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A NOTE ON LUSIN MEASURABILITY IN MEASURE SPACES

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ABSTRACT. If X and Y are Hausdorff topological spaces, $\mathbb{P}(X)$ and $\mathbb{P}(Y)$ the corresponding spaces of Radon probability measures, then any universally Lusin measurable map $f\colon X \to Y$ defines "the image measure map" $\tilde{f}\colon \mathbb{P}(X) \to \mathbb{P}(Y)$. We ask and partially provide answers to the following problems:

(1) When the surjectivity of f implies the surjectivity of $\tilde{f}$?

(2) Under which circumstances is the map $\tilde{f}$ universally Lusin measurable?

It is a known fact that both problems are answered positively if X and Y are Souslin spaces. Our results show that the desired properties are connected more generally with the presence or absence of the measure convexity of the spaces $\mathbb{P}(X)$ and $\mathbb{P}(Y)$.

1. Preliminaries

All topological spaces $X, Y, \dots$ we shall treat here are supposed to be Hausdorff if not stated else. We shall denote by $\mathbb{K}(X)$ and $\mathbb{B}(X)$ the family of all compact and Borel sets in X, respectively and by $\mathbb{P}(X)$ the set of all Radon probability measures defined on X, i.e. the set of all probability measures p on $\mathbb{B}(X)$ such that

$$p(B) = \sup\{p(K) :\ B \supset K \in \mathbb{K}(X)\} \qquad \text{holds for all} \quad B \in \mathbb{B}(X)\,.$$

$\mathbb{P}(X)$ will be always topologized by its *weak topology* i.e. by the coarsest topology on $\mathbb{P}(X)$ for which all maps $p \mapsto p(f) := \int f \, \mathrm{d}p$ from $\mathbb{P}(X)$ to $\mathbb{R}$ are lower semi-continuous as f goes through all bounded lower-semicontinuous real functions on X. Recall that the weak topology of $\mathbb{P}(X)$ is Hausdorff [4; p. 371,

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Proposition 2] and note that for a completely regular space X the weak topology of $\mathbb{P}(X)$ is exactly the initial topology for the set of maps $p \mapsto p(f)$ from $\mathbb{P}(X)$ to $\mathbb{R}$ as f goes through all bounded continuous functions $f\colon X \to \mathbb{R}$ [4; p. 269, Proposition 1].

Having a $p \in \mathbb{P}(X)$ we denote by

$$\mathbb{B}(X)^p := \{A \subset X : \ (\exists\, B_1, B_2 \in \mathbb{B}(X))\,(B_1 \subset A \subset B_2 \quad \& \quad p(B_2 - B_1) = 0)\}$$

the σ-algebra of *p-measurable sets* and by

$$\mathbb{U}(X) := \bigcap\{\mathbb{B}(X)^p : \ p \in \mathbb{P}(X)\}$$

the σ-algebra of *universally measurable sets* in X.

Having a $p \in \mathbb{P}(X)$ and a map $f\colon X \to Y$ recall that the f is called *Lusin p-measurable* if

$$(\forall \varepsilon > 0)\,(\exists\, K \in \mathbb{K}(X))\,\Big(p(K) \ge 1 - \varepsilon \quad \& \quad f|_K \text{ is continuous}\Big),$$

(where $f|_M$ denotes the restriction of f to a set M),
and it is *Borel p-measurable* if $f^{-1}(B)$ is in $\mathbb{B}(X)^p$ for all $B \in \mathbb{B}(Y)$. Finally, a map $f\colon X \to Y$ is called *universally Lusin measurable* (briefly ULM, $f \in$ ULM, $f \in \mathrm{ULM}(X,Y)$) if it is Lusin p-measurable for all $p \in \mathbb{P}(X)$ and it is called *universally Borel measurable* if it is Borel p-measurable for all $p \in \mathbb{P}(X)$ or equivalently if $f^{-1}(\mathbb{B}(Y)) \subset \mathbb{U}(X)$ holds (briefly UBM, $f \in$ UBM, $f \in \mathrm{UBM}(X,Y)$).

The most important information is provided by

LUSIN THEOREM.

(1) $\mathrm{ULM}(X,Y) \subset \mathrm{UBM}(X,Y)$.

(2) $\mathrm{ULM}(X,Y) = \mathrm{UBM}(X,Y)$ *if Y is either a separable metric or Souslin topological space.*

For the non-trivial proofs see [4; p. 26, Theorem 5, p. 129, Theorem 14].

Remark. Any semi-continuous function $f\colon X \to \bar{\mathbb{R}}$ is ULM ([4; p. 28, Corollary]) and $f \to \{0,1\}$ is ULM if and only if the set $\{f = 1\}$ belongs to $\mathbb{U}(X)$. Agree that, having measurable spaces $(X, \mathcal{X})$, $(Y, \mathcal{Y})$, a probability measure p defined on $\mathcal{X}$ and a map f that is measurable as $f^{-1}(\mathcal{Y}) \subset \mathcal{X}$, we denote by $f \circ p$ the probability measure q defined on $\mathcal{Y}$ by $q(B) = p(f^{-1}(B))$ for $B \in \mathcal{Y}$.

Observe that any $p \in \mathbb{P}(X)$ can be uniquely extended to a probability measure $\bar{p}$ on the σ-algebra $\mathbb{U}(X)$. Having moreover a map $f \in \mathrm{UBM}(X,Y)$, $f \circ \bar{p}$ is a well defined Borel probability measure on Y. Assuming that f is universally Lusin measurable we get $f \circ \bar{p} \in \mathbb{P}(Y)$, i.e. as a Radon measure on Y and call it the *image of p under f*. Thus any $f \in \mathrm{ULM}(X,Y)$ yields a map from $\mathbb{P}(X)$

to $\mathbb{P}(Y)$ defined by $p \mapsto f \circ \bar{p}$. Agree to denote it by $\tilde{f}$ or apply the convention $\tilde{f} = F$, $\tilde{g} = G$, etc..

Next lemma lists some almost obvious properties of universally Lusin measurable maps. For any $f: X \to Y$ denote by $\mathbb{K}_f = \mathbb{K}_f(X) = \mathbb{K}_f(X,Y)$ the set of all compacts $K \subset X$ such that $f|_K$ is a continuous map.

Lemma 1.1.

(1) *Any* $f \in \mathrm{ULM}(X,Y)$ *is a measurable map in the sense of* $f^{-1}(\mathbb{U}(Y)) \subset \mathbb{U}(X)$. *Further, for any* $p \in \mathbb{P}(X)$, *denoting* $q := \tilde{f}(p)$, *we get that* $\bar{q}(U) = \bar{p}(f^{-1}(U))$ *holds for any* $U \in \mathbb{U}(Y)$, *i.e. that* $\bar{q}$ *and* $f \circ \bar{p}$ *are identical measures on* $\mathbb{U}(Y)$.

(2) *Consider a Lusin* p*-measurable map* $f: X \to Y$, *an arbitrary map* $g: Y \to Z$ *and a* $p \in \mathbb{P}(X)$. *Then* g *is Lusin* $\tilde{f}(p)$*-measurable if and only if* $g \circ f$ *is Lusin* p*-measurable.*

(3) *If* $f \in \mathrm{ULM}(X,Y)$ *and* $g \in \mathrm{ULM}(Y,Z)$, *then* $g \circ f \in \mathrm{ULM}(X,Z)$ *and* $\widetilde{g \circ f} = \tilde{f} \circ \tilde{g}$.

(4) *If* $f \in \mathrm{ULM}(X,Y)$ *and* $M \subset X$, *then* $f|_M \in \mathrm{ULM}(M,Y)$ *and* $f \in \mathrm{ULM}(X, f(M))$.

(5) *Consider* $f_n: X_n \to Y_n$ *for all* $n \in \mathbb{N}$. *Then*

$$(\forall n \in \mathbb{N})\,(f_n \in \mathrm{ULM}(X_n, Y_n)) \iff \Big(f := (f_1, f_2, \ldots) \in \mathrm{ULM}\Big(\prod_n X_n, \prod_n Y_n\Big)\Big).$$

Especially, if $f_n: X \to Y_n$ *for all* $n \in \mathbb{N}$, *then*

$$(\forall n \in \mathbb{N})\,(f_n \in \mathrm{ULM}(X, Y_n)) \iff \Big(f := (f_1, f_2, \ldots) \in \mathrm{ULM}\Big(X, \prod_n Y_n\Big)\Big).$$

(6) *For any* $f \in \mathrm{ULM}(X,Y)$ *and* $p \in \mathbb{P}(X)$

$$p(B) = \sup\{p(K):\ B \supset K \in \mathbb{K}_f\}, \qquad B \in \mathbb{B}(X).$$

(7) *If* $f \in \mathrm{ULM}(X,Y)$, *then* $\mathrm{Graph}(f) \in \mathbb{U}(X \times Y)$.

P r o o f. The statements (1), (4) and (6) follow directly from definitions and from the first statement of the Lusin Theorem. As for (2) and (3), see [4; p. 35, Theorem 9, p. 36, Corollary 2].

To prove (5), denote $\mathbf{X} := \prod_n X_n$ and $\mathbf{Y} := \prod_n Y_n$, consider a $p \in \mathbb{P}(\mathbf{X})$ and $\varepsilon > 0$. Put $p_n := \tilde{\pi}_n(p) \in \mathbb{P}(X_n)$, where π_n denotes the projection of $\mathbf{X}$ to X_n. If all f_n's are universally Lusin measurable, then there are compacts $K_n \in \mathbb{K}_{f_n}$ such that $p_n(K_n) \geq \varepsilon 2^{-n}$ holds for any $n \in \mathbb{N}$. It follows that $K = \prod_n K_n \in \mathbb{K}_f$

and $p(X \setminus K) \leq \sum_n p_n(X_n \setminus K_n) \leq \varepsilon$ holds. Hence, $f \in \mathrm{ULM}(\mathbf{X}, \mathbf{Y})$. The reverse implication follows from (3) as any f_n is defined by $\pi_n \circ f$, where π_n denotes this time the continuous projection of $\mathbf{Y}$ to Y_n.

To prove the rest in (5), assume that $f_n \in \mathrm{ULM}(X, Y_n)$ for all n. We have already proved that $\mathbf{f} := (f_1, f_2, \ldots) \in \mathrm{ULM}(X^{\mathbb{N}}, \prod_n Y_n)$ and therefore $\mathbf{f}|_M \in \mathrm{ULM}\left(M, \prod_n Y_n\right)$ by (4), where the M denotes the diagonal in $X^{\mathbb{N}}$. Because the sets X and M are homeomorphic, we apply (3) to conclude that $f \in \mathrm{ULM}\left(X, \prod_n Y_n\right)$. The reverse implication follows again by an application of (3).

To prove (7), denote by i the identity map on Y and by D the diagonal set in $Y \times Y$. Then $\mathrm{Graph}(f) = (f, i)^{-1}(D)$ is a set in $\mathbb{U}(X \times Y)$ according to (1) as D is a closed set in $Y \times Y$. □

Remark that the implication (7) cannot be reversed, see Example 2.5.

Any $f \in \mathrm{ULM}(X, Y)$ generates $\tilde{f} \colon \mathbb{P}(X) \to \mathbb{P}(Y)$. We ask for the properties inherited by $\tilde{f}$ from f. The continuity and injectivity can be included into the list.

Lemma 1.2.

(1) *If $f \colon X \to Y$ is an injective universally Lusin measurable map, then $\tilde{f} \colon \mathbb{P}(X) \to \mathbb{P}(Y)$ is also an injection.*
(2) *Any continuous $f \colon X \to Y$ generates continuous $\tilde{f} \colon \mathbb{P}(X) \to \mathbb{P}(Y)$.*
(3) *If $f \colon X \to Y$ is a continuous map, then*

$$\tilde{f}(\mathbb{P}(X)) = \left\{ q \in \mathbb{P}(Y) : \ \sup\{q(f(K)) : \ K \in \mathbb{K}(X)\} = 1 \right\}.$$

The injectivity part is proved in [4; p. 37, Theorem 10] for continuous maps. The argument remains true for any injective $f \in \mathrm{ULM}(X, Y)$. Indeed, let $q_i := \tilde{f}(p_i)$ for a $p_i \in \mathbb{P}(X)$ and $i = 1, 2$. If $q_1 = q_2$, then obviously $p_1(K) = q_1(f(K)) = q_2(f(K)) = p_2(K)$ holds for any compact set $K \subset X$ such that $f|_K$ is continuous ($f(K)$ is a compact in Y). It follows from Lemma 1.1.6 that $p_1 = p_2$ on $\mathbb{B}(X)$. The continuity part is exactly [4; p. 372, Proposition 1] and the equality (3) follows from [4; p. 39, Theorem 12].

Consider a family of compact sets $\mathbb{K} \subset \mathbb{K}(X)$ and a measure $p \in \mathbb{P}(X)$. We shall say that p is a $\mathbb{K}$-*regular* if

$$p(B) = \sup\{p(K) : \ B \supset K \in \mathbb{K}\} \qquad \text{for any} \quad B \in \mathbb{B}(X). \tag{1}$$

Lemma 1.3.1 shows that the $\mathbb{K}$-regularity concept may be simplified in some cases.

Agree to call a $\mathbb{K} \subset \mathbb{K}(X)$ an *ideal* if $K \cap D \in \mathbb{K}$ whenever $K \in \mathbb{K}$ and $D \in \mathbb{K}(X)$. Also denote by $\mathbb{K}^f = \mathbb{K}^f(X) = \mathbb{K}^f(X,Y)$ the set of all $f(K) \subset Y$ where K goes through the set of all compacts, $K \subset X$, such that $f|_K$ is a continuous map, i.e. $\mathbb{K}^f = \{f(K) : K \in \mathbb{K}_f\}$.

LEMMA 1.3.

(1) *Let* $\mathbb{K} \subset \mathbb{K}(X)$ *be an ideal and* p *a measure in* $\mathbb{P}(X)$ *such that* (1) *holds for* $B = X$. *Then* p *is a* $\mathbb{K}$*-regular measure.*

(2) *For any* $f \colon X \to Y$ *the sets* $\mathbb{K}_f \subset \mathbb{K}(X)$ *and* $\mathbb{K}^f \subset \mathbb{K}(Y)$ *are ideals.*

(3) *If* $f \colon X \to Y$ *is a continuous map, then*

$$\tilde{f}(\mathbb{P}(X)) = \{q \in \mathbb{P}(Y) : \ q \text{ is a } \mathbb{K}^f\text{-regular measure}\}.$$

To see that $\mathbb{K}^f$ is an ideal, choose $K \in \mathbb{K}_f$ and $\mathbb{K}(Y) \ni C \subset f(K)$ and check that if $K_1 := f^{-1}(C) \cap K$, then $K_1 \in \mathbb{K}_f$ and $f(K_1) = C$.

2. Surjectivity of the measure image map

Regarding the problems proposed by Abstract we offer first an example and a positive result. Denote by $\mathbb{P}_d(X) \subset \mathbb{P}(X)$ the set of all discrete Borel probability measures on X, i.e. of those Borel probability measures that are supported by an at most countable set in X. Recall that a bijection $f \colon X \to Y$ is called a *Borel isomorphism* of X and Y if $f(B)$ is a Borel set in Y if and only if B is a Borel set in X. Also recall that X is called a *Radon space* if there are no other Borel probability measures on X than those in $\mathbb{P}(X)$. Note, that all Souslin spaces are Radon [4; p. 122, Theorem 10].

EXAMPLE 2.1. Denote by R, S and by D the set of real numbers endowed by the standard topology, by Sorgenfrey topology which has for a base the family of all half-open intervals $[a, b)$ and by the discrete topology, respectively. Obviously, the topology of D is stronger than that of S which is again stronger than that of R.

Recall that S is a fully Lindelöf space (separable, non-metrizable) such that its compacts are at most countable (see [3; p. 59, Example K]). Hence, $\mathbb{B}(R) = \mathbb{B}(S)$ and $\mathbb{P}(S) = \mathbb{P}_d(S)$ and therefore the identity map $i \colon S \to R$ is a Borel isomorphism such that $I \colon \mathbb{P}(S) \to \mathbb{P}(R)$ is not a surjection where $I := \tilde{i}$ as agreed. Note that both i and I are continuous maps by Lemma 1.2.2.

Obviously, D is a non-separable metric space such that $\mathbb{B}(D) = 2^{\mathbb{R}}$ and such that $\mathbb{K}(D)$ consists exactly of finite sets in $\mathbb{R}$. Hence, again $\mathbb{P}(D) = \mathbb{P}_d(D)$, but this time the identity $I \colon \mathbb{P}(D) \to \mathbb{P}(S)$ is even a bijection while the identity $i \colon D \to S$ is everything but Borel isomorphism of S and D.

Note that under the continuum hypotheses there are no other Borel probabilities on D than the discrete ones ([1; p. 266, Theorem 13]), especially D is a Radon space while S does not possess the property. Hence, a continuous image of a Radon space need not be generally a Radon space.

To get some other examples, we apply 1.2.3 that offers a simple equivalent definition of the universal Lusin measurability concept.

2.2. *Let $f\colon X \to Y$ be an arbitrary map and denote the projection of $X \times Y$ onto X by π_X. Then $\widetilde{\pi_X}\colon \mathbb{P}(\mathrm{Graph}(f)) \to \mathbb{P}(X)$ is a surjective map if and only if $f \in \mathrm{ULM}(X,Y)$.*

Proof. According to 1.2.3, $\widetilde{\pi_X}$ is a surjection if and only if any $q \in \mathbb{P}(X)$ is a measure supported by an $\bigcup \pi_X(K_n)$ where $K_n \in \mathbb{K}(\mathrm{Graph}(f))$ for $n \in \mathbb{N}$. This is exactly as to say that any $q \in \mathbb{P}(X)$ is supported by an $\bigcup D_n$ where $D_n \in \mathbb{K}_f$ for $n \in \mathbb{N}$, i.e. that $f \in \mathrm{ULM}(X,Y)$. □

EXAMPLE 2.3. Choosing a $g\colon [0,1] \to [0,1]$ which is not measurable for the Lebesgue measure λ on $[0,1]$ we construct a separable metric space X ($X = \mathrm{Graph}(g) \subset [0,1]^2$) and a continuous open surjection f ($f = \pi_X$) onto a compact metric space Y ($Y = [0,1]$) such that $\tilde{f}\colon \mathbb{P}(X) \to \mathbb{P}(Y)$ is not a surjective map.

Agree to call a map $f\colon X \to Y$ *measure surjective* or a *measure surjection* if $f \in \mathrm{ULM}(X,Y)$ and $\tilde{f}\colon \mathbb{P}(X) \to \mathbb{P}(Y)$ is a surjective map. Note that if an $f\colon X \to Y$ is a measure surjective map, then automatically $f(X) = Y$ holds. Further, any bijection $f\colon X \to Y$ such that both f and f^{-1} are measure surjections will be called a *Lusin isomorphism* of X and Y. According to 2.1 there is a Borel isomorphism that is not a Lusin one and *vice versa* while any homeomorphism $h\colon X \to Y$ is both Borel and Lusin isomorphism. If each of X and Y is either a separable metric space or a Souslin space, then it follows from the second statement of the Lusin Theorem that any Borel isomorphism of the spaces is also their Lusin isomorphism. Example 2.1 and the statement 2.2 offer examples of a continuous or even open continuous bijection $f\colon X \to Y$ that is not measure surjective and therefore cannot be a Lusin isomorphism.

Next lemma offers a simple *measure surjectivity calculus.* Denote by $\mathrm{MS}(X,Y)$ the set of all measure surjective maps $f\colon X \to Y$. Recall that if $f\colon X \to Y$ is a surjection, then any map $s\colon Y \to X$ such that $f \circ s$ is the identity map on Y is called *a section* of the map f.

LEMMA 2.4. (Compare with [4; p. 37, Theorem 11].)

(1) *Consider $f \in \mathrm{MS}(X,Y)$ and $g\colon Y \to Z$ an arbitrary map. Then $g \in \mathrm{ULM}(Y,Z)$ if and only if $g \circ f \in \mathrm{ULM}(X,Z)$ and $g \in \mathrm{MS}(Y,Z)$ if and only if $g \circ f \in \mathrm{MS}(X,Z)$.*

(2) *Let* $f \in \mathrm{ULM}(X,Y)$ *be a surjection and* $s\colon Y \to X$ *a section of* f. *Then*

$$s \in \mathrm{ULM}(Y,X) \implies f \in \mathrm{MS}(X,Y).$$

(3) *Let* $f\colon X \to Y$ *be a bijection. Then* $f \in \mathrm{MS}(X,Y)$ *if and only if* $f^{-1} \in \mathrm{MS}(Y,X)$ *and if and only if* $f \in \mathrm{ULM}(X,Y)$ & $f^{-1} \in \mathrm{ULM}(Y,X)$. *Moreover, any of the above properties is equivalent to the statement that* f *is a Lusin isomorphism of* X *and* Y.

(4) *If* $f\colon X \to Y$ *is a Lusin isomorphism of* X *and* Y *and* $g\colon Y \to Z$ *a bijection, then* g *is a Lusin isomorphism of* Y *and* Z *if and only if* $g \circ f$ *is a Lusin isomorphism of* X *and* Z.

(5) *Let* $f_n\colon X_n \to Y_n$ *for* $n \in \mathbb{N}$. *Then* $f := (f_1, f_2, \ldots)$ *is a Lusin isomorphism of* $\prod\limits_n X_n$ *and* $\prod\limits_n Y_n$ *if and only if* f_n *is a Lusin isomorphism of* X_n *and* Y_n *for any* $n \in \mathbb{N}$.

Proof. Both equivalences in (1) follow by a straightforward application of 1.1.2. If f and s are as in (2) and $q \in \mathbb{P}(Y)$ put $p = \tilde{s}(q) \in \mathbb{P}(X)$ and note that $q = \tilde{f}(p)$ by 1.1.3. The equivalences in (3) are verified applying a combination of (1) and (2) as follows:

$$\begin{aligned} f, f^{-1} \in \mathrm{MS} &\implies f \in \mathrm{MS} \overset{(1)}{\iff} f^{-1} \in \mathrm{MS} \\ &\implies f, f^{-1} \in \mathrm{ULM} \overset{(2)}{\implies} f, f^{-1} \in \mathrm{MS}. \end{aligned}$$

The equivalence (4) is received by the latter equivalence in (1) and by (3). The stability statement (5) follows by (3) (the last equivalent definition) and by the first part of 1.1.5. □

EXAMPLE 2.5. In the setting of 2.1 we proved that the identity map $i\colon S \to R$ is not measure surjective. Hence, by 2.4.2 the $f := i^{-1}$ is a map with a closed graph in $R \times S$ such that $f \notin \mathrm{ULM}(R,S)$.

Remark 2.6. Considering a surjective map $f\colon X \to Y$ we may apply either 2.4.2 or 1.2.3 to receive sufficient conditions for f to be a measure surjection.

(1) If f is a continuous map, then according to 1.2.3 the f is a measure surjection if and only if any $q \in \mathbb{P}(Y)$ is supported by a countable union of sets in $\mathbb{K}^f$. Thus, if X is a Souslin space and f a continuous map, then $f \in \mathrm{MS}(X,Y)$ by [4; p. 126, Lemma 19].

(2) For example, if $M \subset X \times Y$ is a Souslin set, then $\pi_X \in \mathrm{MS}(M, \pi_X(M))$ where π_X denotes the projection of $X \times Y$ onto X. Assertion 2.2 shows that if $M = \mathrm{Graph}(g)$ for a $g\colon X \to Y$, the requirement on M to be a Souslin space can be removed if and only if $g \in \mathrm{ULM}(X,Y)$.

(3) Observing (2) one might be inclined to believe that π_X is a measure surjective map from M onto $\pi_X(M)$ if $M \in \mathbb{U}(X \times Y)$ as it happens when the M is the graph of a $g \in \mathrm{ULM}(X,Y)$ by 1.1.7 and by 2.2. But Example 2.1 disproves the hypotheses:

Let M be the diagonal in $[0,1]^2$. Then M is a closed set both in $R \times S$ and $R \times D$ and neither for $M \subset R \times S$ nor $M \subset R \times D$ it may be true that $f := \pi_R \in \mathrm{MS}(M,R)$ holds because in both cases $\tilde{f}(\mathbb{P}(M)) = \mathbb{P}_d(R)$. Especially, the identity map i on $\mathbb{R}$ does not belong neither to $\mathrm{ULM}(R,S)$ nor to $\mathrm{ULM}(R,D)$ according to 2.2 even though its graph is in both cases universally measurable.

Are there Souslin spaces X, Y and a set $M \in \mathbb{U}(X \times Y)$ such that π_X is not in $MS(M, \pi_X(M))$?

(4) If X, Y are both Souslin spaces and $f\colon X \to Y$ is a Borel surjection, then according to von Neumann theorem [4; p. 127, Theorem 13] there is a section $s \in \mathrm{ULM}(Y,X)$ of f. It follows from 2.4.2 that the map f is measure surjective.

Remark 2.7. Consider an arbitrary $f\colon X \to Y$ and note that it follows from 2.4.3 that 2.2 reads equivalently as

$$f \in \mathrm{ULM}(X,Y) \iff \pi_X\colon \mathrm{Graph}(f) \to X \quad \text{is a Lusin isomorphism of } X \text{ and } \mathrm{Graph}(f).$$

Applying 2.4.3 again it is further equivalent to $\pi_X, \pi_X^{-1} \in \mathrm{ULM}$. These statements hold simultaneously if and only if $\pi_X^{-1} \in \mathrm{ULM}(X, \mathrm{Graph}(f))$ as π_X is a continuous map. Thus,

$$f \in \mathrm{ULM}(X,Y) \iff \pi_X^{-1} \in \mathrm{ULM}(X, \mathrm{Graph}(f)).$$

The equivalence follows also by 1.1.5 because $\pi_X^{-1} = (i,f)$ where i denotes the identity on X.

Recall that Hausdorff topological spaces X_1 and X_2 are called *Radon equivalent*[1] if $X_1 = X_2$ and $i \in \mathrm{ULM}(X_1,X_2) \cap \mathrm{ULM}(X_2,X_1)$, where the i denotes the identity map on $X := X_1 = X_2$. It follows from 2.4.3 that X_1 and X_2 are Radon equivalent if and only if the identity map i is a Lusin isomorphism of X_1 and X_2 and obviously if and only if

$$\mathrm{ULM}(X_1,Y) = \mathrm{ULM}(X_2,Y) \qquad \text{holds for all spaces } Y. \tag{2}$$

The observation that X_1 and X_2 are Radon equivalent if and only if (2) holds follows by 1.1.3 also observing that $i \in \mathrm{ULM}(X_1,X_1) \cap \mathrm{ULM}(X_2,X_2)$. If the

[1] L. Schwarz prefers rather to speak about Radon equivalent topologies of X_1 and X_2 in this case ([4; p. 156]).

topology of X_2 is finer than that of X_1, then X_i's are Radon equivalent if and only if $i \in \mathrm{ULM}(X_1, X_2)$. Moreover, if X_2 is a Souslin space, it follows that X_1 is also Souslin and by 2.6.4 and 2.4.3 that X_1 and X_2 are Radon equivalent spaces.

To state that X_1 and X_2 are Radon equivalent especially means that X_1- and X_2-measurable sets *are identically placed* in $X = X_1 = X_2$ and that $\mathbb{P}(X_1)$ and $\mathbb{P}(X_2)$ *are identical sets.* More precisely,

$$\mathbb{U}(X_1) = \mathbb{U}(X_2)\,, \qquad \bar{\mathbb{P}}(X_1) = \bar{\mathbb{P}}(X_2) \quad \text{where } \bar{\mathbb{P}}(X_i) := \{\bar{p}:\ p \in \mathbb{P}(X_i)\}\,. \quad (3)$$

The former equality follows from the first statement in 1.1.1 because the identity map i on X is both in $\mathrm{ULM}(X_1, X_2)$ and $\mathrm{ULM}(X_2, X_1)$. If $p \in \mathbb{P}(X_1)$, then $p = \tilde{i}(q)$ for a measure $q \in \mathbb{P}(X_2)$ because $i \in \mathrm{MS}(X_2, X_1)$. Hence, according to the second statement in 1.1.1, $\bar{p} = i \circ \bar{q} = \bar{q}$ on $\mathbb{U}(X_1) = \mathbb{U}(X_2)$. It follows that $\bar{p} \in \bar{\mathbb{P}}(X_2)$ and by symmetry the latter equality in (3).

The pair of requirements (3) is necessary and sufficient for X_1 and X_2 to be Radon equivalent if their topologies are comparable.

2.8. *Assume that the topology of X_2 is finer than that of X_1. Then*

(1) *X_1 and X_2 are Radon equivalent,*
(2) *any $p \in \mathbb{P}(X_1)$ is a $\mathbb{K}(X_2)$-regular measure.*
(Note that $\mathbb{K}(X_2) \subset \mathbb{B}(X_1)$),
(3) *any $p \in \mathbb{P}(X_1)$ has an extension to a $q \in \mathbb{P}(X_2)$,*
(4) $\mathbb{U}(X_1) = \mathbb{U}(X_2)$, $\bar{\mathbb{P}}(X_1) = \bar{\mathbb{P}}(X_2)$

are equivalent statements. If X_2 is a separable metric space, then X_1 and X_2 are Radon equivalent if and only if $\mathbb{U}(X_1) = \mathbb{U}(X_2)$.

Generally, *in the setting of* 2.8, *any $p \in \mathbb{P}(X_1)$ has at most one extension* $q \in \mathbb{P}(X_2)$ because if q is a such extension, then $p = \tilde{i}(q)$ holds where i denotes the continuous identity map $X_2 \to X_1$ and according to 1.2.1 the above equation has at most one solution $q \in \mathbb{P}(X_2)$.

Proof. To check the above equivalencies just observe that we have already proved that (1) $\Longrightarrow$ (4); that (4) $\Longrightarrow$ (3) $\Longrightarrow$ (2) are trivial statements and that (2) is the same as to say that the continuous identity map $i\colon X_2 \to X_1$ is measure surjective (by 1.3.3), i.e. that it is Lusin isomorphism of X_1 and X_2. If X_2 is a separable metric space, it follows from (2) in the Lusin Theorem that to state $\mathbb{U}(X_1) = \mathbb{U}(X_2)$ is as to state that the identity map $i\colon X_1 \to X_2$ is universally Lusin measurable. □

For any $f\colon X \to Y$ denote by X_f the topological space such that $X_f = X$ and such that it has for a topological base sets $G \cap f^{-1}(V)$ where $G \subset X$ and $V \subset Y$ are open sets. Obviously, the topology of X_f is equivalently defined as the coarsest topology among those that are finer than the original topology of

X and for which the map f is continuous. Now, 2.2 translates to the following theorem:

THEOREM 2.9. *Let $f\colon X \to Y$ be an arbitrary map. Then $f \in \mathrm{ULM}(X,Y)$ if and only if the spaces X and X_f are Radon equivalent.*

P r o o f . By 2.7 we are to verify that X and X_f are Radon equivalent if and only if $\pi_X\colon \mathrm{Graph}(f) \to X$ is a Lusin isomorphism of X and $\mathrm{Graph}(f)$. It is easy to check that $h := \pi_X^{-1}\colon X_f \to \mathrm{Graph}(f)$ is a homeomorphism, i.e. a Lusin isomorphism of the spaces X_f and $\mathrm{Graph}(f)$, where the latter set inherits its topology as a subset of $X \times Y$. Now, the identity map i on $X = X_f$ may be written as $i = \pi_X \circ h$ and it is a Lusin isomorphism of X and X_f if and only if π_X is a Lusin isomorphism of X and $\mathrm{Graph}(f)$ by 2.4.4. □

A simple complement to 2.9 is the following remark:

Remark 2.10. Consider $f\colon X \to Y$ and assume that Y has a countable topological base. Then $\mathbb{B}(X) = \mathbb{B}(X_f)$ holds for any Borel map f, or equivalently, f is Borel if and only if the identity i on $X = X_f$ is a Borel isomorphism of X and X_f.

Indeed, if β is a countable topological base for Y, then the sets $U_{G,V} := G \cap f^{-1}(V)$, where $G \subset X$ is an open set and $V \in \beta$, constitute a topological base for X_f. Hence, any open set $U \subset X_f$ is a countable union of $U_{G,V}$-sets, hence a set in $\mathbb{B}(X)$ if f is a Borel map.

Theorem 2.9 applies to extend [4; p. 39, Theorem 12] (see also 1.2.3 and 1.3.3).

THEOREM 2.11. *Let f be a map in $\mathrm{ULM}(X,Y)$. Then*

$$\tilde{f}(\mathbb{P}(X)) = \{q \in \mathbb{P}(Y) :\ q \text{ is a } \mathbb{K}^f(X)\text{-regular measure}\}.$$

Hence, an universally Lusin measurable map $f\colon X \to Y$ is measure surjective if and only if each measure in $\mathbb{P}(Y)$ is $\mathbb{K}^f(X)$-regular.

P r o o f . It follows from 2.9 that the identity i on $X = X_f$ is a measure surjective map in $MS(X, X_f)$. Hence, $\tilde{f}(\mathbb{P}(X)) = \tilde{f}(\mathbb{P}(X_f))$ and according to 1.3.3 $\tilde{f}(\mathbb{P}(X_f))$ consists exactly of those $q \in \mathbb{P}(Y)$ which are $\mathbb{K}^f(X_f, Y)$-regular. Finally, it is easy to check that the transfer $X \to X_f$ yields $\mathbb{K}(X_f) = \mathbb{K}_f(X)$ and therefore the equality stated by 2.11 holds because the map $f\colon X_f \to Y$ is continuous. □

COROLLARY 2.12. *If $f_n \in \mathrm{MS}(X_n, Y_n)$ for any $n \in \mathbb{N}$, then $f := (f_1, f_2, \ldots) \in \mathrm{MS}(X,Y)$ where $X := \prod_n X_n$ and $Y := \prod_n Y_n$.*

Proof. Consider $q \in \mathbb{P}(Y)$ and $\varepsilon > 0$. Because all f_n's are measure surjective, it follows from 2.11 that $q_n\big(f_n(K_n)\big) \geq \varepsilon 2^{-n}$ for some $K_n \in \mathbb{K}_{f_n}(X_n)$ where $q_n := \tilde{\pi}_n(q) \in \mathbb{P}(Y_n)$ and π_n stays for the projection of Y to Y_n. Putting $D := \prod_n f_n(K_n) = f\Big(\prod_n K_n\Big)$ we obviously get a set D in $\mathbb{K}^f(X,Y)$ such that $q(D) \geq \varepsilon$ holds. Hence, $\sup\{q(D) : D \in \mathbb{K}^f(X,Y)\} = 1$ and by 1.3.1 and 1.3.2 the measure q is $\mathbb{K}^f(X,Y)$-regular. Because $f \in \mathrm{ULM}(X,Y)$ by 1.1.5, Theorem 2.11 finally applies to prove that the map f is measure surjective. □

We suspect that if we modify 2.12 as

$$(\forall n \in \mathbb{N})\,(f_n \in \mathrm{MS}(X,Y_n)) \implies \Big(f := (f_1, f_2, \ldots) \in \mathrm{MS}\Big(X, \prod_n Y_n\Big)\Big)$$

we get a statement *that is not true*. Because of 2.6.3 we cannot repeat the argument we applied to derive the latter statement of 1.1.5 as a corollary to its former one.

Remark that Lemma 2.4 suggests to introduce a *category* where the objects are all Hausdorff topological spaces $X, Y, Z, \ldots$, where the morphisms from X to Y are exactly the maps in $\mathrm{MS}(X,Y)$, i.e. the measure surjective maps $f : X \to Y$ and finally where the composition law is given by the standard composition of maps. Observe that the definition is legal because it follows from 2.4.1 that $g \circ f \in \mathrm{MS}(X,Z)$ for any $f \in \mathrm{MS}(X,Y)$ and any $g \in \mathrm{MS}(Y,Z)$. Agree to denote this category by $\mathbf{L}$ and observe that a map $f : X \to Y$ is an $\mathbf{L}$-isomorphism if and only if it is Lusin isomorphism of X and Y, i.e. objects X and Y are $\mathbf{L}$-isomorphic if and only if there is an Lusin isomorphism of X and Y. Thus, for example, 2.4.5 and 2.12 are equivalently stated in this context as

> $\prod_n X_n$ and $\prod_n Y_n$ are $\mathbf{L}$-isomorphic spaces if and only if X_n and Y_n are $\mathbf{L}$-isomorphic for all $n \in \mathbb{N}$;
>
> f_n is a morphism from X_n to Y_n for all $n \in \mathbb{N} \implies f := (f_1, f_2, \ldots)$ is a morphism from $\prod_n X_n$ to $\prod_n Y_n$,

respectively.

Considering the functor $X \mapsto \mathbb{P}(X)$ that sends a Hausdorff space X to the Hausdorff space of its Radon probabilities $\mathbb{P}(X)$ we may ask a natural question wether $\mathbf{L}$-isomorphic spaces X and Y yield $\mathbf{L}$-isomorphic spaces of the corresponding Radon probabilities $\mathbb{P}(X)$ and $\mathbb{P}(Y)$. Especially, we may ask under which conditions a Lusin isomorphism f of X and Y yields a Lusin isomorphism $\tilde{f}$ of $\mathbb{P}(X)$ and $\mathbb{P}(Y)$. If f is a Lusin isomorphism of X and Y, then it follows from 1.2.1 that $\tilde{f} : \mathbb{P}(X) \to \mathbb{P}(Y)$ is a bijection. Hence, in view of 2.4.3, the problem is basically reduced to a possibility to verify that $\tilde{f} \in \mathrm{ULM}\big(\mathbb{P}(X), \mathbb{P}(Y)\big)$

if $f \in \mathrm{ULM}(X, Y)$, which is the second problem imposed by our Abstract. The answer is frequently positive:

Remark 2.13. If X is an arbitrary space, Y is either a *separable metric* or a *Souslin space* and $f\colon X \to Y$ a *Borel map* (hence in $\mathrm{ULM}(X, Y)$ by the Lusin Theorem), then $\tilde{f}\colon \mathbb{P}(X) \to \mathbb{P}(Y)$ is again a Borel map (hence in $\mathrm{ULM}(\mathbb{P}(X), \mathbb{P}(Y))$ again by the Lusin Theorem).

The argument involves the following hereditary properties:

Y separable metric $\implies$ $\mathbb{P}(Y)$ separable metric,
see [5; p. 49, Theorem 11.1];

Y Souslin $\implies$ $\mathbb{P}(Y)$ Souslin,
see [4; p. 385, Theorem 7]

that imply in both cases that the Borel σ-algebra of $\mathbb{P}(Y)$ is generated by maps $q \mapsto q(g)$ where g's go through all bounded Borel functions $g\colon X \to \mathbb{R}$. These maps are Borel measurable by [4; p. 387, Theorem 8] and the measurability stated by Remark follows directly.

3. Lusin measurability of the measure image map

First agree to denote the *superspace* of Radon probabilities on $\mathbb{P}(X)$, i.e. the space $\mathbb{P}(\mathbb{P}(X))$, by $\mathbb{PP}(X)$ and further note that for any $M \subset \mathbb{P}(X)$ and $P \in \mathbb{P}(M)$

$$rP(B) := \int_M p(B)\,\mathrm{d}P\,, \qquad B \in \mathbb{B}(X)\,, \tag{4}$$

defines a Borel probability measure on X that we shall call the *barycenter of* P. Definition (4) is correct because $p \mapsto p(f) := \int_X f\,\mathrm{d}p$ is a Borel measurable map from $\mathbb{P}(X)$ to $\mathbb{R}$ for any bounded Borel function $f\colon X \to \mathbb{R}$ by [4; p. 387, Theorem 8].

Recall that a set $M \subset \mathbb{P}(X)$ is called *measure convex* if $rP \in M$ for arbitrary $P \in \mathbb{P}(M)$, i.e. if the barycenter map r given by (4) is a map from $\mathbb{P}(M)$ to M. Agree also to call a Hausdorff space X $\mathbb{P}$-*convex* if $M = \mathbb{P}(X)$ is a measure convex set.

Remark 3.1. Recall that

(1) If X is a $\mathbb{P}$-convex space, then r is a continuous and (obviously) affine map from $\mathbb{PP}(X)$ to $\mathbb{P}(X)$.

We argument as follows: The barycentrical formula (4) can be obviously extended to $rP(f) = \int p(f)\,\mathrm{d}P$ to be valid for all bounded Borel $f\colon X \to \mathbb{R}$.

Hence, if $P_\alpha \to P$ in $\mathbb{PP}(X)$, then $rP_\alpha \to rP$ in $\mathbb{P}(X)$ because $p \mapsto p(f)$ is a lower-semicontinuous function for any bounded lower-semicontinuous function $f\colon X \to \mathbb{R}$.

(2) Radon (in particular Souslin) and locally compact spaces are $\mathbb{P}$-convex, see [6; p. 24, Example 2].

(3) If X is a completely regular topological space, then X is a $\mathbb{P}$-convex space if and only if the closed convex hull of any compact set $D \subset \mathbb{P}(X)$ is a compact subset of $\mathbb{P}(X)$,
see [6; p. 26, 1.2.5 Proposition].

(4) There is a set $X \subset [0,1]^2$ $(X \notin \mathbb{U}([0,1]^2))$ that is not $\mathbb{P}$-convex, see [6; p. 20, Example 5].

We shall need the following lemma:

Lemma 3.2. *Let X be a $\mathbb{P}$-convex space and g a bounded function in* $\mathrm{ULM}(X,\mathbb{R})$. *Then the map $p \mapsto \bar{p}(g)$ is in* $\mathrm{UBM}(\mathbb{P}(X),\mathbb{R})$ *and* (4) *extends to*

$$\big(\forall P \in \mathbb{PP}(X)\big)\Big(\mu := rP \implies \bar{\mu}(g) = \int\limits_{\mathbb{P}(X)} \bar{p}(g)\, \mathrm{d}\overline{P}\Big)\,. \tag{5}$$

Proof. Denote $G(p) := \bar{p}(g)$ for all bounded $g \in \mathrm{ULM}(X,\mathbb{R})$ and all $p \in \mathbb{P}(X)$. Fix such a g and a $P \in \mathbb{PP}(X)$. Then $\mu := rp \in \mathbb{P}(X)$ as X is a $\mathbb{P}$-convex space, and therefore the g is a μ-measurable function. It follows that there are bounded Borel functions $g_1 \le g \le g_2$ such that $\mu[g_1 < g_2] = 0$ holds. Because both G_1 and G_2 are Borel maps $\mathbb{P}(X) \to \mathbb{R}$ by [4; p. 387, Theorem 8] such that $G_1 \le G \le G_2$ holds on $\mathbb{P}(X)$, we may compute that $\int G_2 - G_1\, \mathrm{d}P = \mu(g_2 - g_1) = 0$. Hence $P[G_1 < G_2] = 0$ and we have proved that the G is a P-measurable function for arbitrary $P \in \mathbb{PP}(X)$, hence $G \in \mathrm{UBM}(\mathbb{P}(X),\mathbb{R})$.

What we have already proved shows that

$$\nu(U) := \int\limits_{\mathbb{P}(X)} \bar{p}(U)\, \mathrm{d}\overline{P}\,, \qquad U \in \mathbb{U}(X)\,,$$

defines correctly an extension of μ from $\mathbb{B}(X)$ to $\mathbb{U}(X)$. Hence, $\nu = \bar{\mu}$ on $\mathbb{U}(X)$ which of course verifies (5). □

We shall also need the following lemma:

Lemma 3.3. *Assume that $f\colon X \to Y$ is such that $\tilde{f} \in$* $\mathrm{ULM}(\mathbb{P}(X),\mathbb{P}(Y))$. *Then the map $F := \tilde{f}\colon \mathbb{P}(X) \to \mathbb{P}(Y)$ is measure affine, i.e. such that*

$$\big(\forall P \in \mathbb{PP}(X)\big)\big(F(rP) = r\widetilde{F}(P)\big)\,.$$

Proof. Fix a $P \in \mathbb{PP}(X)$ and note that $Q := \widetilde{F}(P)$ is a measure in $\mathbb{PP}(Y)$. Denote $\mu := rP$ and $\nu := rQ$, consider a set $B \in \mathbb{B}(Y)$. Then

$$F(\mu)(B) = \bar{\mu}(f^{-1}(B)) \overset{(a)}{=} \int_{\mathbb{P}(X)} \bar{p}(f^{-1}(B))\, \mathrm{d}\overline{P}$$

$$\overset{(b)}{=} \int_{\mathbb{P}(X)} F(p)(B)\, \mathrm{d}\overline{P} = \int_{\mathbb{P}(Y)} q(B)\, \mathrm{d}(F \circ \overline{P}) \overset{(c)}{=} \int_{\mathbb{P}(Y)} q(B)\, \mathrm{d}Q = \nu(B),$$

where (a) follows by formula (5) in 3.2, (b) is implied by 1.1.1 that in our notation reads as $f \circ \bar{p} = \overline{F(p)}$ observing that the B is a Borel set and finally (c) follows again from 1.1.1 that gives $F \circ \overline{P} = \overline{Q}$ and by [4; p. 387, Theorem 8] that says that $q \mapsto q(B)$ is a Borel map. □

In contrast to 3.1.4 we may apply 3.2 and 3.3 to get:

3.4. *If X is a $\mathbb{P}$-convex space, then any M in $\mathbb{U}(X)$ inherits the property.*

Proof. Consider an $M \in \mathbb{U}(X)$ and denote by $h\colon M \to X$ the map defined by $h(x) = x$ for all $x \in M$. Hence, $H := \tilde{h}\colon \mathbb{P}(M) \to \mathbb{P}(X)$ is a continuous injection and

$$\begin{aligned}\mathbb{P}(X|M) := H(\mathbb{P}(M)) &= \{p \in \mathbb{P}(X) :\ p \text{ is } \mathbb{K}(M)\text{-regular}\} \\ &= \{p \in \mathbb{P}(X) :\ \bar{p}(M) = 1\}\end{aligned}$$

holds by 1.2.3. It follows from 3.2 that $\mathbb{P}(X|M)$ is a universally measurable subset of $\mathbb{P}(X)$ and therefore, by (5) in 3.2, it is a measure convex set.

Take $Q \in \mathbb{PP}(M)$, denote $\nu := rQ$, apply the barycentric formula 3.3 to get $H(\nu) = r\widetilde{H}(Q)$, which shows that $H(\nu)$ is in $\mathbb{P}(X|M)$ because the latter set is measure convex and the measure $\widetilde{H}(Q)$ is a Radon probability measure on $\mathbb{P}(X|M)$. Thus, $H(\nu)$ is a $\mathbb{K}(M)$-regular measure, which directly implies that $\nu = rQ$ is a Radon measure on M. □

3.5. *If X is a $\mathbb{P}$-convex space and $f\colon X \to Y$ a map such that $\tilde{f} \in \mathrm{MS}(\mathbb{P}(X), \mathbb{P}(Y))$, then also Y is a $\mathbb{P}$-convex space.*

Proof. The requirement on F imposed by 3.5 says that $f \in \mathrm{ULM}(X, Y)$ is such that $F \in \mathrm{ULM}(\mathbb{P}(X), \mathbb{P}(Y))$ and $\widetilde{F}$ is a surjective map from $\mathbb{PP}(X)$ to $\mathbb{PP}(Y)$. Fix a $Q \in \mathbb{PP}(Y)$ and let $P \in \mathbb{PP}(X)$ be a measure such that $Q = \widetilde{F}(P)$ holds. The $\mathbb{P}$-convexity of X and the barycentric formula 3.3 thus imply the P-convexity of Y. □

Having a $\mathbb{K} \subset \mathbb{K}(X)$ we shall say that a set $T \subset \mathbb{P}(X)$ is $\mathbb{K}$-*tight* (simply tight if $\mathbb{K} = \mathbb{K}(X)$) if for every $\varepsilon > 0$ there exists a set $K \in \mathbb{K}$ such that $p(K) \geq 1 - \varepsilon$ holds for every $p \in T$. Denote

$$\mathbb{T}(\mathbb{K}) := \{T \subset \mathbb{P}(X) :\ T \text{ is } \mathbb{K}\text{-tight and closed}\}.$$

Recall that all tight sets $T \subset \mathbb{P}(X)$ are relatively compact in $\mathbb{P}(X)$ by [4; p. 379, Theorem 3] and therefore $\mathbb{T}(\mathbb{K}) \subset \mathbb{K}(\mathbb{P}(X))$ holds for any $\mathbb{K} \subset \mathbb{K}(X)$ and any X. On the other hand the equality $\mathbb{T}(\mathbb{K}(X)) = \mathbb{K}(\mathbb{P}(X))$ does not hold generally. Topological spaces X for which the above equality is valid are called *Prochorov spaces.* Remark that Polish spaces are Prochorov and that there are Souslin spaces that are not Prochorov, distinguished among them being the space all rationals Q.

The tightness and $\mathbb{P}$-convexity concepts are connected.

LEMMA 3.6. *Let $\mathbb{K} \subset \mathbb{K}(X)$ be an ideal. Then*

(1) *For any $P \in \mathbb{PP}(X)$, P is a $\mathbb{T}(\mathbb{K})$-regular measure if and only if rP is a measure that is $\mathbb{K}$-regular.*

(2) *X is a $\mathbb{P}$-convex space if and only if any P in $\mathbb{PP}(X)$ is a $\mathbb{T}(\mathbb{K}(X))$-regular measure.*

(3) *Every Prochorov space is $\mathbb{P}$-convex.*

Proof. (2) follows from (1) putting there $\mathbb{K} = \mathbb{K}(X)$.

We shall prove (1): Let P be a $\mathbb{T}(\mathbb{K})$-regular measure, choose $\varepsilon > 0$, $T \in \mathbb{T}(\mathbb{K})$ with $P(T) \geq 1 - 2^{-1}\varepsilon$ and finally $K \in \mathbb{K}$ such that $p(K) \geq 1 - 2^{-1}\varepsilon$ holds for all $p \in T$. Putting $\mu := rP$ we get

$$\mu(K) = \int\limits_{\mathbb{P}(X)} p(K)\, \mathrm{d}P \geq \int\limits_{T} p(K)\, \mathrm{d}P \geq P(T) - \frac{\varepsilon}{2} \geq 1 - \varepsilon .$$

Hence, $rP = \mu$ is a $\mathbb{K}$-regular measure by 1.3.1 because $\mathbb{K}$ is assumed to be an ideal.

Before we shall proceed to verify ($\Longleftarrow$) in (1) note that if T is a $\mathbb{K}$-tight set, then its closure $\overline{T}$ is a set in $\mathbb{T}(\mathbb{K})$. The tightness of $\overline{T}$ follows directly because $p \mapsto p(K)$ are upper semi-continuous functions on $\mathbb{P}(X)$ for all compacts $K \subset X$.

Let $\mu := rP$ be a $\mathbb{K}$-regular measure. Then there are $K_n \in \mathbb{K}$ such that $\mu(K_n) \to 1$. It follows that $p(K_n) \to 1$ in probability P and therefore for arbitrary $\varepsilon > 0$ there is a Borel set $T \subset \mathbb{P}(X)$ such that $P(T) \geq 1 - \varepsilon$ and $p(K_n) \to 1$ uniformly on T as $n \to \infty$. Obviously, the T is a $\mathbb{K}$-tight set and, as we have already noticed, $\overline{T}$ is a set in $\mathbb{T}(\mathbb{K})$ such that $P(\overline{T}) \geq 1 - \varepsilon$ holds. Hence, P is a $\mathbb{T}(\mathbb{K})$-regular measure by 1.3.1 because $\mathbb{T}(\mathbb{K})$ is is easily seen to be an ideal of compact sets.

The assertion (3) is an easy corollary to (2) and 1.3.1 because again $\mathbb{T}(\mathbb{K}(X)) = \mathbb{K}(\mathbb{P}(X))$ is an ideal. □

Theorem 2.9 offers a necessary and sufficient condition for a universally Lusin measurable $f\colon X \to Y$ to yield a universally Lusin measurable $\tilde{f}\colon \mathbb{P}(X) \to \mathbb{P}(Y)$. It reads that $\tilde{f} \in \mathrm{ULM}(\mathbb{P}(X), \mathbb{P}(Y))$ if and only if the spaces $\mathbb{P}(X)$ and $\mathbb{P}(X)_{\tilde{f}}$ are Radon equivalent. We offer a weaker form of the statement.

THEOREM 3.7. *Let X be a $\mathbb{P}$-convex space. Then $\tilde{f} \in \mathrm{ULM}(\mathbb{P}(X), \mathbb{P}(Y))$ for every $f \in \mathrm{ULM}(X,Y)$ and every Y.*

Proof. Fix Y and $f \in \mathrm{ULM}(X,Y)$. Denote by $i\colon X_f \to X$ the continuous identity which is a Lusin isomorphism of X_f and X by 2.9. We shall be able to apply $\mathbb{P}$-convexity of X to prove that

$$I := \tilde{i}\colon \mathbb{P}(X_f) \to \mathbb{P}(X) \qquad \text{is a Lusin isomorphism of } \mathbb{P}(X_f) \text{ and } \mathbb{P}(X). \quad (6)$$

Because $\tilde{f} = \tilde{f}(I^{-1})$ on $\mathbb{P}(X)$ by 1.1.3 and $\tilde{f}\colon \mathbb{P}(X_f) \to \mathbb{P}(Y)$ is a continuous map by 1.2.2, it follows by (6) that $\tilde{f}\colon \mathbb{P}(X) \to \mathbb{P}(Y)$ is a universally Lusin measurable map which concludes the proof.

We shall prove (6): Because X_f and X are Radon equivalent spaces it follows that $I\colon \mathbb{P}(X_f) \to \mathbb{P}(X)$ is a bijection. Thus, by 2.4.3, (6) holds if and only if $I \in \mathrm{MS}(\mathbb{P}(X_f), \mathbb{P}(X))$. Observing that $\tilde{I}$ is a continuous map it is further equivalent to the requirement that $\tilde{I}\colon \mathbb{PP}(X_f) \to \mathbb{PP}(X)$ is a surjection.

Consider a $P \in \mathbb{PP}(X)$ and denote $\mu := rP$. Because X is $\mathbb{P}$-convex space, we get in this way a $\mu \in \mathbb{P}(X)$ and therefore a μ that is a $\mathbb{K}_f(X)$-regular measure. It follows further from 3.6.1 that the measure P is $\mathbb{T}(\mathbb{K}_f(X))$-regular. Applying the equality $\mathbb{K}_f(X) = \mathbb{K}(X_f)$ we conclude that P is measure regular with respect to the family $\mathcal{K}$ of the tight closed subsets of $\mathbb{P}(X_f)$. It follows from [4; p. 379, Theorem 3] that $\mathcal{K} \subset \mathbb{K}(\mathbb{P}(X_f))$ and we have proved that any $P \in \mathbb{PP}(X)$ is a $\mathbb{K}(\mathbb{P}(X_f))$-regular measure. Hence, $\tilde{I}\colon \mathbb{PP}(X_f) \to \mathbb{PP}(X)$ is a surjective map by 1.3.3. □

Theorem 3.7 applies to prove:

THEOREM 3.8. *If X and Y are $\mathbb{P}$-convex spaces, $f\colon X \to Y$ a Lusin isomorphism of X and Y, then $\tilde{f}\colon \mathbb{P}(X) \to \mathbb{P}(Y)$ is a Lusin isomorphism of $\mathbb{P}(X)$ and $\mathbb{P}(Y)$. In particular, if X and Y are Lusin isomorphic $\mathbb{P}$-convex spaces, then also $\mathbb{P}(X)$ and $\mathbb{P}(Y)$ are Lusin isomorphic.*

Proof. Let $f\colon X \to Y$ is a Lusin isomorphism of X and Y. Then $\tilde{f}\colon \mathbb{P}(X) \to \mathbb{P}(Y)$ is a bijection such that both $\tilde{f}$ and $(\tilde{f})^{-1} = \widetilde{f^{-1}}$ are universally Lusin measurable maps according to 3.7. It follows from 2.4.3 that the map $\tilde{f}$ is a Lusin isomorphism of $\mathbb{P}(X)$ and $\mathbb{P}(Y)$. □

In some cases, $\mathbb{P}$-convexity of X is also a necessary condition for an $f \in \mathrm{ULM}(X,Y)$ to yield $\tilde{f} \in \mathrm{ULM}(\mathbb{P}(X), \mathbb{P}(Y))$.

3.9. *Let $f\colon Y \to X$ be a continuous map and a Lusin isomorphism of Y and X. Suppose that Y is a $\mathbb{P}$-convex space. Then $\tilde{f}\colon \mathbb{P}(Y) \to \mathbb{P}(X)$ is a Lusin isomorphism of $\mathbb{P}(Y)$ and $\mathbb{P}(X)$ if and only if the space X is $\mathbb{P}$-convex. In particular:*

(•) *If $X = Y$ and the topology of Y is finer than that of X, if X and Y are Radon equivalent and finally if Y is a $\mathbb{P}$-convex space, then*

$$\tilde{i}\colon \mathbb{P}(Y) \to \mathbb{P}(X) \text{ is a Lusin isomorphism of } \mathbb{P}(Y) \text{ and } \mathbb{P}(X) \iff \text{the space } X \text{ is } \mathbb{P}\text{-convex,}$$

denoting by i the identity map on $X = Y$.

Proof. If $\tilde{f}$ is a Lusin isomorphism of $\mathbb{P}(Y)$ and $\mathbb{P}(X)$, then X is a $\mathbb{P}$-convex space by 3.5. If X is a $\mathbb{P}$-convex space, then $(\tilde{f})^{-1} = \widetilde{f^{-1}} \in \mathrm{ULM}(\mathbb{P}(X), \mathbb{P}(Y))$ by 3.7 and $\tilde{f}$ is a Lusin isomorphism of $\mathbb{P}(Y)$ and $\mathbb{P}(X)$ by 2.4.3 observing that $\tilde{f}$ is a continuous map by 1.2.2 and a bijective map as the f is assumed to be a Lusin isomorphism. □

We are sorry to admit that *we have no example* of a Lusin isomorphism f of X and Y that would not yield the $\tilde{f}$ as a Lusin isomorphism of $\mathbb{P}(X)$ and $\mathbb{P}(Y)$. Are there any spaces X and Y that satisfy the requirements (•) in 3.9 such that X is not a $\mathbb{P}$-convex space? According to 2.1, $X = S$ and $Y = D$ satisfy (•). Is $\mathbb{P}(S)$ a $\mathbb{P}$-convex space, or equivalently, is the map $\tilde{i}\colon \mathbb{P}(D) \to \mathbb{P}(S)$ a Lusin isomorphism of $\mathbb{P}(D)$ and $\mathbb{P}(S)$ (denoting by i the identity on $\mathbb{R}$)? Are spaces $\mathbb{P}(D)$ and $\mathbb{P}(S)$ Lusin isomorphic?

Our final remark is:

Remark 3.10. The following conditions are equivalent:

$$(\forall U \in \mathbb{U}(X))\,\big(p \mapsto \bar{p}(U) \text{ is a map in } \mathrm{ULM}(\mathbb{P}(X), \mathbb{R})\big)\,. \tag{7}$$

For all spaces Y it holds that

$$(\forall f \in \mathrm{ULM}(X,Y))\,(\forall B \in \mathbb{B}(Y))\,\big(p \mapsto \tilde{f}(p)(B) \text{ is a map in } \mathrm{ULM}(\mathbb{P}(X), \mathbb{R})\big)\,. \tag{8}$$

$$(\forall f \in \mathrm{ULM}(X,\mathbb{R}))\,\big(\tilde{f} \in \mathrm{ULM}(\mathbb{P}(X), \mathbb{P}(\mathbb{R}))\big)\,. \tag{9}$$

To verify (7) $\Longrightarrow$ (8) note that any $f \in \mathrm{ULM}(X,Y)$ is in $\mathrm{UBM}(X,Y)$ by the second statement of the Lusin Theorem and therefore $f^{-1}(B) \in \mathbb{U}(X)$ for every Borel set $B \subset Y$. Observing that $\tilde{f}(p)(B) = \bar{p}(f^{-1}(B))$ holds for all such sets we prove the implication.

To verify (8) $\Longrightarrow$ (9) consider (8) with $Y = \mathbb{R}$. It says that for every Borel set $B \subset \mathbb{R}$ the function $b\colon \mathbb{P}(X) \to \mathbb{R}$ defined by $b(p) := \tilde{f}(p)(B)$ is measurable as $b^{-1}\mathbb{B}(\mathbb{R}) \subset \mathbb{U}(\mathbb{P}(X))$. This proves the implication as the Borel σ-algebra of the space $\mathbb{P}(\mathbb{R})$ is generated by the maps $q \mapsto q(B)$ where B runs in $\mathbb{B}(\mathbb{R})$.

Finally, let us verify (9) $\implies$ (7): Consider a $U \in \mathbb{U}(X)$, put $f := I_U$ and observe that $\bar{p}(U) = \tilde{f}(p)(\{1\})$ for all $p \in \mathbb{P}(X)$. As obviously $f \in \mathrm{ULM}(X, \mathbb{R})$, it follows by (9) that $p \mapsto \bar{p}(U)$ is in $\mathrm{ULM}(\mathbb{P}(X), \mathbb{R})$.

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