\psi(E)>\pi(E)$ *and ψ is nonatomic on E,*

(ii) $\psi(E)\leqslant\pi(E)$ *and π has the Darboux property on A for every $A\in\mathcal{S}_E$.*

Proof. It is sufficient to show that for every number $\alpha\in(0, 2^{-1}\varphi(E)\rangle$ there exists a set $F\in\mathcal{S}$, $F\subset E$ and such that $\varphi(F)=\alpha$. In case of $\alpha\in(2^{-1}\varphi(E), \varphi(E))$, let us denote $\alpha'=\varphi(E)-\alpha$. Then $\alpha'\in(0, 2^{-1}\varphi(E))$ and if $F'\in\mathcal{S}$ is such that $\varphi(F')=\alpha'$, then for the set $F=E-F'$ we have

$$\varphi(F)=\varphi(E-F')=\varphi(E)-\varphi(F')=\varphi(E)-\alpha'=\alpha.$$

Let the condition (i) hold and let $\alpha\in(0, 2^{-1}\varphi(E)\rangle\subset(0, \psi(E))$. According to [4, 1.19] there exists a set $B\in\mathcal{S}$, $B\subset E$ and such that $\pi(E-B)=0$ and $\psi(B)<\varepsilon=\psi(E)-\alpha$. Whence $\alpha<\psi(E)-\psi(B)=\psi(E-B)$. Since by (i) ψ is nonatomic on E, it is nonatomic on $E-B$ as well. Therefore ψ has the Darboux property on $E-B$ ([2, §2, prop. 7]). It follows that there exists a set $A\in\mathcal{S}$, $A\subset E-B$ and such that $\psi(A)=\alpha$. Since $0\leqslant\pi(A)\leqslant\pi(E-B)=0$, we have $\pi(A)=0$ and

$$\varphi(A)=\psi(A)+\pi(A)=\alpha.$$

Now let (ii) hold and let $\alpha\in(0, 2^{-1}\varphi(E)\rangle\subset(0, \pi(E)\rangle$. According to [4, 1.19] there exists a set $C_1\in\mathcal{S}_E$ such that $\pi(C_1)=\pi(E)$ and $\psi(C_1)<\varepsilon_1=\min\{1, \alpha\}$. Since π has the Darboux property on every $A\in\mathcal{S}_E$ and $C_1\in\mathcal{S}_E$, there exists a set $A_1\in\mathcal{S}_E$, $A_1\subset C_1$ and such that $\pi(A_1)=\alpha$. Since $A_1\subset C_1$ we have $\psi(A_1)<\varepsilon_1$ and consequently

$$\alpha \leqslant \psi(A_1) + \pi(A_1) < \alpha + \varepsilon_1 ,$$

whence

$$\alpha \leqslant \varphi(A_1) \leqslant \alpha + \varepsilon_1 .$$

Since π has the Darboux property on A_1, there exists a set $B_1 \in \mathcal{S}$, $B_1 \subset A_1$ and such that $\pi(B_1) = \alpha - \varepsilon_1$. From the fact $B_1 \subset A_1 \subset C_1$ it follows that $\psi(B_1) < \varepsilon_1$. Consequently

$$\alpha - \varepsilon_1 = \pi(B_1) \leqslant \psi(B_1) + \pi(B_1) < \alpha ,$$

whence

$$\alpha - \varepsilon_1 \leqslant \varphi(B_1) < \alpha .$$

Finally we have

$$\pi(A_1 - B_1) = \pi(A_1) - \pi(B_1) = \alpha - \pi(B_1) \geqslant \alpha - \varphi(B_1) > 0 .$$

Summarizing, there exists a number ε_1, $0 < \varepsilon_1 \leqslant 1$ and sets $B_1 \subset A_1$ such that $\pi(A_1 - B_1) \geqslant \alpha - \varphi(B_1)$ and

$$0 \leqslant \alpha - \varepsilon_1 \leqslant \varphi(B_1) < \alpha \leqslant \varphi(A_1) < \alpha + \varepsilon_1 .$$

Suppose now that there exists a number ε_n, $0 < \varepsilon_n \leqslant n^{-1}$ and sets $B_n \subset A_n$ such that $\pi(A_n - B_n) \geqslant \alpha - \varphi(B_n)$ and

$$0 \leqslant \alpha - \varepsilon_n \leqslant \varphi(B_n) < \alpha \leqslant \varphi(A_n) < \alpha + \varepsilon_n .$$

According to [4, 1.19] there exists a set $C_{n+1} \in \mathcal{S}_E$, $C_{n+1} \subset A_n - B_n$ such that $\pi(C_{n+1}) = \pi(A_n - B_n)$ and $\psi(C_{n+1}) < \varepsilon_{n+1} = \min\{(n+1)^{-1}, \alpha - \varphi(B_n)\}$. Since π has the Darboux property on C_{n+1}, there exists a set $A'_{n+1} \subset C_{n+1}$ such that $\pi(A'_{n+1}) = \alpha - \varphi(B_n)$. Since $A'_{n+1} \subset C_{n+1}$, then $\psi(A_{n+1}) < \varepsilon_{n+1}$. Putting $A_{n+1} = B_n \cup A'_{n+1}$, we obtain

$$\alpha = \varphi(B_n) + \pi(A'_{n+1}) \leqslant \varphi(B_n) + \pi(A'_{n+1}) + \psi(A'_{n+1}) < \alpha + \varepsilon_{n+1}$$

and since B_n, A'_{n+1} are disjoint, it follows

$$\alpha \leqslant \varphi(A_{n+1}) < \alpha + \varepsilon_{n+1} .$$

According to the assumption of this theorem, π has the Darboux property on A'_{n+1}. It follows that there exists a set $B'_{n+1} \subset A'_{n+1}$ such that $\pi(B'_{n+1}) = \alpha - \varphi(B_n) - \varepsilon_{n+1}$. Since $B'_{n+1} \subset A'_{n+1} \subset C_{n+1}$, then $\psi(B'_{n+1}) < \varepsilon_{n+1}$, consequently

$$0 \leqslant \alpha - \varepsilon_{n+1} = \varphi(B_n) + \pi(B'_{n+1}) \leqslant \varphi(B_n) + \pi(B'_{n+1}) + \psi(B'_{n+1}) < \alpha .$$

Putting $B_{n+1} = B_n \cup B'_{n+1}$, we obtain, since B_n, B'_{n+1} are disjoint

$$\alpha - \varepsilon_{n+1} \leqslant \varphi(B_{n+1}) < \alpha .$$

Further, we have

$$\pi(A_{n+1}-B_{n+1})=\pi((B_n\cup A'_{n+1})-(B_n\cup B'_{n+1}))=$$
$$=\pi(A'_{n+1}-B'_{n+1})=\pi(A'_{n+1})-\pi(B'_{n+1})=\alpha-\varphi(B_n)-$$
$$-(\alpha-\varphi(B_n)-\varepsilon_{n+1})=\varepsilon_{n+1}\geqslant\alpha-\varphi(B_{n+1})>0.$$

Summarizing, there exists a real number ε_{n+1}, $0<\varepsilon_{n+1}\leqslant(n+1)^{-1}$ and sets A_{n+1}, B_{n+1} such that $B_n\subset B_{n+1}\subset A_{n+1}\subset A_n$, $\pi(A_{n+1}-B_{n+1})\geqslant\alpha-\varphi(B_{n+1})$ and

$$0\leqslant\alpha-\varepsilon_{n+1}\leqslant\varphi(B_{n+1})<\alpha\leqslant\varphi(A_{n+1})<\alpha+\varepsilon_{n+1}.$$

In this way we have found by means of recurrence sequences $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ of sets belonging to $\mathscr{S}_E$ such that

$$B_1\subset B_2\subset\ldots\subset B_n\subset\ldots\subset A_n\subset\ldots\subset A_2\subset A_1$$

and

$$\alpha-\frac{1}{n}\leqslant\varphi(B_n)<\alpha\leqslant\varphi(A_n)<\alpha+\frac{1}{n}$$

for every positive integer n. Denote

$$A=\bigcap_{n=1}^{\infty}A_n.$$

Then $A\in\mathscr{S}_E$ and for every n we have $B_n\subset A\subset A_n$. It follows that

$$\alpha-\frac{1}{n}\leqslant\varphi(B_n)\leqslant\varphi(A)\leqslant\varphi(A_n)<\alpha+\frac{1}{n},$$

for every n and consequently

$$\varphi(A)=\alpha.$$

REFERENCES

[1] BANACH, S.: Théorie des opérations linéaires.
[2] DINCULEANU, N.: Vector measures, Berlin 1966.
[3] TARSKI, A.: Une contribution a la théorie de la measure, Fundam. Math. 15, 1930, 42—50.
[4] YOSHIDA, K., HEWITT, E.: Finitely additive measures, Trans. Amer. Math. Soc. 72, 1952, 46—66.

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СВОЙСТВО ДАРБУ КОНЕЧНО АДДИТИВНОЙ МЕРЫ НА δ-КОЛЬЦЕ

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Резюме

Мера φ определена на кольце $\mathscr{S}$ обладает свойством Дарбу на множестве $E \in \mathscr{S}$, если для произвольного числа α из интервала $(0, \varphi(E))$ существует множество $A \in \mathscr{S}$, $A \subset E$, для которого $\varphi(A) = \alpha$. Говорят что φ обладает свойством Дарбу, если φ обладает свойством Дарбу на E для каждого $E \in \mathscr{S}$.

Теорема 1. *Существует конечно аддитивная мера, которая является неатомической и не обладает свойством Дарбу.*

Если φ конечно аддитивная мера на δ-кольце $\mathscr{S}$ и ψ и π образуют ее разложение на σ-аддитивную и чисто аддитивную часть на σ-алгебре $\mathscr{S}_E = \{A \in \mathscr{S}: A \subset E\}$, то имеет место следующая теорема.

Теорема 2. *Мера φ обладает свойством Дарбу на множестве E, если выполнено по крайней мере одно из следующих условий:*

(и) $\psi(E) > \pi(E)$ *и ψ является неатомической на E,*

(ии) $\psi(E) \leqslant \pi(E)$ *и π обладает свойством Дарбу на A для каждого $A \in \mathscr{S}_E$.*