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POSITIVE SOLUTIONS OF A CERTAIN TYPE OF TWO-POINT BOUNDARY VALUE PROBLEMS

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ABSTRACT. The paper gives sufficient and necessary conditions for the existence of positive solutions for a certain type of two point boundary value problems which depend on a parameter $a \in \mathbf{R}$.

The present paper considers the following problem. We want to find all $a \in \mathbf{R}$ such that the equation

$$\begin{aligned} -u'' &= (f_a(x) + g(u)) \,.\, u - s(u) \,.\, v \\ -v'' &= (a + r(u)) \,.\, v - v^2 \\ u(0) &= u(\pi) = v(0) = v(\pi) = 0 \end{aligned} \tag{1}$$

has a positive solution u, v, i.e. $u(x) > 0$, $v(x) > 0$ for $x \in (0, \pi)$. Generally, the existence of positive solutions of boundary value problems has important applications in ecology. We make use of the *global and local bifurcation theorem of Crandall and Rabinowitz* from [1], where a similar problem is solved.

Let us assume

$$f_.(.) \in C^1(\mathbf{R} \times \mathbf{R}, \mathbf{R}),\ g,\ s,\ r \in C^1(\mathbf{R}, \mathbf{R}),\ \frac{\partial}{\partial a} f_a(.) > 0, f_a(.) \geqq 2,$$

$$g(0) = g'(0) = 0,\ g'(u) < 0 \ \text{for}\ u > 0,\ r(0) = r'(0) = 0,\ s(0) = s'(0) = 0,$$

$$r \langle 0, \infty) \geqq 1,\ r'/(0, \infty) > 0,\ s/(0, \infty) \geqq 0,\ \lim g = -\infty \text{ as } x \to \infty.$$

Theorem 1. *Consider the equation*

$$\begin{aligned} -u'' &= (g(u) + f(x)) \,.\, u \\ u(0),\ u'(0) &= e,\ e > 0, \end{aligned} \tag{1$^+$}$$

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where $f \in C^1$, $f(.) \geqq 2$. *Let* $x_1(e) > 0$ *be a first root of a solution* $u(., e)$ *of* $(1)^+$, *i.e. we assume that* $u(., e)$ *is a solution of* $(1)^+$ *on* $\langle 0, x_1(e) \rangle$, $u(x, e) > 0$ *for* $x \in \in (0, x_1(e))$ *and* $u(x_1(e), e) = 0$. *Then* $x_1(.)$ *is increasing.*

Proof. Denote

$$w(x, e) = \frac{u(x, e)}{e} \quad \text{for} \quad e > 0, \; w(x, 0) = \frac{\partial}{\partial e} u(x, 0).$$

Then

$$-w'' = (g(u(x, e)) + f(x)) . w$$

$$w(0, e) = 0, \; w'(0, e) = 1$$

and

$$-w''(x, 0) = f(x) . w(x, 0).$$

Since $f \geqq 2$ using the *Sturm comparison theorem* there exists an $x_1(e)$ for e small and $0 < \lim_{e \to 0_+} x_1(e) < \pi/\sqrt{2}$. From $u(x_1(e), e) = 0$ we obtain

$$u'(x_1(e), e) . x_1'(e) + \frac{\partial}{\partial e} u(x_1(e), e) = 0. \tag{2}$$

It is clear that $u'(x_1(e), e) < 0$. Let us denote

$$z(x, e) = \frac{\partial}{\partial e} u(x, e);$$

then

$$-z'' = (g'(u) . u + g(u) + f(x)) . z$$

$$z(0, e) = 0, \; z'(0, e) = 1.$$

By the Sturm comparison theorem and $g'(u) . u < 0$ for $u > 0$ we obtain:

$$z(x_1(e), e) > 0.$$

Indeed, we compare the equations

$$-z'' = (g'(u) . u + g(u) + f(x)) . z \quad \text{and} \quad -w'' = (g(u) + f(x)) . w$$

and we know that for $e > 0$, $w(0, e) = w(x_1(e), e) = 0$. Finally, we have by (2) $x_1'(e) > 0$.

Lemma 1. x_1 *is defined only on* $(0, e_0)$ *and* $\lim_{e \to e_0} x_1(e) = \infty$, *where* $\infty \geqq e_0 > 0$.

Proof. Let $P = \{e, \; e > 0$ and $x_1(e)$ exists$\}$ and $e_0 \in \partial P$ be the smallest positive element of ∂P, of course $0 < e_0 \leqq \infty$. We assert that $\lim_{e \to e_0} x_1(e) =$

$= \infty$. Indeed, if $\lim_{e \to e_0} x_1(e) = d < \infty$, then we put for $0 < e < e_0$ $u(x_0(e), e) = \max_{\langle 0, x_1(e) \rangle} u(x, e)$. Further,

$$0 \leqq -u''(x_0(e), e) . u(x_0(e), e) = (g(u(x_0(e), e)) + f(x_0(e))) u^2(x_0(e), e).$$

We note that $-g$ is increasing on $\langle 0, \infty)$, $\lim_{x \to \infty} -g(x) = \infty$ and this gives

$$0 \leqq u(x_0(e), e) \leqq (-g)^{-1}(f(x_0(e)) \leqq (-g)^{-1}(\max_{\langle 0, d \rangle} f(x)) = C$$

Thus for $x \in \langle 0, x_1(e) \rangle$, $e < e_0$ we have

$$|u(x, e)| \leqq C,$$

$$|u''(x, e)| \leqq (\max_{\langle 0, C \rangle} |g(u)| + \max_{\langle 0, d \rangle} f(x)) . C = M,$$

$$|u'(x, e)| \leqq C_2.$$

Since C, M, C_2 are independent of e we obtain $e_0 < \infty$, $\lim_{e \to e_0} u(x, e) = u(x, e_0)$ and $u'(x_1(e_0), e_0) < 0$. This implies $e_0 \in P \setminus \partial P$, which is a contradiction. Since x_1 is increasing, for $e > e_0$ it doesn't exist.

Corollary. *The equation* $(1)^+$ *has a unique positive solution with the boundary condition* $u(0) = u(\pi) = 0$.

Proof. By the proof of Theorem 1 it follows that

$$\lim_{e \to 0_+} x_1(e) < \pi/\sqrt{2} < \pi.$$

We know from Lemma 1 that x_1 is defined only on $(0, e_0)$, $\lim_{e \to e_0} x_1(e) = \infty$ and x_1 is increasing on $(0, e_0)$.

Using the implicit function theorem we obtain

Lemma 2. *The mapping* $F: a \to u_a$, $F: \mathbf{R} \to C^2((0, \pi), \mathbf{R})$ *is* C^1*-smooth, where* u_a *is the unique solution*

$$-u_a'' = (f_a(x) + g(u_a)) . u_a$$

$$u_a(0) = u_a(\pi) = 0, \; u_a(.) > 0 \text{ on } (0, \pi).$$

Lemma 3. $n_a(x) = \frac{\partial}{\partial a} u_a(x) > 0$ *on* $(0, \pi)$.

Proof. Since

$$-n_a'' = (g'(u_a) . u_a + f_a(x) + g(u_a)) . n_a(x) + \frac{\partial}{\partial a} f_a(x) . u_a$$

we obtain

$$-n_a'' - (g(u_a) + f_a + g'(u_a) \, . \, u_a) \, . \, n_a = \frac{\partial}{\partial a} f_a(x) \, . \, u_a > 0.$$

Now we utilize a monotone property of the first eigenvalue of boundary value problems [2]:

$$\lambda_1(-g(u_a) - f_a(.) - g'(u_a) \, . \, u_a) > \lambda_1(-f_a(.) - g(u_a)) = 0.$$

($\lambda_1(q)$ is a first eigenvalue of the equation $v'' + q \, . \, v = 0$, $v(0) = v(\pi) = 0$.) There exists an open interval I, $\langle 0, \pi \rangle \subset I$ such that the first eigenvalue of L,

$$Lv = -v'' - (g(u_a) + f_a + g'(u_a) \, . \, u_a) \, . \, v, \quad v/\partial I = 0$$

is positive and a competent first eigenfunction ϕ and $L\phi$ are positive on $\langle 0, \pi \rangle$. Then by the generalized maximum principle [3] we have: $n_a \ \phi$ has not a non-positive minimum on $\langle 0, \pi \rangle$, i.e. $n_a > 0$.

Lemma 4. *Let* (u, v) *be a positive solution of* (1); *then* $u \leqq u_a$ *and* $v \leqq a + 1$.

Proof. Let $v(x_0) = \max\limits_{0, \pi \rangle} v(x)$ then

$$0 \leqq -v''(x_0) = (a + r(u)) \, . \, v - v^2 = v \, . \, (a + r(u) - v).$$

Hence

$$v(x_0) \leqq a + r(u) \leqq a + 1.$$

Further,

$$-u'' = (f_a + g(u)) \, . \, u - s(u) \, . \, v \leqq (f_a + g(u)) \, . \, u.$$

We take $w = M \, . \, u_a$, $M > 1$, then

$$-w'' = -Mu_a'' = (f_a + g(u_a)) \, . \, M \, . \, u_a > (f_a + g(M \, . \, u_a)) \, . \, M \, . \, u_a.$$

Hence

$$w'' + (f_a + g(w)) \, . \, w \leqslant 0.$$

If M is sufficiently large, then $w > u$ and using [2, pp 96] there exists a positive solution u_1 of the equation

$$u'' + (f_a + g(u)) \, . \, u = 0, \ u(0) = u(\pi) = 0 \quad \text{such that}$$

$$0 < u \leqq u_1 \leqq w \quad \text{and hence} \quad u_1 = u_a, \ u \leqq u_a.$$

Lemma 5. *There function* $a \to \lambda_1(r(u_a))$ *is increasing and continuous on* $\mathbf{R}$.

Proof. Since $a \to u_a$ is continuous, then $a \to \lambda_1(r(u_a))$ is continuous too [2]. Since $a \to u_a$ is increasing, i.e. $\frac{\partial}{\partial a} u_a(.) > 0$, then by [2] the function $\lambda_1(r(u_a))$ is increasing too.

Lemma 6. *The exists a unique* $a_0 \in \mathbf{R}$ *such that*

$$a_0 + \lambda_1(r(u_{a_0})) = 0.$$

Proof. By Lemma 5 $a + \lambda_1(r(u_a))$ is increasing and continuous. Further,

$$1 + \lambda_1(r(u_1)) > 1 + \lambda_1(0) = 0$$

$$0 + \lambda_1(r(u_0)) < \lambda_1(1) = 0.$$

Lemma 7. *If* (1) *has a positive solution then* $a > a_0$.

Proof.

$$-v'' = (a + r(u) - v) . v$$

Hence

$$0 = \lambda_1(r(u) + a - v) < \lambda_1(r(u_a) + a) = \lambda_1(r(u_a)) + a,$$

which implies $a > a_0$.

We have three trivial solutions of (1): (0, 0), $(u_a, 0)$, $(0, v_a)$, where

$$-v_a'' = (a - v_a) . v_a$$

$$v_a(0) = v_a(\pi) = 0,\ v_a/(0, \pi) > 0.$$

This equation has a unique positive solution iff $a > 1$ by [1] and these solutions bifurcate from the trivial solution $v = 0$. Using the *Crandall–Rabinowitz theorem* [1] we find local bifurcations of (1) at points $(a, 0, 0)$, $(a, 0, v_a)$ and a global bifurcation from $(a, u_a, 0)$. Now we shall find local bifurcations from the branch $\{(a, 0, 0)\}_{a\in\mathbf{R}}$:

a) *The branch* $\{(a, 0, 0)\}_{a\in\mathbf{R}}$.

Denote $H\colon \mathbf{R} \times X \times X \to X \times X$

$$H = I - T,\ T = (K((f_a + g(u) . u - s(u) . v),\ K((a + r(u)) . v - v^2)),$$

where $K = \Delta^{-1}$, $\Delta u = -u''$, $u(0) = u(\pi) = 0$. We know that $K\colon X \to X$ is a compact operator, $X = \{u \in C^0(\langle 0, \pi\rangle, \mathbf{R}),\ u(0) = u(\pi) = 0$. We have

$$\mathrm{D}_{u,v} H(a, 0, 0)(u_1, v_1) = (u_1 - K(f_a(x) . u_1,\ v_1 - Kav_1)$$

and

$$(u_1, v_1) \in \operatorname{Ker} \mathrm{D}_{u,v} H(a, 0, 0) \quad \text{iff} \quad -u_1'' = f_a . u_1,\ -v_1'' = a . v_1$$

$$u_1(0) = u_1(\pi) = v_1(0) = v_1(\pi) = 0.$$

We look for positive bifurcations and use the following property: If (0, 0) is a bifurcation point of $H(a, u)$ (see [1]), where $Hu = u - T(a, u)$, $T\colon \mathbf{R} \times Z \to Z$, Z is a Banach space, T is a compact, continuously differentiable operator and $T(a, u) = K(a)u + R(a, u)$, $R_u(a, 0) = 0$, then

i) $K(0)$ has an eigenvalue 1.

ii) If $\{(a_n, u_n)\}$ is a sequence of nontrivial solutions such that $a_n \to 0$ and $u_n \to 0$, then there is a subsequence of $\{u_n\}$, again denoted by $\{u_n\}$, such that $u_n/|u_n| \to u_0$, where u_0 is an eigenvector of $K(0)$ corresponding to the eigenvalue 1.

Now we shall find local nonnegative bifurcations from the branch $\{(a, 0, 0)\}_{a \in \mathbf{R}}$. Hence in our case a point $(a, 0, 0)$ is the bifurcation point if

$$-u_1'' = f_a . u_1, \quad u_1(0) = u_1(\pi) = 0, \quad u_1 \geqq 0$$

$$-v_1'' = a . v_1, \quad v_1(0) = v_1(\pi) = 0, \quad v_1 \geqq 0, \quad (u_1, v_1) \neq (0, 0),$$

By $f_a \geqq 2$ and the Sturm comparison theorem $u_1 = 0$. For the second equation the point $a = 1$ is a unique bifurcation point.
It is clear that

$$\operatorname{Ker} \mathrm{D}_{u,v} H(1, 0, 0) = \{(0, \sin .)\}.$$

We compute

$$\mathrm{D}_{a,u,v} H(1, 0, 0)(1, 0, \sin x) = (0, -K \sin x).$$

If there exist u_1, v_1 such that

$$\mathrm{D}_{u,v} H(1, 0, 0)(u_1, v_1) = (0, K \sin x), \text{ then}$$

$$v_1 - Kv_1 = -K \sin x, \ -v_1'' - v_1 = -\sin x, \ v_1(0) = v_1(\pi) = 0$$

and

$$0 = \int_0^\pi (-v_1'' - v_1) . \sin x \, \mathrm{d}x = -\int_0^\pi \sin^2 x \, \mathrm{d}x$$

and this is a contradiction.
Hence

$$\mathrm{D}_{a,u,v} H(1, 0, 0)(0, \sin x) \notin \operatorname{Im} \mathrm{D}_{u,v} H(1, 0, 0)$$

and the conditions for the local bifurcation [1] are satisfied. But we know that $\{(a, 0, v_a)\}_{a \geq 1}$ bifurcates from the trivial solutions $\{(a, 0, 0)\}_{a \in \mathbf{R}}$. Hence we have just found that from the branch $\{(a, 0, 0)\}_{a \in \mathbf{R}}$ there is only the local nonnegative bifurcation $\{(a, 0, v_a)\}_{a > 1}$.

b) *The branch* $\{(a, 0, v_a)\}_{a \geq 1}$.
We compute

$$\mathrm{D}_{u,v} H(a, 0, v_a)(u_1, v_1) = (u_1 - K((f_a) . u_1), v_1 - K((a - 2v_a) v_1)).$$

If $u_1 \geqq 0$, $v_1 \geqq 0$ and $u_1, v_1 \in \operatorname{Ker} \mathrm{D}_{u,v} H(a, 0, v_a)$, then $-u_1'' = f_a . u_1$, $u_1(0) = u_1(\pi) = 0$ and from this it follows that $u_1 = 0$, $-v_1'' + (a - 2v_a) . v_1 = 0$, $v_1(0) = v_1(\pi) = 0$ and hence $\lambda_1(a - 2v_a) = \lambda_1(a - v_a - v_a) < \lambda_1(a - v_a) = 0$ and from this there follows $v_1 = 0$.
We see that from the branch $\{(a, 0, v_a)\}_{a \geqq 1}$ we have not the bifurcation of nonnegative solutions.

c) *The global bifurcations for* $\{(a, u_a, 0)\}_{a \in \mathbf{R}}$.
Compute

$$\mathrm{D}_{u,v} H(a, u_a, 0)(u_1, v_1) = (u_1 - K((g'(u_a) . u_a + f_a + g(u_a)) u_1 - s(u_a) . v_1, \\ v_1 - K((a + r(u_a)) . v_1)).$$

If $u_1 \geqq 0$, $v_1 \geqq 0$ and $u_1, v_1 \in \operatorname{Ker} \mathrm{D}_{u,v} H(a, u_a, 0)$, then

$$-u_1'' - (g'(u_a) . u_a + f_a + g(u_a)) . u_1 = s(u_a) . v_1$$
$$v_1'' + (a + r(u_a)) . v_1 = 0$$
$$u_1(0) = u_1(\pi) = v_1(0) = v_1(\pi) = 0.$$

For $v_1 \neq 0$ we obtain $0 = \lambda_1(a + r(u_a)) = a + \lambda_1(r(u_a))$ and this implies $a = a_0$. For $v_1 = 0$ we have $u_1 = 0$. Hence we can have positive bifurcations only from the point $a = a_0$. Now we verify conditions for the global bifurcation of the Crandall–Rabinowitz theorem (see [1]). In our case

$$K(a) = I - \mathrm{D}_{u,v} H(a, u_a, 0).$$

Let $a < a_0$ and we look for all $s > 1$ such that s is the eigenvalue of $K(a)$:

$$s . (u_1, v_1) - K(a)(u_1, v_1) = 0$$

so that

$$-s . u_1'' - (g'(u_a) . u_a + f_a + g(u_a)) . u_1 = s(u_a) . v_1$$
$$-s . v_1'' + (a + r(u_a)) . v_1 = 0$$
$$u_1(0) = u_1(\pi) = v_1(0) = v_1(\pi) = 0.$$

Then

$$v_1'' + (a + r(u_a)) . v_1/s = 0.$$

Hence for $a < a_0$

$$\lambda_1((a + r(u_a))/s) < \lambda_1((a + r(u_a)/1) < 0$$

and this implies $v_1 = 0$ and $u_1 = 0$.
We have computed that the index of $K(a)$ at 0 for $a < a_0$ is 0, i.e.,

$$i(K(a)) = 0 \quad \text{(see [1])}.$$

Let $1 > a > a_0$ and we look for all $s > 1$ such that s is the eigenvalue of $K(a)$:

$$-s . u_1'' - (g'(u_a) . u_a + f_a + g(u_a)) . u_1 = s(u_a) . v_1$$
$$v_1'' + (a + r(u_a)) . v_1/s = 0$$
$$u_1(0) = u_1(\pi) = v_1(0) = v_1(\pi) = 0.$$

Then

$$-1 = \lambda_1(0) < \lambda_1((a + r(u_a))/s) \quad \text{and} \quad \lambda_1(a + r(u_a)) > 0$$

$$\lambda_2((a + r(u_a))/s) \leqq \lambda_2(a + r(u_a)) = \lambda_2(2) = -2.$$

We obtain

$$\lambda_2((a + r(u_a))/s) \leqq -2 < -1 < \lambda_1((a + r(u_a))\ s).$$

Hence there exists a unique $t > 1$ such that $v_1'' + (a + r(u_a)) \,.\, v_1/s = 0$ has a nontrivial solution and

$$\lambda_1((a + r(u_a))/t) = 0.$$

Now we compute the algebraic multiplicity of t (see [1]).

Lemma 8. $\operatorname{Ker}(t \,.\, I - K(a)) \cap \operatorname{Im}(t \,.\, I - K(a)) = \{0\}$.

Proof. Indeed, we have by the above results:

$$\operatorname{Ker}(t \,.\, I - K(a)) = \operatorname{span}\{(a_1, b_1)\}.$$

If

$$t \,.\, (u_1, v_1 - K(a)(u_1, v_1) = (a_1, b_1), \text{ then}$$

$$t \,.\, v_1'' + (a + r(u_a)) \,.\, v_1/t = b_1''/t$$

$$v_1(0) = v_1(\pi) = 0.$$

Hence

$$0 = \int_0^\pi (v_1'' \,.\, b_1'' + (a + r(u_a) \,.\, v_1 \,.\, b_1/t)\, \mathrm{d}x = \int_0^\pi b_1' \,.\, b_1/t\, \mathrm{d}x = -\int_0^\pi (b_1')^2\, t\, \mathrm{d}x.$$

This implies that $b_1 = 0$. In the same way $a_1 = 0$ and this is a contradiction.

We have proved that $i(K(a)) = -1$ for $1 > a > a_0$. By the global bifurcation theorem [1] there is a global branch of solutions, which bifurcates from $(a_0, u_{a_0}, 0)$. If we denote this branch by $\{(a, \bar{u}_a, \bar{v}_a)\}$, then $\bar{u}_a > 0$, $\bar{v}_a > 0$ for a near to a_0. Indeed, we have

$$-v_a'' = (a + r(\bar{u}_a) - \bar{v}_a) \,.\, \bar{v}_a,\ \bar{v}_a(0) = \bar{v}_a(\pi) = 0 \text{ and}$$

$$\lambda_2(a + r(\bar{u}_a) - \bar{v}_a) < \lambda_2(5/2) = -3/2 < 0.$$

Hence $\lambda_1(a + r(\bar{u}_a) - \bar{v}_a) = 0$ and $\bar{v}_a$ is the first eigenfunction and this implies $\bar{v}_a > 0$ on $(0, \pi)$.

Let us assume that for some $a_1 > a_0$, $\bar{u}_{a_1}$, $\bar{v}_{a_1}$ are not positive. We take the smallest one. Then either $\bar{u}_{a_1} = 0$ or $\bar{v}_{a_1} = 0$ which is impossible by the case a) and b). Hence for all $a > a_0$, $\bar{u}_a > 0$, $\bar{v}_a > 0$. By the Crandall–Rabinowitz theorem $\{(a, \bar{u}_a, \bar{v}_a)\} \to \infty$ and since $\bar{u}_a \leqq u_a$, $\bar{v}_a \leqq a + 1$ we obtain $a \to \infty$. Summing up we conclude the proof of the main theorem

Theorem 2. *If* (1) *satisfies our assumptions, then* (1) *has positive solutions if and only if* $a > a_0$.

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