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RATIONALS WITH EXOTIC CONVERGENCES

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We study properties of rational numbers equipped with various sequential and filter convergence structures coarser than the usual metric convergence. In particular, we show that rational numbers can be equipped with both sequential and filter convergence structures compatible with a given algebraic structure (group, ring, vector structure over Q) such that no two rational numbers can be separated by disjoint neighbourhoods. Consequently, all continuous mappings of the resulting space are constant. After proving the "sequential" results directly, the corresponding "filter" results will follow immediately by applying the coarse first countable filter modification functor, recently introduced by R. Beattie and H.-P. Butzmann.

1. Introduction

Background information about sequential convergence groups and rings can be found in [15], [12], [18] and [4]. Coarse sequential convergence groups (coarse means that there is no strictly coarser sequential group convergence for the group in question) are studied in [8], [5] and [6]. Observe that different authors use different notation and different axioms of convergence. Fortunately, if we work with convergences with unique limits and satisfying the Urysohn axiom (as it is done throughout this paper), then no confusion can arise.

As a rule, R, Q, Z and N, denote the real numbers, rational numbers, integers and natural numbers (positive integers), respectively, MON denotes the strictly monotone mappings of N into N and if $S = \langle S(n)\rangle \in X^{\mathrm{N}}$ is a sequence of points of X and $s \in$ MON, then $S \circ s = \langle S(s(n))\rangle$ denotes the subsequence of S, the n-th term of which is $(S \circ s)(n) = S(s(n))$. Recall that, for a convergence $\mathfrak{L}$, a set U is a neighbourhood of a point x if S is almost contained in U whenever $(S, x) \in \mathfrak{L}$.

Consider the field Q of all rational numbers equipped with the usual metric sequential convergence $\mathfrak{M}$. It is a sequential (i.e. FLUSH-) convergence group (recall that L stands for the compatibility of the convergence with the group structure and H for the uniqueness of limits) and since $\mathfrak{M}$ is also compatible with the ring and vector structure over the scalar field Q, it can be studied as a sequential convergence ring and a sequential convergence vector space over Q. According to [8], $\mathfrak{M}$ can be enlarged to a coarse sequential group convergence for Q. Let $\mathfrak{G}$ be a coarse sequential group convergence for Q coarser than $\mathfrak{M}$. As proved in [6], Q equipped with $\mathfrak{G}$ is a complete group (each Cauchy sequence converges) no two points of which can be separated by disjoint neighbourhoods. Further (cf. [8]), $\mathfrak{G}$ satisfies the following coarseness criterion:

(*C*) *For each sequence* $S \in \mathrm{Q}^{\mathrm{N}}$ *either*

(*C*1) *For some* $s \in \mathrm{MON}$ *we have* $(S \circ s, 0) \in \mathfrak{G}$;

or

(*C*2) *There are* $k \in \mathrm{N}$, $z(i) \in \mathrm{Z}$, $s(i) \in \mathrm{MON}$, $i = 1, \ldots, k$, *and* $x \neq 0$ *such that*

$$\left(\sum_{i=1}^{k} z(i)\, S \circ s(i),\ x\right) \in \mathfrak{G}.$$

holds true.

Proposition 1. *Let* $(T, 0) \in \mathfrak{G}$ *and* $y \in \mathrm{Q}$. *Then* $(\langle y \rangle T, 0) \in \mathfrak{G}$.

Proof. Suppose that, on the contrary, $(\langle y \rangle T, 0) \notin \mathfrak{G}$. Then $0 \neq y = p/q$, where $p \in \mathrm{Z}$, $q \in \mathrm{N}$, $p \neq 0 \neq q$. Since $\mathfrak{G}$ satisfies the Urysohn axiom, there exists $t \in \mathrm{MON}$ such that for each $s \in \mathrm{MON}$ ve have $(\langle y \rangle T \circ t \circ s, 0) \notin \mathfrak{G}$. It follows from (*C*) that there are $k \in \mathrm{N}$, $z(i) \in \mathrm{Z}$, $s(i) \in \mathrm{MON}$, $i = 1, \ldots, k$, and $x \neq 0$ such that $\left(\sum_{i=1}^{k} z(i) \langle p \rangle T \circ t \circ s(i),\ qx\right) \in \mathfrak{G}$. But $qx \neq 0$ contradicts $(T, 0) \in \mathfrak{G}$. This completes the proof.

2. Q equipped with a coarse ring convergence

Since $\mathfrak{M}$ is also compatible with the multiplication in Q, Q equipped with $\mathfrak{M}$ is a sequential convergence ring. As usual, using the Kuratowski – Zorn lemma, it can be shown that $\mathfrak{M}$ can be enlarged to a coarse sequential ring convergence (with unique limits) for Q. Let $\mathfrak{R}$ be a coarse sequential ring convergence for Q such that $\mathfrak{M} \subset \mathfrak{R}$. For $S \in \mathrm{Q}^{\mathrm{N}}$ define $\mathcal{N}(S) \subset \mathrm{Q}^{\mathrm{N}}$ as follows: $T \in \mathcal{N}(S)$ iff either $(T, 0) \in \mathfrak{R}$ or T is of the form $\langle x \rangle S \circ s$, where $x \in \mathrm{Q}$ and $s \in \mathrm{MON}$. A straightforward calculation shows that $\mathfrak{R}$ satisfies the following coarseness criterion:

(*CR*) *For each* $S \in \mathrm{Q}^{\mathrm{N}}$ *either*

(*CR*1) *For some* $s \in \mathrm{MON}$ *we have* $(S \circ s, 0) \in \mathfrak{R}$;

or

(*CR*2) *There are* $r \in \mathrm{Q}$, $r \neq 0$, l, $k(j) \in \mathrm{N}$, $T(j, i) \in \mathcal{N}(S)$, $j = 1, \ldots, l$, $i = 1, \ldots, k(j)$, *such that*

$$\langle r \rangle = \sum_{j=1}^{l} T(j, 1) \ldots T(j, k(j)) \quad \textit{holds true.}$$

We shall prove that Q equipped with $\mathfrak{R}$ has some exotic properties.

Proposition 2. *Let S be a Cauchy sequence in* Q *equipped with* $\mathfrak{R}$. *Then S is a bounded sequence.*

Proof. If $(T, 0) \in \mathfrak{R}$, then T is bounded. Indeed, otherwise for some $t \in \mathrm{MON}$ we would have $((T \circ t)^{-1}, 0) \in \mathfrak{M} \subset \mathfrak{R}$, a contradiction with $(T \circ t)(T \circ t)^{-1} = \langle 1 \rangle$. Since $(S \circ s - S \circ t, 0) \in \mathfrak{R}$ for each $s, t \in \mathrm{MON}$, we easily infer that S is a bounded sequence.

Proposition 3. *Let $y \in \mathrm{R} \backslash \mathrm{Q}$ be an algebraic number over* Q *and let S be a sequence of rational numbers converging in the real line to y. Then for each $r \in \mathrm{Q}$ and each $s \in \mathrm{MON}$ we have $(S \circ s, r) \notin \mathfrak{R}$.*

Proof. Let $P(x) = a(n)x^n + \ldots + a(1)x + a(0)$ be a polynomial over Q with the smallest possible degree n such that $P(y) = 0$. Suppose that, on the contrary, for some $r \in \mathrm{Q}$ and $s \in \mathrm{MON}$ we have $(S \circ s, r) \in \mathfrak{R}$. Then $\left(\sum_{i=0}^{n} a(i) \times (S \circ s)^i, \sum_{i=0}^{n} a(i) r^i\right) \in \mathfrak{R}$. Since $\left(\sum_{i=0}^{n} a(i)(S \circ s)^i, 0\right) \in \mathfrak{M} \subset \mathfrak{R}$, we have $P(r) = 0$. Dividing $P(x)$ by $x - r$ we get a polynomial $P'(x)$ over Q such that $P'(y) = 0$ and the degree of $P'(x)$ is less than n. This is a contradiction.

Corollary 1. Q *equipped with $\mathfrak{R}$ is not complete.*

Proposition 4. $\mathfrak{M} \neq \mathfrak{R}$.

Proof. Contrariwise, suppose that $\mathfrak{M} = \mathfrak{R}$. To arrive at a contradiction it suffices to find a sequence $S \in \mathrm{Q}^{\mathrm{N}}$ which does not satisfy (CR). Let S be a sequence of rational numbers converging in the real line to a transcendental number $y \in \mathrm{R} \backslash \mathrm{Q}$. Since S cannot satisfy $(CR1)$, it satisfies $(CR2)$. From $\langle r \rangle = \sum_{j=1}^{l} T(j, 1) \ldots T(j, k(j))$ it follows that there is a polynomial $P(x) = a(n)x^n + \ldots + a(1)x + a(0)$ over Q, $a(n) \neq 0$, $n > 0$, such that $\sum_{j=1}^{l} T(j, 1) \ldots T(j, k(j))$ converges in the real line to $P(y) = r \in \mathrm{Q}$. This contradicts the fact that y is a transcendental number.

Proposition 5. *Let $(T, x) \in \mathfrak{R} \backslash \mathfrak{M}$. Then there are $t \in \mathrm{MON}$ and a transcendental number y such that $T \circ t$ converges in the real line to y.*

Proof. According to Proposition 2, T is a bounded sequence. Then there exists $t \in \mathrm{MON}$ such that $T \circ t$ converges in the real line to $y \in \mathrm{R} \backslash \mathrm{Q}$. It follows from Proposition 3 that y is a transcendental number.

Proposition 6. *In* Q *equipped with $\mathfrak{R}$ no two points can be separated by disjoint neighbourhoods.*

Proof. It follows from Proposition 4 and Proposition 5 that there are a sequence $T \in \mathrm{Q}^{\mathrm{N}}$, $x \in \mathrm{Q}$ and a transcendental number y such that $(T, x) \in \mathfrak{R}$ and T converges in the real line to y. Put $S = T - \langle x \rangle$. Then $(S, 0) \in \mathfrak{R}$ and S converges in the real line to $y - x \neq 0$. For each $z \in \mathrm{Z}$ and each $k \in \mathrm{N}$ we have

$(\langle z/k \rangle S, 0) \in \mathfrak{R}$. For each $r \in \mathrm{Q}$ we can choose $z \in \mathrm{Z}$ and $k \in \mathrm{N}$ such that the sequence $\langle z/k \rangle S$ is almost contained in the neighbourhood of r. The assertion follows immediately.

Via an example, it is shown in [4] that in a commutative sequential convergence ring (with identity) the product of a sequence converging to zero and a Cauchy sequence need not converge to zero. Consequently, not every commutative sequential convergence ring has a completion. Another such example is provided by Q equipped with $\mathfrak{R}$.

Proposition 7. *Let* $(T, 0) \in \mathfrak{R} \backslash \mathfrak{M}$. *Then there is* $t \in \mathrm{MON}$ *such that* $(T \circ t)^{-1}$ *is an* $\mathfrak{R}$*-Cauchy sequence.*

Proof. The assertion follows immediately from Proposition 5.

Corollary 2. Q *equipped with* $\mathfrak{R}$ *cannot be embedded into a complete sequential convergence ring.*

3. Convergence vector spaces over Q

By a scalar field F we understand either Q or R and by $\mathfrak{R}$ we denote the usual metric convergence of sequences in F.

Definition. *Let X be a vector space over the scalar field* F. *Let* $\mathfrak{L}$ *be a sequential group convergence for X such that the following conditions are satisfied*:

($Lv1$) *If* $(\langle r(n) \rangle, r) \in \mathfrak{R}$ *and* $x \in X$, *then* $(\langle r(n)x \rangle, rx) \in \mathfrak{L}$;

($Lv2$) *If* $(\langle x(n) \rangle, x) \in \mathfrak{L}$ *and* $r \in \mathrm{F}$, *then* $(\langle rx(n) \rangle, rx) \in \mathfrak{L}$.

Then X equipped with $\mathfrak{L}$ *is said to be a weak sequential convergence vector space.*

Weak sequential convergence vector spaces over R are investigated, e.g., in [13], [14] and [11]. In particular, in [11] fine and coarse convergences are used to describe the relationship between various conditions concerning sequential convergence in vector spaces.

We shall show that Q equipped with $\mathfrak{G}$ and $\mathfrak{R}$, respectively, are weak sequential convergence vector spaces over Q and hence do inherit the peculiar properties of $\mathfrak{G}$ and $\mathfrak{R}$. Further, via a modification functor, these properties can be carried over to filter convergence vector spaces over Q.

If we replace conditions ($Lv1$) and ($Lv2$) by the following stronger one:

(Lv) *If* $(\langle r(n) \rangle, r) \in \mathfrak{M}$ *and* $(\langle x(n) \rangle, x) \in \mathfrak{L}$, *then* $(\langle r(n) \times (n) \rangle, rx) \in \mathfrak{L}$;
then we speak of sequential convergence vector spaces.

For additional information on sequential and filter convergence vector spaces the reader is referred to [1], [7], [16] and [9], respectively.

Proposition 8. Q *equipped with* $\mathfrak{G}$ *is a weak sequential convergence vector space over* Q. *It is complete and no two points of it can be separated by disjoint neighbourhoods.*

P r o o f. Condition ($Lv1$) follows from $\mathfrak{M} \subset \mathfrak{G}$ and condition ($Lv2$) is an immediate consequence of Proposition 1. The rest follows from Proposition 3.1 and Proposition 3.3 in [6].

The obvious proofs of the next two propositions are omitted.

Proposition 9. Q *equipped with* $\mathfrak{G}$ *does not satisfy condition* (Lv).

Proposition 10. Q *equipped with* $\mathfrak{R}$ *is a sequential convergence vector space over* Q *no two points of which can be separated by disjoint neighbourhoods.*

Let X be a set. As shown in [2], to each sequential FUSH-convergence $\mathfrak{L}$ for X there corresponds the so-called coarse first countable filter modification $\gamma(\mathfrak{L})$. It is the coarsest first countable filter convergence for X having the same convergent sequences as $\mathfrak{L}$ (by a filter convergence here we understand a separated pseudotopology in the sense of [9] and first countable means that for each convergent filter there is a coarser filter having a countable base and converging to the same limit). If X is equipped with an algebraic structure (group, ring, vector structure over F) and if $\mathfrak{L}$ is compatible with the algebraic structure of X, then $\gamma(\mathfrak{L})$ is compatible with the algebraic structure of X, too (see also [3]). Further, if X is a group, then X equipped with a sequential group (FLUSH-) convergence $\mathfrak{L}$ is complete iff its coarse first countable filter modification, i.e., X equipped with $\gamma(\mathfrak{L})$, is complete. Consequently (cf. Corollary 2), Q equipped with $\gamma(\mathfrak{R})$ has no ring completion. Observe that $\gamma(\mathfrak{M})$ is the usual topology for F. If we speak of a filter convergence vector space over Q, then the scalar field Q is equipped with $\gamma(\mathfrak{M})$.

The considerations above yield the following proposition.

Proposition 11. Q *equipped with* $\gamma(\mathfrak{R})$ *is a filter convergence vector space (i.e. a separated pseudotopological vector space) no two points of which can be separated by disjoint neighbourhoods.*

In the same way as in Remark 3.4 in [6] it can be shown that the closure (adherence) operators for Q derived from $\mathfrak{G}$ and $\mathfrak{R}$, respectively, (the same operators are derived from $\gamma(\mathfrak{G})$ and $\gamma(\mathfrak{R})$, respectively) are not idempotent. In fact, as recently proved by J. G e r l i t s, the sequential order of Q equipped with $\mathfrak{G}$ is ω_1, i.e., all countable iterations of the closure operator derived from $\mathfrak{G}$ are distinct. The same holds for Q equipped with $\mathfrak{R}$.

Let X be a vector space over an arbitrary scalar field F. Then X equipped with a group topology such that the multiplication by each scalar $r \in \mathrm{R}$ is a continuous self-mapping of X is called a topological vector group (cf. [17]). Accordingly, if X is equipped with a sequential group convergence $\mathfrak{L}$ satisfying condition ($Lv2$), then X is said to be a sequential vector group. In [10], several vector group topologies for R are investigated. It might be interesting to find out more about sequential vector group convergences and their relationship to vector group topologies.

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ЭКЗОТИЧЕСКИЕ СХОДИМОСТИ РАЦИОНАЛЬНЫХ ЧИСЕЛ

Роман Фрич

Резюме

В работе исследуются рациональные числа, снабженные различными структурами сходимости более грубыми, чем метрическая сходимость. Показано, что существуют структуры сходимости, совместимые с алгебраическими структурами (группа, кольцо, векторное пространство) так, что никакие две точки неотделимы.