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THE CSÁKÁNY THEORY OF REGULARITY FOR FINITE ALGEBRAS

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ABSTRACT. If an algebra A has at most five elements, then A is congruence regular if and only if there exists a ternary functions compatible with $\operatorname{Con} A$ such that $p(x,y,z) = z$ if and only if $x = y$. If A has six elements, the assertion does not hold.

A. Pixley [5] posed the following problem: If some congruence property is characterized by a Mal'cev condition in varieties of algebras, can this Mal'cev condition (modified in a natural way) be used also for characterizing this congruence property in the case of a single algebra? For arithmeticity, he solved himself this problem affirmatively in [5]. Since every congruence identity can be characterized in varieties by a Mal'cev condition (see [6]), H.-P. Gumm asked for which other congruence identity there exists a Mal'cev theory in the case of a single algebra. The answer is "for none" in a general case, see [4]. However, for small algebras, permutability of congruences can be characterized by a Mal'cev theory, see e.g. [1] for at most four-element algebras, and [2] for at most eight-element algebras (the answer is negative for at least 25-element algebra). This motivated our effort to proceed similar investigations for congruence regularity (which is not a congruence identity). Although some Mal'cev-type characterizations of regular varieties are known, see e.g. [7], we prefer another but more simple term condition given by B. Csákány in [3]. At first we recall:

DEFINITION. An algebra A is *regular* if $\theta = \phi$ for $\theta, \phi \in \operatorname{Con} A$ whenever they have a congruence class in common. A variety $\mathcal{V}$ is regular if each $A \in \mathcal{V}$ has this property.

CSÁKÁNY'S THEOREM. ([3]) *For a variety $\mathcal{V}$, the following conditions are equivalent:*

(1) *$\mathcal{V}$ is regular;*

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(2) *there exist ternary terms* $p_1(x,y,z),\dots,p_n(x,y,z)$ *such that*

$$\bigl[p_1(x,y,z)=z\wedge\cdots\wedge p_n(x,y,z)=z\bigr] \iff x=y\,.$$

We are going to investigate if such Csákány-type conditions can characterize regularity of a single algebra.

Let A be an algebra, $\theta\in\operatorname{Con}A$ and $f\colon A^n\to A$ be an n-ary function. We say that f is *compatible with* θ if $\bigl\langle f(a_1,\dots,a_n),\,f(b_1,\dots,b_n)\bigr\rangle\in\theta$ whenever $\langle a_i,b_i\rangle\in\theta$ for $i=1,\dots,n$.

Denote by ω the least and by ι the greatest congruence of A.

LEMMA 1. *Let A be an at least two-element algebra, and $\theta\in\operatorname{Con}A$, $\theta\neq\omega$. If θ has a one-element congruence class, then there does not exist a ternary function $p\colon A^3\to A$ compatible with θ such that*

$$p(x,y,z)=z \iff x=y\,.$$

P r o o f. Suppose $[c]_\theta=\{c\}$ for some $c\in A$. Let $p(x,y,z)$ be a ternary function compatible with θ such that

$$p(x,y,z)=z \iff x=y\,.$$

Since $\theta\neq\omega$, there exists a congruence class B of θ containing at least two different elements, say a and b. Since $p(a,a,c)=c$, we have

$$\bigl\langle p(a,b,c),\,c\bigr\rangle=\bigl\langle p(a,b,c),\,p(a,a,c)\bigr\rangle\in\theta\,,$$

whence $p(a,b,c)=c$, which is a contradiction.

THEOREM. *Let A be a finite algebra with* $\operatorname{card}A\leq 5$. *The following conditions are equivalent:*

(1) *A is regular,*
(2) *there exists a ternary function $p\colon A^3\to A$ compatible with every congruence of A such that*

$$p(x,y,z)=z \iff x=y\,.$$

P r o o f. For A with $\operatorname{card}A\leq 2$, the assertion is trivial.

(a) Suppose $\operatorname{card}A=3$, i.e. $A=\{a,b,c\}$. If A is regular, then evidently $\operatorname{Con}A=\{\omega,\iota\}$. Define $p\colon A^3\to A$ by the rules

$$p(x,y,z)=\begin{cases} z & \text{if } x=y\,,\\ x & \text{if } x\neq y \text{ and } x\neq z\,,\\ y & \text{if } x\neq y \text{ otherwise}\,.\end{cases}$$

Trivially, p is compatible with every congruence of $\operatorname{Con} A$ and satisfies (2).

Conversely, let A fail to be regular. Without loss of generality, suppose the existence of $\theta \in \operatorname{Con} A$ such that θ has two classes, namely $\{c\}$ and $\{a, b\}$. By Lemma 1, we obtain a contradiction with (2).

(b) Let $\operatorname{card} A = 4$, $A = \{a, b, c, d\}$. If A is regular, the desired compatible function can be defined by the rule

$$p(x, y, z) \begin{cases} = z & \text{for } x = y\,, \\ \in [z]_{\theta(x,y)} - \{z\} & \text{otherwise}\,, \end{cases}$$

since $\operatorname{Con} A \subseteq \{\omega, \iota, \theta_1, \theta_2, \theta_3\}$, where

$$\begin{array}{lll} \theta_1 & \text{has classes} & \{a,b\}\,,\ \{c,d\}\,, \\ \theta_2 & \text{has classes} & \{a,c\}\,,\ \{b,d\}\,, \\ \theta_3 & \text{has classes} & \{a,d\}\,,\ \{b,c\}\,. \end{array}$$

It is easy to show that p is compatible with every congruence of $\operatorname{Con} A$.

If A fails to be regular, then there exists $\theta \in \operatorname{Con} A$ such that $\theta \neq \omega$, and θ has a one-element class. By Lemma 1, we obtain a contradiction.

(c) Let $\operatorname{card} A = 5$, $A = \{a, b, c, d, e\}$. If A is regular, then the lattice $\operatorname{Con} A$ cannot include any congruence θ, $\theta \neq \omega$, having a one-element class, i.e. $\operatorname{Con} A \subseteq \{\omega, \iota, \theta_i\}$, where every θ_i has one two-element and one three-element class. There exist 10 of such θ_i on the underlying set of A, however, since A is regular, $\operatorname{Con} A$ contains at most one of them (because for $i \neq j$, $\theta_i \cap \theta_j \neq \omega$, and $\theta_i \cap \theta_j$ contains a one element class). Suppose that θ_1 has classes $C = \{a, b, c\}$ and $D = \{d, e\}$. Define $p_1 \colon A^3 \to A$ by the rules

$$\begin{array}{ll} p_1(x,x,z) = z\,, & \\ \left.\begin{array}{l} p_1(x,y,d) = e \\ p_1(x,y,e) = d \end{array}\right\} & \text{for } x \neq y\,,\ x, y \in C\,, \\ p_1(x_1,x_2,x_3) = p_1\big(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}\big) & \text{for } x_1, x_2 \in C\,,\ x_3 \in D\,, \\ & \text{and every permutation } \pi \text{ of } \{1,2,3\}, \\ \left.\begin{array}{l} p_1(x,y,a) = b \\ p_1(x,y,b) = c \\ p_1(x,y,c) = a \end{array}\right\} & \text{for } x, y \in D\,,\ x \neq y\,. \end{array}$$

For $x, y, z \in C$ we put

$$\begin{array}{ll} p_1(x,y,z) = x & \text{for } z = y\,, \\ p_1(x,y,z) = v & \text{for } z \neq y\,, \\ & \text{where } v \in C\,,\ z \neq v \neq y\,. \end{array}$$

Since D has only two elements, the case $x, y, z \in D$ yields $p(x, y, z) = p(x, x, z)$, which was solved before.

It is a routine calculation to verify that p_1 is compatible with θ_1 (and, trivially, also with ω, ι). Permuting the elements a, b, c, d, e, we obtain the functions p_i for each θ_i $(i = 1, \dots, 10)$.

If A is not regular, then again $\operatorname{Con} A$ has to contain a congruence θ, $\theta \neq \omega$, with a one-element class; thus we obtain a contradiction by Lemma 1. □

For algebras with more than 5 elements, the conditions (1), (2) of our Theorem need not be equivalent. The essential part of this statement is contained in the following:

LEMMA 2. *There exists a six-element non-regular algebra with a ternary function $p\colon A^3 \to A$ satisfying* (2) *of Theorem.*

Proof. Let $A = \{a, b, c, d, e, f\}$ and p be a ternary operation on A as follows:

$$p(x, x, z) = z \qquad \text{for each} \quad x, z \in A\,,$$

and for each $x, y \in A$, $x \neq y$, we put

$$\begin{aligned} p(x,y,a) &= b\,, \qquad & p(x,y,c) &= d\,, \qquad & p(x,y,e) &= f\,,\\ p(x,y,b) &= a\,, & p(x,y,d) &= c\,, & p(x,y,f) &= e\,. \end{aligned}$$

Let θ, ϕ be equivalences on A determined by their partitions:

$$\begin{aligned} &\theta \quad \text{has classes} \quad \{a,b\}\,,\ \{c,d\}\,,\ \{e,f\}\,,\\ &\phi \quad \text{has classes} \quad \{a,b\}\,,\ \{c,d,e,f\}\,. \end{aligned}$$

Then θ, ϕ are congruences on the algebra (A, p), and $p(x, y, z)$ satisfies (2) of Theorem (trivially, p is compatible with every congruence on (A, p) because it is the operation of this algebra). Moreover, (A, p) is not regular because two different congruences θ, ω have a common class $\{a, b\}$. □

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