Math. Slovaca, **51** (2001), No. 4, 421–439

Mathematica Slovaca

SOME FAMILIES OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

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(*Communicated by Michal Zajac*)

ABSTRACT. By making use of the familiar Sălăgean derivatives, the authors introduce and study a certain subclass $\mathcal{T}_{n,k}^{\lambda}(\alpha,\beta,\gamma)$ of normalized analytic functions with negative coefficients. In addition to finding a necessary and sufficient (and sharp) condition for a function to belong to the class $\mathcal{T}_{n,k}^{\lambda}(\alpha,\beta,\gamma)$, a number of other potentially useful properties and characteristics of functions in this class are investigated rather systematically. Finally, several applications involving an integral operator and some fractional calculus operators are also considered.

1. Introduction and definitions

Denote by $\mathcal{A}_k$ the class of functions of the form:

$$f(z) = z + \sum_{j=k+1}^{\infty} a_j z^j \qquad (k \in \mathbb{N} := \{1,2,3,\dots\}), \tag{1.1}$$

which are *analytic* in the *open* unit disk

$$\mathcal{U} := \{z : \ z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Also let the operator:

$$\mathcal{D}^n \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})$$

be defined, for a function $f \in \mathcal{A}_k$, by

$$\mathcal{D}^0 f(z) = f(z),$$
$$\mathcal{D}^1 f(z) = z f'(z),$$

2000 Mathematics Subject Classification: Primary 30C45; Secondary 26A33, 33B15.
Key words: analytic function, Sălăgean derivative, Hadamard product (or convolution), integral operator, fractional calculus operator, distortion theorem.

The present investigation was supported, in part, by the *Natural Sciences and Engineering Research Council of Canada* under Grant OGP0007353.

and

$$\mathcal{D}^n f(z) = \mathcal{D}\big(\mathcal{D}^{n-1} f(z)\big) \qquad (n \in \mathbb{N}).$$

The operator $\mathcal{D}^n$ is known as the Sălăgean derivative operator of order $n \in \mathbb{N}_0$ (cf. [9]; see also [6], where it was used recently in determining several interesting criteria for univalence of analytic functions).

For a function $f(z)$ given by (1.1), it follows from the above definition that

$$\mathcal{D}^n f(z) = z + \sum_{j=k+1}^{\infty} j^n a_j z^j \qquad (n \in \mathbb{N}_0\,). \tag{1.2}$$

With the help of the operator $\mathcal{D}^n$, we say that a function $f \in \mathcal{A}_k$ is in the class $\mathcal{A}^{\lambda}_{n,k}(\alpha,\beta,\gamma)$ if and only if

$$\left|\frac{F_{n,\lambda}(z) - 1}{\gamma F_{n,\gamma}(z) + 1 - (1+\gamma)\alpha}\right| < \beta \tag{1.3}$$

$$(z \in \mathcal{U};\ \ n \in \mathbb{N}_0\,;\ \ 0 \leqq \lambda \leqq 1;\ \ 0 \leqq \alpha < 1;\ \ 0 < \beta \leqq 1;\ \ 0 \leqq \gamma \leqq 1),$$

where, for convenience,

$$F_{n,\lambda}(z) := \frac{(1-\lambda)z\big(\mathcal{D}^n f(z)\big)' + \lambda z\big(\mathcal{D}^{n+1} f(z)\big)'}{(1-\lambda)\mathcal{D}^n f(z) + \lambda \mathcal{D}^{n+1} f(z)} =: \frac{\phi_{n,\lambda}(z)}{\psi_{n,\lambda}(z)}\,.$$

Let $\mathcal{T}_k$ denote the subclass of $\mathcal{A}_k$ consisting of functions of the form:

$$f(z) = z - \sum_{j=k+1}^{\infty} a_j z^j \qquad (a_j \geqq 0;\ \ j = k+1,\, k+2,\, k+3, \dots;\ \ k \in \mathbb{N}) \tag{1.4}$$

and define the class $\mathcal{T}^{\lambda}_{n,k}(\alpha,\beta,\gamma)$ by

$$\mathcal{T}^{\lambda}_{n,k}(\alpha,\beta,\gamma) = \mathcal{A}^{\lambda}_{n,k}(\alpha,\beta,\gamma) \cap \mathcal{T}_k\,. \tag{1.5}$$

We note that, by specializing the parameters k, λ, α, β, γ, and n, we can obtain the following subclasses studied by various authors.

(i) $\mathcal{T}^{\lambda}_{0,k}(\alpha,1,1) = \mathcal{P}(k,\lambda,\alpha)$ (Altintaş [1]),

(ii) $\mathcal{T}^{0}_{0,1}(\alpha,1,1) = \mathcal{T}^{*}(\alpha)$ and $\mathcal{T}^{1}_{0,1}(\alpha,1,1) = \mathcal{T}^{0}_{1,1}(\alpha,1,1) = \mathcal{C}(\alpha)$ (Silverman [11]),

(iii) $\mathcal{T}^{0}_{0,k}(\alpha,1,1) = \mathcal{T}_{\alpha}(k)$ and $\mathcal{T}^{1}_{0,k}(\alpha,1,1) = \mathcal{T}^{0}_{1,k}(\alpha,1,1) = \mathcal{C}_{\alpha}(k)$ (Chatterjea [4] and Srivastava et al. [15]),

(iv) $\mathcal{T}^{\lambda}_{n,k}(\alpha,1,1) = \mathcal{P}(k,\lambda,\alpha,n)$ (Aouf and Srivastava [3]), where $\mathcal{P}(k,\lambda,\alpha,n)$ represents the class of functions $f \in \mathcal{A}_k$ which satisfy the inequality [3; p. 763, Equation (1.5)]:

$$\Re\left(\frac{(1-\lambda)z\big(\mathcal{D}^n f(z)\big)' + \lambda z\big(\mathcal{D}^{n+1} f(z)\big)'}{(1-\lambda)\mathcal{D}^n f(z) + \lambda \mathcal{D}^{n+1} f(z)}\right) > \alpha \tag{1.6}$$
$$(z \in \mathcal{U};\ \ n \in \mathbb{N}_0;\ \ 0 \leqq \lambda \leqq 1;\ \ 0 \leqq \alpha < 1).$$

The present paper aims at providing a systematic investigation of the various interesting properties and characteristics of the general class $\mathcal{T}_{n,k}^{\lambda}(\alpha,\beta,\gamma)$, which we have introduced here. Our results involving the class $\mathcal{T}_{n,k}^{\lambda}(\alpha,\beta,\gamma)$ provide improvements and generalizations of those given by (for example) the aforecited earlier authors.

2. Coefficient inequalities and other basic properties of the class $\mathcal{T}_{n,k}^{\lambda}(\alpha,\beta,\gamma)$

THEOREM 1. *Let the function f be defined by* (1.4). *Then $f \in \mathcal{T}_{n,k}^{\lambda}(\alpha,\beta,\gamma)$ if and only if*

$$\sum_{j=k+1}^{\infty} j^n(1-\lambda+\lambda j)\{(j-1)(1+\beta\gamma)+\beta(1+\gamma)(1-\alpha)\}a_j \leqq \beta(1+\gamma)(1-\alpha). \tag{2.1}$$

The result is sharp.

Proof. Assume that the inequality (2.1) holds true. Then, for $|z| = r < 1$, we observe that

$$\begin{aligned}
&\big|\phi_{n,\lambda}(z) - \psi_{n,\lambda}(z)\big| - \beta\big|\gamma\phi_{n,\lambda}(z) + \{1-(1+\gamma)\alpha\}\psi_{n,\lambda}(z)\big| \\
&= \left|-\sum_{j=k+1}^{\infty} j^n(1-\lambda+\lambda j)(j-1)a_j z^{j-1}\right| \\
&\qquad - \beta\left|(1+\gamma)(1-\alpha) - \sum_{j=k+1}^{\infty} j^n(1-\lambda+\lambda j)\{(1-\alpha)+(j-\alpha)\gamma\}a_j z^{j-1}\right| \\
&\leqq \sum_{j=k+1}^{\infty} j^n(1-\lambda+\lambda j)(j-1)a_j \\
&\qquad - \beta\left[(1+\gamma)(1-\alpha) - \sum_{j=k+1}^{\infty} j^n(1-\lambda+\lambda j)\{(1-\alpha)+(j-\alpha)\gamma\}a_j\right]
\end{aligned}$$

$$\leqq \sum_{j=k+1}^{\infty} j^n(1-\lambda+\lambda j)\{(j-1)(1+\beta\gamma)+\beta(1+\gamma)(1-\alpha)\}a_j - \beta(1+\gamma)(1-\alpha)$$
$$\leqq 0,$$

where we have used the inequality (2.1). Hence, by the *Maximum Modulus Theorem* (cf., e.g., [5]) and (1.3), $f \in \mathcal{T}_{n,k}^{\lambda}(\alpha,\beta,\gamma)$.

Conversely, we assume that the function f is in the class $\mathcal{T}_{n,k}^{\lambda}(\alpha,\beta,\gamma)$. Then we have

$$\left|\frac{F_{n,\lambda}(z)-1}{\gamma F_{n,\lambda}(z)+1-(1+\gamma)\alpha}\right|$$
$$= \left|\frac{-\sum_{j=k+1}^{\infty} j^n(1-\lambda+\lambda j)(j-1)a_j z^{j-1}}{(1+\gamma)(1-\alpha)-\sum_{j=k+1}^{\infty} j^n(1-\lambda+\lambda j)\{(1-\alpha)+(j-\alpha)\gamma\}a_j z^{j-1}}\right|$$
$$< \beta \qquad (z \in \mathcal{U}).$$

Since $|\mathfrak{R}(z)| \leqq |z|$ for all z, we obtain the inequality:

$$\mathfrak{R}\left(\frac{\sum_{j=k+1}^{\infty} j^n(1-\lambda+\lambda j)(j-1)a_j z^{j-1}}{(1+\gamma)(1-\alpha)-\sum_{j=k+1}^{\infty} j^n(1-\lambda+\lambda j)\{(1-\alpha)+(j-\alpha)\gamma\}a_j z^{j-1}}\right) < \beta$$
$$(z \in \mathcal{U}). \tag{2.2}$$

Now choose values of z on the real axis so that $F_{n,\lambda}(z)$ is real. Upon clearing the denominator in (2.2) and letting $z \to 1-$ through real values, we find that

$$\sum_{j=k+1}^{\infty} j^n(1-\lambda+\lambda j)(j-1)a_j$$
$$\leqq \beta(1+\gamma)(1-\alpha) - \beta\sum_{j=k+1}^{\infty} j^n(1-\lambda+\lambda j)\{(1-\alpha)+(j-\alpha)\gamma\}a_j,$$

which leads us readily to the inequality (2.1).

Finally, by noting that the function f given by

$$f(z) = z - \frac{\beta(1+\gamma)(1-\alpha)}{j^n(1-\lambda+\lambda j)\{(j-1)(1+\beta\gamma)+\beta(1+\gamma)(1-\alpha)\}}z^j \tag{2.3}$$
$$(j \geqq k+1;\quad k \in \mathbb{N})$$

is an extremal function for the assertion of Theorem 1, we complete our proof of Theorem 1. □

COROLLARY 1. *Let the function f defined by* (1.4) *be in the class* $\mathcal{T}_{n,k}^{\lambda}(\alpha,\beta,\gamma)$. *Then*

$$a_j \leqq \frac{\beta(1+\gamma)(1-\alpha)}{j^n(1-\lambda+\lambda j)\{(j-1)(1+\beta\gamma)+\beta(1+\gamma)(1-\alpha)\}} \tag{2.4}$$
$$(j \geqq k+1;\ \ k \in \mathbb{N}).$$

The equality in (2.4) *is attained for the function* $f(z)$ *given by* (2.3).

Remark 1. Since

$$1-\lambda+\lambda j \leqq 1-\mu+\mu j \qquad (j \geqq k+1;\ \ k \in \mathbb{N};\ \ 0 \leqq \lambda \leqq \mu \leqq 1),$$

we have the inclusion property:

$$\mathcal{T}_{n,k}^{\mu}(\alpha,\beta,\gamma) \subseteq \mathcal{T}_{n,k}^{\lambda}(\alpha,\beta,\gamma) \qquad (0 \leqq \lambda \leqq \mu \leqq 1).$$

Furthermore, for $0 \leqq \alpha_1 \leqq \alpha_2 < 1$, it is easily verified that

$$\frac{(j-1)(1+\beta\gamma)+\beta(1+\gamma)(1-\alpha_1)}{1-\alpha_1} \leqq \frac{(j-1)(1+\beta\gamma)+\beta(1+\gamma)(1-\alpha_2)}{1-\alpha_2},$$

so that, with the aid of Theorem 1, we obtain the inclusion property:

$$\mathcal{T}_{n,k}^{\lambda}(\alpha_2,\beta,\gamma) \subseteq \mathcal{T}_{n,k}^{\lambda}(\alpha_1,\beta,\gamma) \qquad (0 \leqq \alpha_1 \leqq \alpha_2 < 1).$$

THEOREM 2. *For each* $n \in \mathbb{N}_0$,

$$\mathcal{T}_{n+1,k}^{\lambda}(\alpha,\beta,\gamma) \subset \mathcal{T}_{n,k}^{\lambda}(\xi,\beta,\gamma),$$

where

$$\xi := \frac{(1+\beta\gamma)(k+\alpha)+\beta(1+\gamma)(1-\alpha)}{(1+\beta\gamma)(k+1)+\beta(1+\gamma)(1-\alpha)}.$$

The result is sharp.

P r o o f. Suppose that the function f defined by (1.4) belongs to the class $\mathcal{T}_{n+1,k}^{\lambda}(\alpha,\beta,\gamma)$. Then, by Theorem 1,

$$\sum_{j=k+1}^{\infty} j^{n+1}(1-\lambda+\lambda j)\{(j-1)(1+\beta\gamma)+\beta(1+\gamma)(1-\alpha)\}a_j \leqq \beta(1+\gamma)(1-\alpha). \tag{2.5}$$

To prove that $f \in \mathcal{T}_{n,k}^{\lambda}(\xi,\beta,\gamma)$, it is sufficient to find the largest ξ such that

$$\sum_{j=k+1}^{\infty} j^{n}(1-\lambda+\lambda j)\{(j-1)(1+\beta\gamma)+\beta(1+\gamma)(1-\xi)\}a_j \leqq \beta(1+\gamma)(1-\xi). \tag{2.6}$$

In view of (2.5), (2.6) will hold true if

$$\frac{(j-1)(1+\beta\gamma)+\beta(1+\gamma)(1-\xi)}{1-\xi} \leqq \frac{j\left[(j-1)(1+\beta\gamma)+\beta(1+\gamma)(1-\alpha)\right]}{1-\alpha}$$
$$(j \geqq k+1;\quad k \in \mathbb{N}),$$

that is, if

$$\xi \leqq \frac{(1+\beta\gamma)(j-1+\alpha)+\beta(1+\gamma)(1-\alpha)}{(1+\beta\gamma)j+\beta(1+\gamma)(1-\alpha)} \qquad (j \geqq k+1;\quad k \in \mathbb{N}). \tag{2.7}$$

Since the right-hand side of (2.7) is an increasing function of j, letting $j = k+1$ in (2.7), we obtain

$$\xi \leqq \frac{(1+\beta\gamma)(k+\alpha)+\beta(1+\gamma)(1-\alpha)}{(1+\beta\gamma)(k+1)+\beta(1+\gamma)(1-\alpha)},$$

which proves the main assertion of Theorem 2.

Finally, by taking the function f given by

$$f(z) = z - \frac{\beta(1+\gamma)(1-\alpha)}{(k+1)^n(1+\lambda k)\left\{(1+\beta\gamma)k+\beta(1+\gamma)(1-\alpha)\right\}} z^{k+1} \qquad (k \in \mathbb{N}), \tag{2.8}$$

we can see that the result of Theorem 2 is sharp. □

Remark 2. Since $\xi > \alpha$, it follows from Remark 1 that

$$\mathcal{T}^{\lambda}_{n,k}(\xi,\beta,\gamma) \subset \mathcal{T}^{\lambda}_{n,k}(\alpha,\beta,\gamma) \qquad (n \in \mathbb{N}_0)$$

and hence that

$$\mathcal{T}^{\lambda}_{n+1,k}(\alpha,\beta,\gamma) \subset \mathcal{T}^{\lambda}_{n,k}(\xi,\beta,\gamma) \subset \mathcal{T}^{\lambda}_{n,k}(\alpha,\beta,\gamma) \qquad (n \in \mathbb{N}_0),$$

where ξ is defined with Theorem 2.

THEOREM 3. *Let* $0 \leqq \alpha_j < 1$ $(j = 1,2)$ *and* $0 < \beta_j \leqq 1$ $(j = 1,2)$. *Then*

$$\mathcal{T}^{\lambda}_{n,k}(\alpha_1,\beta_1,1) = \mathcal{T}^{\lambda}_{n,k}(\alpha_2,\beta_2,1) \qquad (n \in \mathbb{N}_0) \tag{2.9}$$

if and only if

$$\frac{\beta_1(1-\alpha_1)}{1+\beta_1} = \frac{\beta_2(1-\alpha_2)}{1+\beta_2}. \tag{2.10}$$

In particular, if $0 \leqq \alpha < 1$ *and* $0 < \beta \leqq 1$, *then*

$$\mathcal{T}^{\lambda}_{n,k}(\alpha,\beta,1) = \mathcal{T}^{\lambda}_{n,k}\left(\frac{1-\beta+2\alpha\beta}{1+\beta},1,1\right) = \mathcal{P}\left(k,\lambda,\frac{1-\beta+2\alpha\beta}{1+\beta},n\right) \tag{2.11}$$
$$(n \in \mathbb{N}_0).$$

P r o o f. Let us first assume that the function f defined by (1.4) is in the class $\mathcal{T}_{n,k}^{\lambda}(\alpha_1,\beta_1,1)$ and let the condition (2.10) hold true. Then, by (2.1), we get

$$\sum_{j=k+1}^{\infty} \frac{j^n(1-\lambda+\lambda j)\{(j-1)(1+\beta_2)+2\beta_2(1-\alpha_2)\}}{2\beta_2(1-\alpha_2)} a_j$$
$$= \sum_{j=k+1}^{\infty} \frac{j^n(1-\lambda+\lambda j)\{(j-1)(1+\beta_1)+2\beta_1(1-\alpha_1)\}}{2\beta_1(1-\alpha_1)} a_j \leqq 1,$$

which shows that $f \in \mathcal{T}_{n,k}^{\lambda}(\alpha_2,\beta_2,1)$, again with the aid of Theorem 1.

Reversing the above steps, we can similarly prove that, under the condition (2.10),

$$f \in \mathcal{T}_{n,k}^{\lambda}(\alpha_2,\beta_2,1) \implies f \in \mathcal{T}_{n,k}^{\lambda}(\alpha_1,\beta_1,1).$$

Conversely, the assertion (2.9) can easily be shown to imply the condition (2.10). The proof of Theorem 3 is thus completed by observing that (2.11) is a special case of (2.9) when

$$\alpha_1 = \alpha, \qquad \beta_1 = \beta, \qquad \beta_2 = 1.$$

□

Similarly, we can prove the following theorem.

THEOREM 4. *Let* $0 \leqq \alpha < 1$, $0 < \beta_j \leqq 1$, *and* $0 \leqq \gamma_j \leqq 1$ $(j = 1,2)$. *Then*

$$\mathcal{T}_{n,k}^{\lambda}(\alpha,\beta_1,\gamma_1) = \mathcal{T}_{n,k}^{\lambda}(\alpha,\beta_2,\gamma_2) \qquad (n \in \mathbb{N}_0) \tag{2.12}$$

if and only if

$$\frac{\beta_1(1+\gamma_1)}{1-\beta_1} = \frac{\beta_2(1+\gamma_2)}{1-\beta_2}.$$

In particular, if $0 < \beta \leqq 1$ *and* $0 \leqq \gamma \leqq 1$, *then*

$$\mathcal{T}_{n,k}^{\lambda}(\alpha,\beta,\gamma) = \mathcal{T}_{n,k}^{\lambda}\left(\alpha, \frac{\beta(1+\gamma)}{2-\beta+\beta\gamma}, 1\right) \qquad (n \in \mathbb{N}_0). \tag{2.13}$$

3. Inclusion properties associated with modified Hadamard products

Let $f(z)$ be defined by (1.4) and let

$$g(z) = z - \sum_{j=k+1}^{\infty} b_j z^j \qquad (b_j \geqq 0;\ j = k+1, k+2, k+3, \dots;\ k \in \mathbb{N}). \tag{3.1}$$

Then the *modified* Hadamard product (or convolution) of $f(z)$ and $g(z)$ is defined here by

$$(f * g)(z) := z - \sum_{j=k+1}^{\infty} a_j b_j z^j \qquad (a_j \geqq 0;\ b_j \geqq 0;\ j = k+1, k+2, k+3, \dots;\ k \in \mathbb{N}). \tag{3.2}$$

We now prove the following theorem.

THEOREM 5. *Let the function f defined by* (1.4) *and the function g defined by* (3.1) *belong to the class $\mathcal{T}_{n,k}^{\lambda}(\eta,\beta,\gamma)$. Then the modified Hadamard product $f * g$ defined by* (3.2) *belongs to the class $\mathcal{T}_{n,k}^{\lambda}(\eta,\beta,\gamma)$, where*

$$\eta := \frac{(k+1)^n(1+\lambda k)\{(1+\beta\gamma)k + \beta(1+\gamma)(1-\alpha)\}^2 - \beta(1+\gamma)(1-\alpha)^2\{(1+\beta\gamma)k + \beta(1+\gamma)\}}{(k+1)^n(1+\lambda k)\{(1+\beta\gamma)k + \beta(1+\gamma)(1-\alpha)\}^2 - \{\beta(1+\gamma)(1-\alpha)\}^2}.$$

The result is sharp.

Proof. Employing the technique used earlier by Schild and Silverman [10], we need to find the largest η such that

$$\sum_{j=k+1}^{\infty} j^n(1-\lambda+\lambda j)\{(j-1)(1+\beta\gamma)+\beta(1+\gamma)(1-\eta)\}a_j b_j \leqq \beta(1+\gamma)(1-\eta). \tag{3.3}$$

Since

$$\sum_{j=k+1}^{\infty} j^n(1-\lambda+\lambda j)\{(j-1)(1+\beta\gamma)+\beta(1+\gamma)(1-\alpha)\}a_j \leqq \beta(1+\gamma)(1-\alpha)$$

and

$$\sum_{j=k+1}^{\infty} j^n(1-\lambda+\lambda j)\{(j-1)(1+\beta\gamma)+\beta(1+\gamma)(1-\alpha)\}b_j \leqq \beta(1+\gamma)(1-\alpha),$$

by the Cauchy-Schwarz inequality, we have

$$\sum_{j=k+1}^{\infty} \frac{j^n(1-\lambda+\lambda j)\{(j-1)(1+\beta\gamma)+\beta(1+\gamma)(1-\alpha)\}}{\beta(1+\gamma)(1-\alpha)}\sqrt{a_j b_j} \leqq 1. \tag{3.4}$$

Thus it is sufficient to show that

$$\frac{(j-1)(1+\beta\gamma)+\beta(1+\gamma)(1-\eta)}{1-\eta}a_j b_j \leqq \frac{(j-1)(1+\beta\gamma)+\beta(1+\gamma)(1-\alpha)}{1-\alpha}\sqrt{a_j b_j} \qquad (j \geqq k+1;\ k \in \mathbb{N}),$$

that is, that

$$\sqrt{a_j b_j} \leqq \frac{(1-\eta)\{(j-1)(1+\beta\gamma)+\beta(1+\gamma)(1-\alpha)\}}{(1-\alpha)\{(j-1)(1+\beta\gamma)+\beta(1+\gamma)(1-\eta)\}} \qquad (j \geqq k+1;\ \ k \in \mathbb{N}).$$

Since (3.4) implies that

$$\sqrt{a_j b_j} \leqq \frac{\beta(1+\gamma)(1-\alpha)}{j^n(1-\lambda+\lambda j)\{(j-1)(1+\beta\gamma)+\beta(1+\gamma)(1-\alpha)\}}$$
$$(j \geqq k+1;\ \ k \in \mathbb{N}),$$

we need only to prove that

$$\frac{\beta(1+\gamma)(1-\alpha)}{j^n(1-\lambda+\lambda j)\{(j-1)(1+\beta\gamma)+\beta(1+\gamma)(1-\alpha)\}}$$
$$\leqq \frac{(1-\eta)\{(j-1)(1+\beta\gamma)+\beta(1+\gamma)(1-\alpha)\}}{(1-\alpha)\{(j-1)(1+\beta\gamma)+\beta(1+\gamma)(1-\alpha)\}} \qquad (j \geqq k+1;\ \ k \in \mathbb{N})$$

or, equivalently, that

$$\eta \leqq \frac{\begin{array}{c} j^n(1-\lambda+\lambda j)\{(j-1)(1+\beta\gamma)+\beta(1+\gamma)(1-\alpha)\}^2 \\ -\beta(1+\gamma)(1-\alpha)^2\{(j-1)(1+\beta\gamma)+\beta(1+\gamma)\} \end{array}}{j^n(1-\lambda+\lambda j)\{(j-1)(1+\beta\gamma)+\beta(1+\gamma)(1-\alpha)\}^2-\{\beta(1-\gamma)(1-\alpha)\}^2}$$
$$(j \geqq k+1;\ \ k \in \mathbb{N}). \tag{3.5}$$

Since the right-hand side of (3.5) is an increasing function of j, by letting $j = k+1$ in (3.5), we obtain

$$\eta \leqq \frac{\begin{array}{c} (k+1)^n(1+\lambda k)\{(1+\beta\gamma)k+\beta(1+\gamma)(1-\alpha)\}^2 \\ -\beta(1+\gamma)(1-\alpha)^2\{(1+\beta\gamma)k+\beta(1+\gamma)\} \end{array}}{(k+1)^n(1+\lambda k)\{(1+\beta\gamma)k+\beta(1+\gamma)(1-\alpha)\}^2-\{\beta(1+\gamma)(1-\alpha)\}^2},$$

which proves the main assertion of Theorem 5.

The sharpness of the result of Theorem 5 follows if we take

$$f(z) = g(z)$$
$$= z - \frac{\beta(1+\gamma)(1-\alpha)}{(k+1)^n(1+\lambda k)\{(1+\beta\gamma)k+\beta(1+\gamma)(1-\alpha)\}} z^{k+1} \qquad (k \in \mathbb{N}). \tag{3.6}$$

□

THEOREM 6. *If each of the functions f and g belongs to the same class $\mathcal{T}_{n,k}^{\lambda}(\alpha,\beta,\gamma)$, then $(f*g)(z)$ belongs to the class $\mathcal{T}_{n,k}^{\lambda}(\rho,1,1)$ or, equivalently, $\mathcal{P}(k,\lambda,\rho,n)$, where*

$$\rho := \frac{(k+1)^n(1+\lambda k)\{(1+\beta\gamma)k+\beta(1+\gamma)(1-\alpha)\}^2-(k+1)\{\beta(1+\gamma)(1-\alpha)\}^2}{(k+1)^n(1+\lambda k)\{(1+\beta\gamma)k+\beta(1+\gamma)(1-\alpha)\}^2-\{\beta(1+\gamma)(1-\alpha)\}^2}. \tag{3.7}$$

The result is the best possible for the functions $f(z)$ and $g(z)$ defined by (3.6).

P r o o f. Proceeding as in the proof of Theorem 5, we get

$$\rho \leqq \frac{j^n(1-\lambda+\lambda j)\{(1+\beta\gamma)(j-1)+\beta(1+\gamma)(1-\alpha)\}^2-j\{\beta(1+\gamma)(1-\alpha)\}^2}{j^n(1-\lambda+\lambda j)\{(1+\beta\gamma)(j-1)+\beta(1+\gamma)(1-\alpha)\}^2-\{\beta(1+\gamma)(1-\alpha)\}^2} \qquad (j\geqq k+1;\ k\in\mathbb{N}). \tag{3.8}$$

The right-hand side of (3.8) being an increasing function of j, setting $j=k+1$ in (3.8), we obtain (3.7).

This completes the proof of Theorem 6. □

THEOREM 7. *Let the function f defined by* (1.4) *and the function g defined by* (3.1) *be in the same class $\mathcal{T}_{n,k}^{\lambda}(\alpha,\beta,\gamma)$. Then the function $h(z)$ defined by*

$$h(z) := z - \sum_{j=k+1}^{\infty}(a_j^2+b_j^2)z^j$$

belongs to the class $\mathcal{T}_{n,k}^{\lambda}(\sigma,\beta,\gamma)$, where

$$\sigma := \frac{\begin{array}{c}(k+1)^n(1+\lambda k)\{(1+\beta\gamma)k+\beta(1+\gamma)(1-\alpha)\}^2\\ -2\beta(1+\gamma)(1-\alpha)^2\{(1+\beta\gamma)k+\beta(1+\gamma)\}\end{array}}{(k+1)^n(1+\lambda k)\{(1+\beta\gamma)k+\beta(1+\gamma)(1-\alpha)\}^2-2\{\beta(1+\gamma)(1-\alpha)\}^2}.$$

The result is sharp for the functions $f(z)$ and $g(z)$ defined by (3.6).

P r o o f. By virtue of Theorem 1, we obtain

$$\sum_{j=k+1}^{\infty}\left(\frac{j^n(1-\lambda+\lambda j)\{(1+\beta\gamma)(j-1)+\beta(1+\gamma)(1-\alpha)\}}{\beta(1+\gamma)(1-\alpha)}\right)^2 a_j^2 \leqq \left(\sum_{j=k+1}^{\infty}\frac{j^n(1-\lambda+\lambda j)\{(1+\beta\gamma)(j-1)+\beta(1+\gamma)(1-\alpha)\}}{\beta(1+\gamma)(1-\alpha)}a_j\right)^2 \leqq 1. \tag{3.9}$$

Similarly, we have

$$\sum_{j=k+1}^{\infty}\left(\frac{j^n(1-\lambda+\lambda j)\{(1+\beta\gamma)(j-1)+\beta(1+\gamma)(1-\alpha)\}}{\beta(1+\gamma)(1-\alpha)}\right)^2 b_j^2 \leqq 1. \tag{3.10}$$

It follows from (3.9) and (3.10) that

$$\sum_{j=k+1}^{\infty} \frac{1}{2}\left(\frac{j^n(1-\lambda+\lambda j)\{(1+\beta\gamma)(j-1)+\beta(1+\gamma)(1-\alpha)\}}{\beta(1+\gamma)(1-\alpha)}\right)^2 (a_j^2+b_j^2) \leqq 1 .$$

Therefore, we need to find the largest σ such that

$$\frac{j^n(1-\lambda+\lambda j)\{(1+\beta\gamma)(j-1)+\beta(1+\gamma)(1-\sigma)\}}{\beta(1+\gamma)(1-\sigma)} \leqq \frac{1}{2}\left(\frac{j^n(1-\lambda+\lambda j)\{(1+\beta\gamma)(j-1)+\beta(1+\gamma)(1-\alpha)\}}{\beta(1+\gamma)(1-\alpha)}\right)^2$$

$$(j \geqq k+1; \quad k \in \mathbb{N}),$$

that is,

$$\sigma \leqq \frac{\begin{array}{c} j^n(1-\lambda+\lambda j)\{(1+\beta\gamma)(j-1)+\beta(1+\gamma)(1-\alpha)\}^2 \\ -2\beta(1+\gamma)(1-\alpha)^2\{(1+\beta\gamma)(j-1)+\beta(1+\gamma)\} \end{array}}{j^n(1-\lambda+\lambda j)\{(1+\beta\gamma)(j-1)+\beta(1+\gamma)(1-\alpha)\}^2 - 2\{\beta(1+\gamma)(1-\alpha)\}^2}$$

$$(j \geq k+1). \tag{3.11}$$

Since the right-hand side of (3.11) is an increasing function of j, we readily have

$$\sigma \leqq \frac{\begin{array}{c} (k+1)^n(1+\lambda k)\{(1+\beta\gamma)k+\beta(1+\gamma)(1-\alpha)\}^2 \\ -2\beta(1+\gamma)(1-\alpha)^2\{(1+\beta\gamma)k+\beta(1+\gamma)\} \end{array}}{(k+1)^n(1+\lambda k)\{(1+\beta\gamma)k+\beta(1+\gamma)(1-\alpha)\}^2 - 2\{\beta(1+\gamma)(1-\alpha)\}^2}, \tag{3.12}$$

and Theorem 7 follows at once. □

4. A family of integral operators

THEOREM 8. *Let the function f defined by* (1.4) *be in the class $\mathcal{T}_{n,k}^{\lambda}(\alpha,\beta,\gamma)$, and let c be a real number such that $c > -1$. Then the function $F(z)$ defined by*

$$F(z) := \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)\, \mathrm{d}t \qquad (c > -1; \quad f \in \mathcal{A}_k) \tag{4.1}$$

belongs to the class $\mathcal{T}_{n,k}^{\lambda}(\kappa,\beta,\gamma)$, where

$$\kappa := \frac{(1+\beta\gamma)\{k+(c+1)\alpha\}+\beta(1+\gamma)(1-\alpha)}{(1+\beta\gamma)(k+c+1)+\beta(1+\gamma)(1-\alpha)} .$$

The result is sharp for the function $f(z)$ *defined by* (2.8).

P r o o f . From the representation (4.1) of $F(z)$, it follows that

$$F(z) = z - \sum_{j=k+1}^{\infty} \left(\frac{c+1}{c+j}\right) a_j z^j .$$

We need to find the largest κ such that

$$\frac{\{(1+\beta\gamma)(j-1)+\beta(1+\gamma)(1-\kappa)\}(c+1)}{(1-\kappa)(c+j)} \leqq \frac{(1+\beta\gamma)(j-1)+\beta(1+\gamma)(1-\alpha)}{1-\alpha} \qquad (j \geqq k+1;\ \ k \in \mathbb{N})$$

or, equivalently,

$$\kappa \leqq \frac{(1+\beta\gamma)\{\alpha(c+1)+(j-1)\}+\beta(1+\gamma)(1-\alpha)}{(1+\beta\gamma)(c+j)+\beta(1+\gamma)(1-\alpha)} \qquad (j \geqq k+1;\ \ k \in \mathbb{N}). \tag{4.2}$$

The right-hand side of (4.2) being an increasing function of j, setting $j = k+1$ in (4.2), we obtain

$$\kappa \leqq \frac{(1+\beta\gamma)\{k+(c+1)\alpha\}+\beta(1+\gamma)(1-\alpha)}{(1+\beta\gamma)(k+c+1)+\beta(1+\gamma)(1-\alpha)} ,$$

which completes the proof of Theorem 8. □

Proceeding as in the proof of Theorem 8, we can deduce:

THEOREM 9. *If* $f \in \mathcal{T}_{n,k}^{\lambda}(\alpha,\beta,\gamma)$, *then the function* $F(z)$ *defined by* (4.1) *belongs to the class* $\mathcal{T}_{n,k}^{\lambda}(\mu,1,1)$ *or, equivalently,* $\mathcal{P}(k,\lambda,\mu,n)$, *where*

$$\mu := \frac{(1+\beta\gamma)(c+k+1)-c\beta(1+\gamma)(1-\alpha)}{(1+\beta\gamma)(c+k+1)+\beta(1+\gamma)(1-\alpha)} .$$

The result is sharp, the extremal function $f(z)$ *being given by* (2.8).

THEOREM 10. *Let the function* $F(z)$ *given by*

$$F(z) = z - \sum_{j=k+1}^{\infty} d_j z^j \qquad (d_j \geqq 0;\ \ j = k+1, k+2, k+3, \dots;\ \ k \in \mathbb{N})$$

be in the class $\mathcal{T}_{n,k}^{\lambda}(\alpha,\beta,\gamma)$, *and let* c *be a real number such that* $c > -1$. *Then the function* $f(z)$ *defined by* (4.1) *is univalent in* $|z| < R$, *where*

$$R := \inf_{j \geqq k+1} \left(\frac{j^{n-1}(1-\lambda+\lambda j)\{(1+\beta\gamma)(j-1)+\beta(1+\gamma)(1-\alpha)\}(c+1)}{\beta(1+\gamma)(1-\alpha)(c+j)}\right)^{1/(j-1)} \tag{4.3}$$

The result is sharp.

P r o o f . We find from (4.1) that

$$f(z)=\frac{z^{1-c}\left(z^{c}F(z)\right)'}{c+1}=z-\sum_{j=k+1}^{\infty}\left(\frac{c+j}{c+1}\right)d_j z^j .$$

In order to obtain the desired result, it suffices to show that

$$|f'(z)-1|<1 \qquad \text{whenever} \quad |z|<R,$$

where R is given by (4.3). Now

$$|f'(z)-1|\leqq\sum_{j=k+1}^{\infty}\frac{j(c+j)}{c+1}d_j|z|^{j-1} .$$

Thus we have $|f'(z)-1|<1$ if

$$\sum_{j=k+1}^{\infty}\frac{j(c+j)}{c+1}d_j|z|^{j-1}<1 . \tag{4.4}$$

But, by Theorem 1, we know that

$$\sum_{j=k+1}^{\infty}\frac{j^n(1-\lambda+\lambda j)\{(1+\beta\gamma)(j-1)+\beta(1+\gamma)(1-\alpha)\}}{\beta(1+\gamma)(1-\alpha)}d_j\leqq 1 .$$

Hence (4.4) will be satisfied if

$$\frac{j(c+j)}{c+1}|z|^{j-1}<\frac{j^n(1-\lambda+\lambda j)\{(1+\beta\gamma)(j-1)+\beta(1+\gamma)(1-\alpha)\}}{\beta(1+\gamma)(1-\alpha)} ,$$

that is, if

$$|z|<\left(\frac{j^{n-1}(1-\lambda+\lambda j)\{(1+\beta\gamma)(j-1)+\beta(1+\gamma)(1-\alpha)\}(c+1)}{\beta(1+\gamma)(1-\alpha)(c+j)}\right)^{1/(j-1)}$$
$$(j\geqq k+1;\ \ k\in\mathbb{N}). \tag{4.5}$$

Therefore, the function $f(z)$ given by (4.1) is univalent in $|z|<R$, where R is defined by (4.3). The sharpness of the result follows if we take

$$f(z)=z-\frac{\beta(1+\gamma)(1-\alpha)(c+j)}{j^n(1-\lambda+\lambda j)\{(1+\beta\gamma)(j-1)+\beta(1+\gamma)(1-\alpha)\}(c+1)}z^j \tag{4.6}$$
$$(j\geqq k+1;\ \ k\in\mathbb{N}).$$

□

5. Applications of fractional calculus

Many essentially equivalent definitions of fractional calculus (that is, fractional derivatives and fractional integrals) have been given in the literature (cf., e.g., [8] and [13]). We find it to be convenient to recall here the following definitions which were used earlier by Owa [7] (and, more recently, by Srivastava and Aouf [12]; see also Aouf [2]).

DEFINITION 1. The *fractional integral of order* μ is defined, for a function $f(z)$, by

$$D_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\mu}}\, \mathrm{d}\mu \qquad (\mu > 0), \tag{5.1}$$

where $f(z)$ is an analytic function in a simply-connected region of the complex z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\mu-1}$ is removed, by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

DEFINITION 2. The *fractional derivative of order* μ is defined, for a function $f(z)$, by

$$D_z^{\mu} f(z) = \frac{1}{\Gamma(1-\mu)} \frac{\mathrm{d}}{\mathrm{d}z} \int_0^z \frac{f(\zeta)\,\mathrm{d}\zeta}{(z-\zeta)^{\mu}} \qquad (0 \leqq \mu < 1), \tag{5.2}$$

where $f(z)$ is constrained, and the multiplicity of $(z-\zeta)^{-\mu}$ is removed, as in Definition 1.

DEFINITION 3. Under the hypotheses of Definition 2, the *fractional derivative of order* $n+\mu$ is defined, for a function $f(z)$, by

$$D_z^{n+\mu} f(z) = \frac{\mathrm{d}^n}{\mathrm{d}z^n}\{D_z^{\mu} f(z)\} \qquad (0 \leqq \mu < 1;\ n \in \mathbb{N}_0). \tag{5.3}$$

THEOREM 11. *Let the function f defined by* (1.4) *be in the class $\mathcal{T}_{n,k}^{\lambda}(\alpha,\beta,\gamma)$. Then*

$$\begin{aligned} &|D_z^{-\mu}(\mathcal{D}^i f(z))| \\ &\geqq \frac{r^{1+\mu}}{\Gamma(2+\mu)}\left(1 - \frac{\beta(1+\gamma)(1-\alpha)\Gamma(k+2)\Gamma(2+\mu)}{(k+1)^{n-i}(1+\lambda k)\{(1+\beta\gamma)k+\beta(1+\gamma)(1-\alpha)\}\Gamma(k+2+\mu)} r^k\right) \\ &\qquad (|z| = r < 1;\ \mu > 0;\ i \in \{0,1,\dots,n\}) \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} &|D_z^{-\mu}(\mathcal{D}^i f(z))| \\ &\leqq \frac{r^{1+\mu}}{\Gamma(2+\mu)}\left(1 + \frac{\beta(1+\gamma)(1-\alpha)\Gamma(k+2)\Gamma(2+\mu)}{(k+1)^{n-i}(1+\lambda k)\{(1+\beta\gamma)k+\beta(1+\gamma)(1-\alpha)\}\Gamma(k+2+\mu)} r^k\right) \\ &\qquad (|z| = r < 1;\ \mu > 0;\ i \in \{0,1,\dots,n\}). \end{aligned} \tag{5.5}$$

Each of the assertions (5.4) *and* (5.5) *is sharp.*

P r o o f . We observe that

$$f(z) \in \mathcal{T}_{n,k}^{\lambda}(\alpha,\beta,\gamma) \iff \mathcal{D}^i f(z) \in \mathcal{T}_{n-i,k}^{\lambda}(\alpha,\beta,\gamma)$$

and that (cf. Equation (1.2))

$$\mathcal{D}^i f(z) = z - \sum_{j=k+1}^{\infty} j^i a_j z^j \qquad (i \in \mathbb{N}_0\,).$$

In view of Theorem 1, we have

$$\begin{aligned}&(k+1)^{n-i}(1+\lambda k)\big\{(1+\beta\gamma)k + \beta(1+\gamma)(1-\alpha)\big\} \sum_{j=k+1}^{\infty} j^i a_j \\ &\leqq \sum_{j=k+1}^{\infty} j^n(1-\lambda+\lambda j)\big\{(1+\beta\gamma)(j-1) + \beta(1+\gamma)(1-\alpha)\big\} a_j \\ &\leqq \beta(1+\gamma)(1-\alpha)\,,\end{aligned}$$

so that

$$\sum_{j=k+1}^{\infty} j^i a_j \leqq \frac{\beta(1+\gamma)(1-\alpha)}{(k+1)^{n-i}(1+\lambda k)\big\{(1+\beta\gamma)k+\beta(1+\gamma)(1-\alpha)\big\}}\,. \tag{5.6}$$

Consider the function $G(z)$ defined by

$$\begin{aligned}G(z) &:= \Gamma(2+\mu) z^{-\mu} D_z^{-\mu}\big(\mathcal{D}^i f(z)\big) \\ &= z - \sum_{j=k+1}^{\infty} \frac{\Gamma(j+1)\Gamma(2+\mu)}{\Gamma(j+1+\mu)} j^i a_j z^j \\ &= z - \sum_{j=k+1}^{\infty} \Phi(j) j^i a_j z^j\,,\end{aligned}$$

where

$$\Phi(j) := \frac{\Gamma(j+1)\Gamma(2+\mu)}{\Gamma(j+1+\mu)} \qquad (j \geqq k+1\,;\ \ k \in \mathbb{N};\ \ \mu > 0\,).$$

Since $\Phi(j)$ is a decreasing function of j, we get

$$0 < \Phi(j) \leqq \Phi(k{+}1) = \frac{\Gamma(k+2)\Gamma(2+\mu)}{\Gamma(k+2+\mu)} \qquad (j \geqq k{+}1\,;\ \ k \in \mathbb{N};\ \ \mu > 0\,). \tag{5.7}$$

Thus, by using (5.6) and (5.7), we see that

$$|G(z)| \geqq r - \Phi(k+1)r^{k+1} \sum_{j=k+1}^{\infty} j^i a_j$$

$$\geqq r - \frac{\beta(1+\gamma)(1-\alpha)\Gamma(k+2)\Gamma(2+\mu)}{(k+1)^{n-i}(1+\lambda k)\{(1+\beta\gamma)k+\beta(1+\gamma)(1-\alpha)\}\Gamma(k+2+\mu)} r^{k+1}$$

$$(\,|z| = r < 1; \ \ \mu > 0; \ \ i \in \{0, 1, \ldots, n\}\,)$$

and

$$|G(z)| \leqq r + \Phi(k+1)r^{k+1} \sum_{j=k+1}^{\infty} j^i a_j$$

$$\leqq r + \frac{\beta(1+\gamma)(1-\alpha)\Gamma(k+2)\Gamma(2+\mu)}{(k+1)^{n-i}(1+\lambda k)\{(1+\beta\gamma)k+\beta(1+\gamma)(1-\alpha)\}\Gamma(k+2+\mu)} r^{k+1}$$

$$(\,|z| = r < 1; \ \ \mu > 0; \ \ i \in \{0, 1, \ldots, n\}\,),$$

which prove the inequalities (5.4) and (5.5) of Theorem 11.

The equalities in (5.4) and (5.5) are attained for the function $f(z)$ given by

$$\mathcal{D}^i f(z) = z - \frac{\beta(1+\gamma)(1-\alpha)}{(k+1)^{n-1}\{(1+\beta\gamma)k+\beta(1+\gamma)(1-\alpha)\}(1+\lambda k)} z^{k+1} \qquad (5.8)$$

$$(\,k \in \mathbb{N}).$$

This completes the proof of Theorem 11. □

Setting $i = 0$ in Theorem 11, we obtain:

COROLLARY 2. *Let the function f defined by* (1.4) *be in the class $\mathcal{T}_{n,k}^{\lambda}(\alpha, \beta, \gamma)$. Then*

$$|D_z^{-\mu} f(z)|$$

$$\geqq \frac{r^{1+\mu}}{\Gamma(2+\mu)} \left(1 - \frac{\beta(1+\gamma)(1-\alpha)\Gamma(k+2)\Gamma(2+\mu)}{(k+1)^n(1+\lambda k)\{(1+\beta\gamma)k+\beta(1+\gamma)(1-\alpha)\}\Gamma(k+2+\mu)} r^k\right)$$

$$(\,|z| = r < 1; \ \ \mu > 0\,) \qquad (5.9)$$

and

$$|D_z^{-\mu} f(z)|$$

$$\leqq \frac{r^{1+\mu}}{\Gamma(2+\mu)} \left(1 + \frac{\beta(1+\gamma)(1-\alpha)\Gamma(k+2)\Gamma(2+\mu)}{(k+1)^n(1+\lambda k)\{(1+\beta\gamma)k+\beta(1+\gamma)(1-\alpha)\}\Gamma(k+2+\mu)} r^k\right)$$

$$(\,|z| = r < 1; \ \ \mu > 0\,). \qquad (5.10)$$

The estimates in (5.9) *and* (5.10) *are sharp for the function $f(z)$ given by* (5.8) *with $i = 0$.*

THEOREM 12. *Let the function f defined by* (1.4) *be in the class $\mathcal{T}_{n,k}^{\lambda}(\alpha,\beta,\gamma)$. Then*

$$\begin{aligned}&\left|D_z^{\mu}\big(\mathcal{D}^i f(z)\big)\right| \\ &\geqq \frac{r^{1+\mu}}{\Gamma(2+\mu)}\left(1-\frac{\beta(1+\gamma)(1-\alpha)\Gamma(k+2)\Gamma(2+\mu)}{(k+1)^{n-i}(1+\lambda k)\big\{(1+\beta\gamma)k+\beta(1+\gamma)(1-\alpha)\big\}\Gamma(k+2+\mu)}r^k\right)\\ &\qquad (\,|z|=r<1;\ \ 0\leqq\mu<1;\ \ i\in\{0,1,\dots,n-1\}\,)\end{aligned} \tag{5.11}$$

and

$$\begin{aligned}&\left|D_z^{\mu}\big(\mathcal{D}^i f(z)\big)\right| \\ &\leqq \frac{r^{1+\mu}}{\Gamma(2+\mu)}\left(1+\frac{\beta(1+\gamma)(1-\alpha)\Gamma(k+2)\Gamma(2+\mu)}{(k+1)^{n-i}(1+\lambda k)\big\{(1+\beta\gamma)k+\beta(1+\gamma)(1-\alpha)\big\}\Gamma(k+2+\mu)}r^k\right)\\ &\qquad (\,|z|=r<1;\ \ 0\leqq\mu<1;\ \ i\in\{0,1,\dots,n-1\}\,).\end{aligned} \tag{5.12}$$

Each of the assertions (5.11) *and* (5.12) *is sharp.*

Proof. Consider the function $H(z)$ defined by

$$\begin{aligned}H(z) &:= \Gamma(2-\mu)z^{\mu}D_z^{\mu}\big(\mathcal{D}^i f(z)\big)\\ &= z-\sum_{j=k+1}^{\infty}\frac{\Gamma(j+1)\Gamma(2-\mu)}{\Gamma(j+1-\mu)}j^i a_j z^j = z-\sum_{j=k+1}^{\infty}\Psi(j)j^{i+1}a_j z^j\,,\end{aligned}$$

where

$$\Psi(j) := \frac{\Gamma(j)\Gamma(2-\mu)}{\Gamma(j+1-\mu)} \qquad (\,j\geqq k+1;\ \ k\in\mathbb{N};\ \ 0\leqq\mu<1\,). \tag{5.13}$$

It is easily seen from (5.13) that

$$0<\Psi(j)\leqq\Psi(k+1)=\frac{\Gamma(k+1)\Gamma(2-\mu)}{\Gamma(k+2-\mu)} \qquad (\,j\geqq k+1;\ \ k\in\mathbb{N};\ \ 0\leqq\mu<1\,). \tag{5.14}$$

Consequently, with the aid of (5.6) and (5.14), we have

$$\begin{aligned}|H(z)| &\geqq r-\Psi(k+1)r^{k+1}\sum_{j=k+1}^{\infty}j^{i+1}a_j\\ &\geqq r-\frac{\beta(1+\gamma)(1-\alpha)\Gamma(k+2)\Gamma(2-\mu)}{(k+1)^{n-i}(1+\lambda k)\big\{(1+\beta\gamma)k+\beta(1+\gamma)(1-\alpha)\big\}\Gamma(k+2-\mu)}r^{k+1}\\ &\qquad (\,|z|=r<1;\ \ 0\leqq\mu<1;\ \ i\in\{0,1,\dots,n-1\}\,)\end{aligned} \tag{5.15}$$

and

$$|H(z)| \leqq r + \Psi(k+1)r^{k+1} \sum_{j=k+1}^{\infty} j^{i+1} a_j$$
$$\leqq r + \frac{\beta(1+\gamma)(1-\alpha)\Gamma(k+2)\Gamma(2-\mu)}{(k+1)^{n-i}(1+\lambda k)\{(1+\beta\gamma)k+\beta(1+\gamma)(1-\alpha)\}\Gamma(k+2-\mu)} r^{k+1}$$
$$(\,|z| = r < 1;\;\; 0 \leqq \mu < 1;\;\; i \in \{0,1,\dots,n-1\}\,). \tag{5.16}$$

The estimates in (5.11) and (5.12) follow from (5.15) and (5.16), respectively. Each of these estimates is sharp for the function $f(z)$ given by (5.8). □

Letting $i = 0$ in Theorem 12, we have:

COROLLARY 3. *Let the function f defined by* (1.4) *be in the class* $\mathcal{T}^{\lambda}_{n,k}(\alpha,\beta,\gamma)$. *Then*

$$|D^{\mu}_{z} f(z)|$$
$$\geqq \frac{r^{1-\mu}}{\Gamma(2-\mu)} \left(1 - \frac{\beta(1+\gamma)(1-\alpha)\Gamma(k+2)\Gamma(2-\mu)}{(k+1)^{n}(1+\lambda k)\{(1+\beta\gamma)k+\beta(1+\gamma)(1-\alpha)\}\Gamma(k+2-\mu)} r^{k}\right)$$
$$(\,|z| = r < 1;\;\; 0 \leqq \mu < 1\,) \tag{5.17}$$

and

$$|D^{\mu}_{z} f(z)|$$
$$\leqq \frac{r^{1-\mu}}{\Gamma(2-\mu)} \left(1 + \frac{\beta(1+\gamma)(1-\alpha)\Gamma(k+2)\Gamma(2-\mu)}{(k+1)^{n}(1+\lambda k)\{(1+\beta\gamma)k+\beta(1+\gamma)(1-\alpha)\}\Gamma(k+2-\mu)} r^{k}\right)$$
$$(\,|z| = r < 1;\;\; 0 \leqq \mu < 1\,). \tag{5.18}$$

The estimates in (5.17) *and* (5.18) *are sharp for the function* $f(z)$ *given by* (5.8) *with* $i = 0$.

Remark 3. Many of the results of this section can suitably (and fairly easily) be extended to hold true for such generalized fractional calculus operators as those with the Gauss hypergeometric function kernel, which were considered earlier by Aouf and Srivastava [3; p. 786, Definition 4] (see also Srivastava and Owa [13]).

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Received May 22, 2000

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