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DOMATIC NUMBER AND LINEAR ARBORICITY OF CACTI

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A cactus is a connected undirected graph with at least two vertices and with the property that each of its edges is contained in at most one of its circuits. Every tree is a cactus, but not conversely. In this paper we shall prove two theorems on numerical invariants of cacti.

The domatic number of a graph was introduced by E. J. Cockayne and S. T. Hedetniemi [2]. A dominating set in an undirected graph G is a subset D of the vertex set $V(G)$ of G with the property that to each vertex $x \in V(G) - D$ there exists a vertex $y \in D$ adjacent to x. A domatic partition of G is a partition of $V(G)$, all of whose classes are dominating sets in G. The maximal number of classes of a domatic partition of G is called the domatic number of G and denoted by $d(G)$.

The domatic number of G may be defined also as the maximal number of colours of a domatic colouring of G. A domatic colouring of G is a colouring of vertices of G with the property that to each vertex x of G and to each clour c distinct from the colour of x there exists a vertex of G having the colour c and adjacent to x. (Two vertices of the same colour may be adjacent.) Evidently both definitions are equivalent. The set of all vertices coloured by a given colour is a class of the corresponding domatic partition of G.

The linear arboricity of a graph was introduced by J. Akiyama, G. Exoo and F. Harary [1]. A linear forest is an undirected graph, all of whose connected components are paths. The linear arboricity $\Xi(G)$ of a graph G is the minimum number of edge-disjoint subgraphs of G which are linear forests and whose union is whole the graph G.

Before formulating theorems, we shall explain briefly the structure of cacti. Each block of a cactus is either a circuit, or an edge with its end vertices. A cactus consisting of one block will be called trivial; other cacti will be called non-trivial. A block of a non-trivial cactus G which contains only one articulation of G will be called a terminal block of G. Evidently each finite non-trivial cactus has at least two terminal blocks.

A path in a graph G whose inner vertices (if any) have degree 2 and whose terminal vertices have degrees different from 2 will be called a simple path. If a circuit C in a non-trivial cactus G does not form a terminal block, then it contains at least two articulations and is the union of at least two edge-disjoint simple paths; each of these paths connectes two articulations of G. The set of these paths will be denoted by $\mathcal{S}(G)$.

For trivial cacti the domatic number is well known. If such a cactus consists of one edge with its end vertices, then evidently its domatic number is 2. If such a cactus G is a circuit, then $d(G)=3$ if and only if the length of this circuit is divisible by 3, otherwise $d(G)=2$; this was proved by E. J. Cockayne and S. T. Hedetniemi. Thus it remains to consider non-trivial cacti.

Theorem 1. *Let G be a finite non-trivial cactus. Then the following two assertions are equivalent:*

(i) *Each terminal block of G is a circuit of a length divisible by 3 and for any circuit C in G not forming a terminal block the set $\mathcal{S}(C)$ contains either at least one path of length 1, or the number of paths of $\mathcal{S}(C)$ with lengths non-divisible by 3 is different from 1.*

(ii) *$d(G)=3$ and there exists a domatic partition of G with 3 classes such that each vertex is adjacent to at most one vertex of the same class and any edge joining two vertices of the same class belongs to a circuit.*

If (i) *does not hold, then $d(G)=2$.*

Proof. First we prove that $2\leqq d(G)\leqq 3$ for any finite cactus G. The inequality $2\leqq d(G)$ follows from the fact that a cactus has no isolated vertices [2]. The inequality $d(G)\leqq 3$ follows from the fact that any cactus contains at least one vertex of degree 1 or 2 (in a non-trivial cactus such a vertex is in its terminal block) and thus the minimal degree $\delta(G)\leqq 2$; in [2] it was proved that $d(G)\leqq \delta(G)+1$. Now we prove the equivalence of (i) and (ii).

(i) $\Rightarrow$ (ii). The proof will be done by induction according to the number k of non-terminal blocks of G. Let $k=0$. Then G has only one articulation a which is common to all blocks of G. We shall construct a domatic colouring of G. The vertex a will be coloured by the colour 1. Now let B be a block of G; the block B is a terminal one, hence (if we suppose (i)) it is a circuit of a length divisible by 3. We colour its vertices subsequently by 1, 2, 3, 1, 2, 3, ..., starting at a. If we do this with each block of G, we obtain a domatic colouring by 3 colours and $d(G)=3$.

Now let $k=k_0\geqq 1$ and suppose that the assertion is true for $k=k_0-1$. Choose a non-terminal block B_0 of G and contract all its vertices (the obtained loops are omitted). We obtain a cactus G_0 with $k=k_0-1$. The contraction of B_0 does not change other blocks; thus if G satisfies (i), so does G_0. Then there exists a domatic colouring of G_0 by three colours satisfying (ii). Let w be the vertex of G_0 obtained by contracting the block B_0; without loss of generality we may suppose that it is coloured by the colour 1. Suppose that B_0 consists of one edge with its end vertices u, v. At least one of them, say u, is adjacent in G to a vertex coloured in the mentioned colouring of G_0 by 2. Then u will be coloured by 1 and v by 3. All vertices of G which are separated from v by u will have the same colours as in the colouring of G_0. If v is adjacent to a vertex coloured by 3, then for all vertices separated from u by v the colours 1 and 2 are mutually interchanged and the colour 3 is preserved. In the opposite case the colour 2 is changed to 3, the colour 1 to 2 and the colour 3 to 1. The obtained colouring is a domatic colouring of G by three colours.

Now suppose that B_0 is a circuit. Then either $\mathcal{S}(B_0)$ contains at least one path of the length 1, or the number of paths of lengths non-divisible by 3 is different from 1. In the first case B_0 contains two adjacent articulations u, v of G. Then we go along B_0 starting at u and ending at v (omitting the edge uv) and colour the vertices of B_0 subsequently by 1, 2, 3, 1, 2, 3, ... In the second case we colour first all articulations in such a way that any two articulations connected by a path from $\mathcal{S}(B_0)$ have equal (or different) colours if such a path has a length divisible (or non-divisible, respectively) by 3. The reader himself may verify that under the above mentioned condition this is possible. Further we colour all other vertices of B_0. Let $P \in \mathcal{S}(B_0)$, let its vertices be $u_0, u_1, \ldots, u_m$ and edges u_iu_{i+1} for $i=0, \ldots, m-1$. The vertices u_0, u_m are articulations of G. If m is divisible by 3, then u_0 and u_m have the same colour. The vertices of P will be coloured so that two vertices u_i, u_j $(0 \leqq i \leqq m,\ 0 \leqq j \leqq m)$ have the same colour if and only if $i \equiv j \pmod 3$. If m is not divisible by 3, then u_0 and u_m have different colours. The vertices of P will be coloured so that two vertices u_i, u_j for $0 \leqq i \leqq m-1$, $0 \leqq j \leqq m-1$ have again the same colour if and only if $i \equiv j \pmod 3$ and further u_{m-1} has another colour than u_m. Thus we obtain a colouring of B_0 in which any vertex is adjacent to a vertex of another colour.

Now let again w be the vertex of G_0 obtained by contracting B_0 and suppose that there exists a domatic partition of G_0 satisfying (ii); without loss of generality let w have the colour 1 in it. Now let u be an articulation of G belonging to B_0 and let i be its colour in the described colouring of B_0. Then u is adjacent to a vertex of B_0 which has the colour $j \neq i$ in this colouring. From the assertion (ii) for G_0 it follows that w is adjacent to a vertex of the colour $k \neq 1$ in G_0. If $k \neq j$, then all vertices of G separated by u from other vertices of B_0 will be coloured so that the colours 1 and k in the colouring of G_0 are mutually interchanged. If $k=j$ and $i=1$, then we interchange mutually the colour j and the colour l which is different from both j and i and if $k=j$ and $i \neq 1$, then also i and 1 are interchanged. Thus a domatic colouring of G by three colours is obtained; the corresponding domatic partition satisfies (ii).

(ii) $\Rightarrow$ (i). Suppose that $d(G)=3$ and consider a domatic colouring of G with three colours. Then each vertex of G of degree 2 must be adjacent to two vertlices whose colours are mutually different and different from its own colour. If a terminal block is a circuit, all of its vertices except one have degree 2 and thus its length must be divisible by 3. Similarly, the terminal vertices of a simple path of a length at least 2 have the same colour if and only if the length of this path is divisible by 3. Suppose that a circuit C of G which does not form a terminal block has the property that $\mathcal{S}(C)$ contains no path of the length 1 and exactly one path of a length non-divisible by 3. Denote by $a_1, \ldots, a_m$ the articulations in C in such a way that the pairs a_i, a_{i+1} for $i=1, \ldots, m-1$ are connected by simple paths of lengths divisible by 3 and a_m, a_0 are connected by a simple path of a length non-divisible by 3. Then according to the above mentioned assertion we obtain (inductively) that all the vertices $a_1, \ldots, a_m$ have the same colour, but on the other hand a_m and a_1 have different colours, which is a contradiction. Thus (i) must hold.

A numerical invariant of a graph which is closely related to the domatic number is the idomatic number of G. An idomatic partition of G is a partition of $V(G)$, each of whose classes is a set which is simultaneously dominating and independent in G. If there exists at least one idomatic partition of G, then the maximal number of classes of such a partition is called the idomatic number of G and denoted by $id(G)$. If no idomatic partition of G exists, then we put $id(G)=0$.

Theorem 2. *Let G be a finite non-trivial cactus. Then the following two assertions are equivalent:*

(i) *Each terminal block of G is a circuit of a length divisible by 3 and for any circuit C in G not forming a terminal block the number of paths of $S(C)$ with lengths non-divisible by 3 is different from 1.*

(ii) $id(G)=3$.

Proof. (i)$\Rightarrow$(ii). The domatic partition constructed in the first part of the proof of Theorem 1 without using the assumption that for a circuit C in G not forming a terminal block the set $\mathcal{S}(C)$ contains at least one path of length 1 is in fact an idomatic partition. This implies the assertion.

(ii)$\Rightarrow$(i). If $id(G)=3$, then evidently also $d(G)=3$ and (ii) from Theorem 1 is satisfied. Hence (i) from Theorem 1 holds. If a circuit C in G not forming a terminal block has the property that in the set $\mathcal{S}(C)$ there exists exactly one path of a length non-divisible by 3, then this length must be 1. By the consideration from the end of the proof of Theorem 1 we prove that the terminal vertices of such a path must have the same colour in any domatic colouring of G with three colours and thus no domatic partition of G with three classes is idomatic.

Theorem 3. *Let G be a finite non-trivial cactus not satisfying the condition* (i) *from Theorem 2. Then $id(G)=2$ if and only if G is bipartite; otherwise $id(G)=0$.*

Proof is straightforward.

Now we shall prove a theorem concerning the linear arboricity of cacti. The symbol $]x[$ denotes the least integer greater than or equal to x and $\Delta(G)$ denotes the maximum degree of a vertex of G. In [1] it is proved that for every tree T the equality $\Xi(T)=]\frac{1}{2}\Delta(T)[$ holds. Further evidently $\Xi(G)\geqq]\frac{1}{2}\Delta(G)[$ for every graph G, because each linear forest of the required decomposition can contain at most two edges incident with a given vertex. In [1] it is conjectured that for a regular graph G of the degree r the equality $\Xi(G)=]\frac{1}{2}(r+1)[$ holds. As a non-regular cactus G can be embedded into a regular graph of the degree $\Delta(G)$, the following result is related to this conjecture.

Theorem 4. *Let G be a finite non-trivial cactus, let $\Delta(G)$ be the maximum degree of a vertex of G. Then*

$$\Xi(G)=]\tfrac{1}{2}\Delta(G)[\,.$$

Proof. We shall carry out the proof by induction according to the number $b(G)$ of blocks of G; as G is a non-trivial cactus, we have $b(G)\geqslant 2$. Let $b(G)=2$. Then G consists of two blocks. If both these blocks are edges with their end vertices, then $\Delta(G)=2$ and G is a path, hence $\Xi(G)=1=\frac{1}{2}\Delta(G)$. If at least one of the blocks is a circuit, then G is the union of two edge-disjoint paths and $\Xi(G)=2$, while $\Delta(G)=3$ or $\Delta(G)=4$. Now let $b(G)=k\geqq 3$ and suppose that the assertion is true for $b(G)=k-1$. Let G be decomposed into edge-disjoint linear forests and let u be a vertex of G of degree $\Delta(G)$. Each of the forests of the decomposition can contain at most two edges incident with u, hence u is contained in at least $]\frac{1}{2}\Delta(G)[$ such forests and $\Xi(G)\geqq]\frac{1}{2}\Delta(G)[$. Let B_0 be a terminal block of G, let a be the articulation of G contained in B_0. Let G_0 be the graph obtained from G by deleting all vertices of B_0 except a; then G_0 is a finite non-trivial cactus and $b(G_0)=k-1$. According to the induction hypothesis $\Xi(G_0)=]\frac{1}{2}\Delta(G_0)[$. Let $\mathscr{L}$ be a decomposition of G_0 into $]\frac{1}{2}\Delta(G_0)[$ linear forests. First suppose that B_0 consists of one edge e with its end vertices. If the degree of a in G is even, then in G_0 it is odd and there exists at least one forest from $\mathscr{L}$ which contains exactly one edge adjacent to a. Then we add B_0 to this forest and obtain a decomposition of G into $]\frac{1}{2}\Delta(G_0)[$ edge-disjoint linear forests and evidently $]\frac{1}{2}\Delta(G_0)[\leqq]\frac{1}{2}\Delta(G)[$. If the degree of a in G is odd, then in G_0 it is even. Let $\delta(a)$ be the degree of a in G_0. If there is no forest from L containing exactly one edge adjacent to u, then there are $\frac{1}{2}\delta(a)$ forests from $\mathscr{L}$, each from which contains two edges adjacent to a. If $\delta(a)<\Delta(G_0)$, then $\frac{1}{2}\delta(a)<]\frac{1}{2}\Delta(G_0)[$ and there exists at least one forest from $\mathscr{L}$ not containing a; we add B_0 to this forest and again obtain a decomposition of G into $]\frac{1}{2}\Delta(G_0)[$ linear forests. If $\delta(a)=\Delta(G_0)$, then $\Delta(G)=\delta(a)+1=\Delta(G_0)+1$. As $\delta(a)=\Delta(G_0)$ is even, we have $]\frac{1}{2}\Delta(G)[=]\frac{1}{2}\Delta(G_0)[+1$. To $\mathscr{L}$ we add B_0 as a new forest and we obtain a decomposition of G into $]\frac{1}{2}\Delta(G)[$ edge-disjoint linear forests.

Now suppose that B_0 is a circuit. If the degree of a in G is even, then it is even also in G_0. If there are two forests F_1, F_2 from $\mathscr{L}$ such that a is incident at most with one edge from each of them, then we decompose B_0 into two edge-disjoint paths, each of which has a terminal vertex a, and add one of them to F_1 and the other to F_2; we obtain a decomposition of G into $]\frac{1}{2}\Delta(G_0)[$ edge-disjoint linear forests. If there is only one such forest F, then (as the degree of a is even) it contains no edge incident with a. Let P_1 be the path whose edges are the two edges of B_0 incident with a and let P_2 be the path in B_0 with the same terminal vertices as P_1 and edge-disjoint with P_1. We add P_1 to F and P_2 to an arbitrary other forest from $\mathscr{L}$ and we obtain a decomposition of G into $]\frac{1}{2}\Delta(G_0)[$ edge-disjoint linear forests. If there is no forest with the required property, then L contains $\frac{1}{2}\delta(a)$ forests and $\frac{1}{2}\delta(a)=]\frac{1}{2}\Delta(G_0)[$, which implies $\delta(a)=\Delta(G_0)$. The degree of a in G is $\delta(a)+2=\Delta(G_0)+2$ and evidently $\Delta(G)=\Delta(G_0)+2$, which implies $]\frac{1}{2}\Delta(G)[=]\frac{1}{2}\Delta(G_0)[+1$. We use again the paths P_1 and P_2. The path P_2 will be added to an arbitrary forest from $\mathscr{L}$ and the path P_1 will form a new forest; thus a required decomposition of G is obtained. If the degree of a in G is odd, then it is odd also in G. There exists at leasts one forest F from $\mathscr{L}$ which contains exactly one edge incident with a. If there is another forest F' from $\mathscr{L}$ which has at most one edge incident with a, we choose a vertex $b \neq a$ of B_0 and take two edge-disjoint paths P, P' both connecting a with b. We add P to F and P' to F' and obtain a required decomposition of G. If there is no such forest F', then there are $\frac{1}{2}(\delta(a)-1)$ forests of $\mathscr{L}$ having two edges incident with a and the forest F and hence $\Xi(G_0)=]\frac{1}{2}\Delta(G_0)[=\frac{1}{2}(\delta(a)+1)$. Then $\Delta(G_0)=\delta(a)$ or $\Delta(G_0)=\delta(a)+1$. The degree of a in G is $\delta(a)+2$ and this is $\Delta(G_0)+2$ or $\Delta(G_0)+1$. Evidently also $\Delta(G)=\delta(a)+2$. We have $]\frac{1}{2}\Delta(G)[=\frac{1}{2}(\delta(a)+3)=\Xi(G_0)+1$. We add P to F and P' will be a new forest; thus a required decomposition of G is constructed. □

Remark. The assertion of Theorem 4 does not hold for trivial cacti which are circuits; for such a cactus G we have $\Delta(G)=2$ and $\Xi(G)=2$. For other trivial cacti the assertion is true.

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ДОМАТИЧЕСКОЕ ЧИСЛО И ЛИНЕЙНАЯ ДРЕВЕСНОСТЬ КАКТУСОВ

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Резюме

Кактус есть связный неориентированный граф G по меншей мере с двумя вершинами, обладающий тем свойством, что каждое ребро из G содержится по большей мере в одном контуре графа G. В статье исследованы доматическое число, идоматическое число и линейная древесность кактусов.