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FULL SUBDIRECT AND WEAK DIRECT PRODUCTS OF ALGEBRAS

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ABSTRACT. In this paper we give a common generalization of full subdirect product and weak direct product of given algebras.

Let A_i $(i \in I)$ be a family of similar algebras, and let $B = \prod(A_i : i \in I)$ denote the direct product of A_i, $i \in I$. For two elements $x, y \in B$ we define

$$I(x,y) = \{i \in I : x(i) \neq y(i)\}.$$

A *weak direct product* of the algebras A_i $(i \in I)$ is a subalgebra A of B satisfying the following two conditions:

(i) if $x, y \in A$, then $I(x,y)$ is finite,

(ii) if $x \in A$, $y \in B$, and $I(x,y)$ is finite, then $y \in A$.

Let A be a subdirect product of A_i, $i \in I$. We say that A is a *full subdirect product* of A_i $(i \in I)$ if the following condition is satisfied:

(iii) for any $i \in I$ and any $x, y \in A$ there is an element $z \in A$ such that $z(i) = x(i)$, $z(j) = y(j)$ for each $j \in I - \{i\}$.

Let I be a nonvoid set. $\mathcal{P}(I)$ and $\mathcal{F}(I)$ denote the set of all subsets of I and the set of all finite subsets of I, respectively. We denote by $P(I)$ the Boolean algebra $\langle \mathcal{P}(I), \cap, \cup, ', \emptyset, I \rangle$. A common generalization of full subdirect and weak direct products of algebras is the following concept:

DEFINITION. *Let A_i $(i \in I)$ be similar algebras and let $\mathcal{L}$ be an ideal of $P(I)$. We say that a subalgebra A of the direct product $\prod(A_i : i \in I)$ is an $\mathcal{L}$-restricted full subdirect product of algebras A_i, $i \in I$, and write $A = \prod_{\mathcal{L}}(A_i : i \in I)$ if and only if the following two conditions hold:*

(iv) *A is a full subdirect product of A_i, $i \in I$,*

(v) *for every $x, y \in A$, $I(x,y) \in \mathcal{L}$.*

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PROPOSITION. *Let A be a subalgebra of the direct product $B = \prod(A_i : i \in I)$ of algebras A_i, $i \in I$.*

(a$_1$) *A is a full subdirect product of A_i ($i \in I$) if and only if*

$$A = \prod\nolimits_{\mathcal{P}(I)}(A_i : i \in I).$$

(a$_2$) *A is a weak direct product of A_i ($i \in I$) if and only if*

$$A = \prod\nolimits_{\mathcal{F}(I)}(A_i : i \in I).$$

Proof. Statement (a$_1$) is obvious.

To prove the second statement, first assume that A is a weak direct product of algebras A_i ($i \in I$). Then A is a full subdirect product of A_i ($i \in I$), and therefore,

$$A = \prod\nolimits_{\mathcal{F}(I)}(A_i : i \in I).$$

Conversely, assume that A is an $\mathcal{F}(I)$-restricted full subdirect product of A_i, $i \in I$. Obviously, the condition (i) is satisfied. To prove (ii), let $x \in A$ and $y \in B$. Suppose that the set $I(x, y)$ contains only one element i_1. Since A is a subdirect product of A_i ($i \in I$), there is $t \in A$ such that $t(i_1) = y(i_1)$. Further, it follows from (iii) that there exists $z \in A$ satisfying $z(i_1) = t(i_1)$, $z(i) = x(i)$ for each $i \in I$, $i \neq i_1$. Clearly $y = z$, thus $y \in A$. From this, we get by induction that (ii) holds. Then A is a weak direct product of algebras A_i ($i \in I$).

Let A and A_i ($i \in I$) be similar algebras. Let f be an embedding of A into $B = \prod(A_i : i \in I)$ and let $\mathcal{L}$ be an ideal of $P(I)$. We write

$$f\colon A \cong \prod\nolimits_{\mathcal{L}}(A_i : i \in I) \iff f(A) = \prod\nolimits_{\mathcal{L}}(A_i : i \in I).$$

We denote by p_i the ith projection function of B. If $f(A)$ is a subdirect product of the algebras A_i, $i \in I$, then the mapping $f_i = p_i \circ f$ is a homomorphism of A onto A_i. This mapping f_i will be referred to as the ith *f-projection*.

We shall now correlate $\mathcal{L}$-restricted factorizations of an algebra A with congruence relations on A. Let $\mathrm{Con}(A)$ denote the set of all congruences on A. Then $\mathrm{Con}(A)$ forms a complete lattice with 0_A and 1_A, the smallest and the largest congruence relation, respectively. Let θ_i, $i \in I$, be congruences on A, and let $\mathcal{L}$ be an ideal of $P(I)$. For any set $M \in \mathcal{L}$ we define a congruence relation $\theta(M)$ of A by

$$\theta(M) = \bigwedge(\theta_j : j \notin M).$$

For $i \in I$ we set $\overline{\theta}_i = \bigwedge(\theta_j : j \in I - \{i\})$. For some $\alpha \in \mathrm{Con}(A)$ we write

$$\alpha = \prod\nolimits_{\mathcal{L}}(\theta_i : i \in I)$$

if and only if the following conditions hold:

(a) $\alpha = \bigwedge(\theta_i : i \in I)$,

(b) $1_A = \bigvee(\theta(M) : M \in \mathcal{L})$,

(c) for all $i \in I$, $1_A = \theta_i \circ \overline{\theta}_i$ (i.e. congruences θ_i and $\overline{\theta}_i$ permute and their join is 1_A).

THEOREM 1. *Let A be an algebra and A_i ($i \in I$) be a family of algebras. Let $\mathcal{L}$ be an ideal of $P(I)$. Then A is isomorphic to an $\mathcal{L}$-restricted full subdirect product of algebras A_i, $i \in I$, if and only if there exists a family θ_i, $i \in I$, of congruences on A such that $0_A = \prod_{\mathcal{L}}(\theta_i : i \in I)$ and $A/\theta_i \cong A_i$ for every $i \in I$.*

Proof.

Necessity. Let $f \colon A \cong \prod_{\mathcal{L}}(A_i : i \in I)$, and let θ_i ($i \in I$) be the kernel of the ith f-projection f_i that is the binary relation $\{\langle x, y\rangle \in A^2 : f_i(x) = f_i(y)\}$. By assumption, the mapping f is one-to-one, and hence $0_A = \bigwedge(\theta_i : i \in I)$.

To prove (b), let $x, y \in A$. Clearly,

$$M = \{i \in I : f_i(x) \neq f_i(y)\} = I(f(x), f(y)) \in \mathcal{L}$$

and $\langle x, y\rangle \in \theta(M)$. Then $\langle x, y\rangle \in \bigvee(\theta(M) : M \in \mathcal{L})$, and hence (b) holds. Condition (c) immediately follows from (iii).

Finally, it is obvious that $A/\theta_i \cong A_i$ for each $i \in I$.

Sufficiency. We define the mapping f from A to $\prod(A/\theta_i : i \in I)$ by setting $f(x) = \langle x/\theta_i : i \in I\rangle$ [1]. The fact that f is an embedding is easy to check. Of course, the mapping $f_i = p_i \circ f$ is onto for each $i \in I$. Now, from (c) we obtain (iii). Therefore, $f(A)$ is a full subdirect product of algebras A/θ_i, $i \in I$.

Now, let $x, y \in A$. By (b), $\langle x, y\rangle \in \bigvee(\theta(M) : M \in \mathcal{L})$. Then there exists a finite number of sets $M_1, M_2, \dots, M_n \in \mathcal{L}$ such that $\langle x, y\rangle \in \theta(M_1) \vee \dots \cdots \vee \theta(M_n)$. Observe that

$$\{i \in I : f_i(x) \neq f_i(y)\} \subseteq M_1 \cup \cdots \cup M_n . \tag{1}$$

Indeed, let $f_i(x) \neq f_i(y)$ for some $i \in I$, and suppose on the contrary that $i \notin M_1 \cup \cdots \cup M_n$. Then $\theta(M_1) \vee \cdots \vee \theta(M_n) \leq \theta_i$, and hence $\langle x, y\rangle \in \theta_i$, i.e. $f_i(x) = f_i(y)$, which is a contradiction.

From (1), by the definition of ideal, we conclude that $\{i : f_i(x) \neq f_i(y)\} \in \mathcal{L}$, which was to be proved. Therefore the proof of Theorem 1 is complete.

[1] x/θ_i is the congruence class containing x

LEMMA 1. *Let I, J be two sets of indices and $\mathcal{L}_1$, $\mathcal{L}_2$ ideals of the Boolean algebras $P(I)$, $P(J)$, respectively. Let A be an algebra with $\mathrm{Con}(A)$ distributive. If*

$$0_A = \prod\nolimits_{\mathcal{L}_1}(\alpha_i :\ i \in I) = \prod\nolimits_{\mathcal{L}_2}(\beta_j :\ j \in J) \tag{2}$$

for congruences α_i, β_j on A, then there exist congruences δ_{ij} ($i \in I$, $j \in J$) such that, for all i and j,

$$\alpha_i = \prod\nolimits_{\mathcal{L}_2}(\delta_{ij} :\ j \in J) \quad \text{and} \quad \beta_j = \prod\nolimits_{\mathcal{L}_1}(\delta_{ij} :\ i \in I).$$

P r o o f . For $i \in I$ and $j \in J$ we put $\delta_{ij} = \alpha_i \vee \beta_j$. Let i be a fixed but arbitrary element of I. First we show that

$$\alpha_i = \bigwedge(\delta_{ij} :\ j \in J). \tag{3}$$

By distributivity of $\mathrm{Con}(A)$, for any j we have

$$\overline{\alpha}_i \wedge \delta_{ij} = \overline{\alpha}_i \wedge (\alpha_i \vee \beta_j) = \overline{\alpha}_i \wedge \beta_j \leq \beta_j .$$

Hence,

$$\overline{\alpha}_i \wedge \bigwedge(\delta_{ij} :\ j \in J) = \bigwedge(\overline{\alpha}_i \wedge \delta_{ij} :\ j \in J) \leq \bigwedge(\beta_j :\ j \in J) = 0_A .$$

Therefore, using distributivity, we get

$$\bigwedge(\delta_{ij} :\ j \in J) = \bigwedge(\delta_{ij} :\ j \in J) \wedge (\alpha_i \vee \overline{\alpha}_i) = \alpha_i \wedge \bigwedge(\delta_{ij} :\ j \in J) = \alpha_i .$$

i.e. (3) is satisfied.

For $M \in \mathcal{L}_2$ we set $\delta(M) = \bigwedge(\delta_{ij} :\ j \notin M)$. Now we prove that

$$1_A = \bigvee\big(\delta(M) :\ M \in \mathcal{L}_2\big). \tag{4}$$

Let $x, y \in A$. By (2), $\langle x, y\rangle \in \bigvee\big(\beta(M) :\ M \in \mathcal{L}_2\big)$. Hence, we can choose a finite number of sets $M_1, M_2, \dots, M_n \in \mathcal{L}_2$ such that $\langle x, y\rangle \in \beta(M_1) \vee \dots \vee \beta(M_n)$. We set $M = \big\{j \in J :\ \langle x, y\rangle \notin \delta_{ij}\big\}$. Observe that $M \subseteq M_1 \cup \dots \cup M_n$. Indeed, let $j \in M$ and $j \notin M_1 \cup \dots \cup M_n$. It is obvious that $\beta(M_k) \leq \beta_j$ for each $k = 1, 2 \dots, n$. Therefore, $\beta(M_1) \vee \dots \vee \beta(M_n) \leq \beta_j \leq \delta_{ij}$. Then $\langle x, y\rangle \in \delta_{ij}$, which gives us a contradiction. Consequently, $M \subseteq M_1 \cup \dots \cup M_n$, and hence $M \in \mathcal{L}_2$. Thus $\langle x, y\rangle \in \delta(M)$, and we conclude that (4) holds.

For each $j \in J$, let us write $\overline{\delta}_{ij}$ for $\bigwedge\big(\delta_{ik} :\ k \in J - \{j\}\big)$. Clearly, $\delta_{ij} \geq \beta_j$ and $\overline{\delta}_{ij} \geq \overline{\beta}_j$. Since $1_A = \beta_j \circ \overline{\beta}_j$, we have

$$1_A = \delta_{ij} \circ \overline{\delta}_{ij} \tag{5}$$

for all $j \in J$. From (3), (4) and (5) it follows that $\alpha_i = \prod_{\mathcal{L}_2}(\delta_{ij} :\ j \in J)$. The proof that $\beta_j = \prod_{\mathcal{L}_1}(\delta_{ij} :\ i \in I)$ is similar.

THEOREM 2. *Under the assumptions of Lemma* 1, *if*

$$A \cong \prod\nolimits_{\mathcal{L}_1}(A_i : i \in I) \quad \textit{and} \quad A \cong \prod\nolimits_{\mathcal{L}_2}(B_j : j \in J),$$

then there exist algebras D_{ij} $(i \in I,\ j \in J)$ *such that, for all* i *and* j,

$$A_i \cong \prod\nolimits_{\mathcal{L}_2}(D_{ij} : j \in J) \quad \textit{and} \quad B_j \cong \prod\nolimits_{\mathcal{L}_1}(D_{ij} : i \in I).$$

Proof. Let $f\colon A \cong \prod_{\mathcal{L}_1}(A_i : i \in I)$ and $g\colon A \cong \prod_{\mathcal{L}_2}(B_j : j \in J)$. Let α_i $(i \in I)$ and β_j $(j \in J)$ be the kernels of the f-projections f_i and the g-projections g_j, respectively. By the proof of Theorem 1,

$$0_A = \prod\nolimits_{\mathcal{L}_1}(\alpha_i : i \in I) = \prod\nolimits_{\mathcal{L}_2}(\beta_j : j \in J).$$

For $i \in I$ and $j \in J$, we set $\delta_{ij} = \alpha_i \vee \beta_j$. From Lemma 1 it follows that

$$\alpha_i = \prod\nolimits_{\mathcal{L}_2}(\delta_{ij} : j \in J) \quad \text{and} \quad \beta_j = \prod\nolimits_{\mathcal{L}_1}(\delta_{ij} : i \in I).$$

Then $\alpha_i/\alpha_i = \prod_{\mathcal{L}_2}(\delta_{ij}/\alpha_i : j \in J)$ [2)]. Hence, by Theorem 1,

$$A/\alpha_i \cong \prod\nolimits_{\mathcal{L}_2}(A/\delta_{ij} : j \in J).$$

Therefore, $A_i \cong \prod_{\mathcal{L}_2}(D_{ij} : j \in J)$, where $D_{ij} = A/\delta_{ij}$.

Similarly, $B_j \cong \prod_{\mathcal{L}_1}(D_{ij} : i \in I)$.

It is easy to prove the following:

LEMMA 2. *Let* $\mathcal{L}$ *be an ideal of the Boolean algebra* $P(I)$. *If an algebra* A *is directly indecomposable and if* $f\colon A \cong \prod_{\mathcal{L}}(A_i : i \in I)$, *then there is an index* $i \in I$ *for which* $f_i\colon A \cong A_i$, *where* f_i *is the* i *th* f*-projection.*

THEOREM 3. *Under the assumptions of Lemma* 1, *if*

$$f\colon A \cong \prod\nolimits_{\mathcal{L}_1}(A_i : i \in I) \quad \textit{and} \quad g\colon A \cong \prod\nolimits_{\mathcal{L}_2}(B_j : j \in J),$$

where the algebras A_i $(i \in I)$ *and* B_j $(j \in J)$ *are directly indecomposable, then there is a bijection* $\sigma\colon I \to J$ *for which the following conditions hold:*

(a$_1$) *for each* $i \in I$, *there exists an isomorphism* $h_i\colon A_i \to B_{\sigma(i)}$ *such that* $h_i \circ f_i = g_{\sigma(i)}$,

(a$_2$) $\sigma\big(I(f(x), f(y))\big) = J\big(g(x), g(y)\big)$ *for all* $x, y \in A$.

2) For $\phi, \psi \in \mathrm{Con}(A)$ with $\phi \subseteq \psi$, $\psi/\phi = \{\langle x/\phi, y/\phi\rangle : \langle x, y\rangle \in \psi\}$.

Proof. Let α_i $(i \in I)$ and β_j $(j \in J)$ be the kernels of f_i and g_j, respectively. For each $i \in I$ and each $j \in J$, set

$$\delta_{ij} = \alpha_i \vee \beta_j \quad \text{and} \quad D_{ij} = A/\delta_{ij} .$$

By Theorem 2, $A_i \cong \prod_{\mathcal{L}_2}(D_{ij} : j \in J)$ and $B_j \cong \prod_{\mathcal{L}_1}(D_{ij} : i \in I)$. Since A_i is directly indecomposable, it follows from Lemma 2 (see also the proof of Theorem 1) that there exists an index $\sigma(i) = j \in J$ such that the map

$$f_i(x) \mapsto x/\delta_{ij} \qquad (x \in A)$$

defines an isomorphism of A_i with D_{ij}. Therefore,

$$A/\alpha_i \cong A_i \cong D_{ij} = A/\alpha_i \vee \beta_j .$$

Then $\alpha_i = \alpha_i \vee \beta_j$, and hence $\alpha_i \geq \beta_j$. Since B_j is directly indecomposable, we conclude that there is an index $\tau(j) = i' \in I$ such that the map

$$g_j(x) \mapsto x/\delta_{i'j} \qquad (x \in A)$$

defines an isomorphism from B_j onto $D_{i'j}$. Now we infer similarly that $\beta_j \geq \alpha_{i'}$. Consequently, $\alpha_i \geq \beta_j \geq \alpha_{i'}$. Observe that $i = i'$. Indeed, if $i \neq i'$, then $\overline{\alpha}_i \leq \alpha_{i'} \leq \alpha_i$, and hence $\alpha_i = 1_A$, contrary to the fact that A_i is directly indecomposable. Therefore, $\tau\sigma(i) = i$ for all $i \in I$, and similarly $\sigma\tau(j) = j$ for all $j \in J$. Then τ is a two-sided inverse of σ, and this proves that σ is a bijection.

If $\sigma(i) = j$, then we have $A_i \cong D_{ij} \cong B_j$, and it is easy to see that the mapping h_i defined on A_i by

$$h_i\big(f_i(x)\big) = g_j(x)$$

is an isomorphism of A_i with B_j.

To prove (a_2), let $x, y \in A$. We have

$$\begin{aligned} i \in I\big(f(x), f(y)\big) &\longleftrightarrow f_i(x) \neq f_i(y) \longleftrightarrow h_i \circ f_i(x) \neq h_i \circ f_i(y) \\ &\longleftrightarrow g_{\sigma(i)}(x) \neq g_{\sigma(i)}(y) \longleftrightarrow \sigma(i) \in J\big(g(x), g(y)\big) . \end{aligned}$$

Therefore, (a_2) is satisfied.

A congruence $\alpha \in \mathrm{Con}(A)$ is called a *decomposition congruence* if and only if there is $\beta \in \mathrm{Con}(A)$ such that $\alpha \wedge \beta = 0_A$ and $\alpha \circ \beta = 1_A$. $\mathrm{DCon}(A)$ denotes the set of all decomposition congruences of A.

From [2; Theorem 6.2] it follows:

LEMMA 3. *Let A be an algebra with* $\mathrm{Con}(A)$ *distributive. Then* $\mathrm{DCon}(A)$ *is a Boolean sublattice of* $\mathrm{Con}(A)$ *and every element of* $\mathrm{DCon}(A)$ *is permutable with any congruence on A.*

LEMMA 4. *Let A be an algebra whose congruence lattice is distributive. If θ is a coatom of* $\mathrm{DCon}(A)$*, then A/θ is directly indecomposable.*

Proof. Suppose on the contrary that there exist two congruences α, β such that $\theta < \alpha$, $\beta < 1_A$, $\alpha \circ \beta = 1_A$ and $\alpha \wedge \beta = \theta$. Let θ' be a congruence satisfying $0_A = \theta \wedge \theta'$ and $1_A = \theta \circ \theta'$. Obviously

$$\alpha \wedge (\beta \wedge \theta') = 0_A . \tag{6}$$

Observe that

$$\alpha \circ (\beta \wedge \theta') = 1_A . \tag{7}$$

Indeed, $\alpha \circ (\beta \wedge \theta') \supseteq \alpha$, and by Lemma 3, and using distributivity we get

$$\alpha \circ (\beta \wedge \theta') \supseteq \theta \circ (\beta \wedge \theta') = \theta \vee (\beta \wedge \theta') = (\theta \vee \beta) \wedge (\theta \vee \theta') = \beta .$$

Therefore, $\alpha \circ (\beta \wedge \theta') \supseteq \alpha \circ \beta = 1_A$, and hence we obtain (7). From (6) and (7) it follows that $\alpha \in \mathrm{DCon}(A)$, contradicting that θ is a coatom of $\mathrm{DCon}(A)$. Then A/θ is directly indecomposable.

We call a sublattice of a complete lattice $\vee$-*closed* whenever it is closed under arbitrary joins.

THEOREM 4. *Let A be an algebra with* $\mathrm{Con}(A)$ *distributive. If* $\mathrm{DCon}(A)$ *is $\vee$-closed in* $\mathrm{Con}(A)$*, then there exists a family A_i ($i \in I$) of directly indecomposable algebras such that $A \cong \prod_{\mathcal{L}}(A_i : i \in I)$, where $\mathcal{L}$ is an ideal of $P(I)$ containing all finite subsets of I.*

Proof. By Lemma 3, $\mathrm{DCon}(A)$ is a Boolean sublattice of $\mathrm{Con}(A)$ and from the proof of [2; Lemma 4.3] it follows that $\mathrm{DCon}(A)$ is atomic. Let $\{\alpha_i : i \in I\}$ be the set of all atoms of $\mathrm{Dcon}(A)$.

By [4; Lemma 4.83], we conclude that $1_A = \bigvee(\alpha_i : i \in I)$.

For $i \in I$, we set

$$\theta_i = \bigvee(\alpha_j : j \in I - \{i\}) \quad \text{and} \quad \overline{\theta}_i = \bigwedge(\theta_j : j \in I - \{i\}) .$$

Now we prove that for each $i \in I$,

$$0_A = \theta_i \wedge \overline{\theta}_i . \tag{8}$$

It is a well-known fact that distributivity of $\mathrm{Con}(A)$ implies infinite distributivity. Then we have

$$\theta_i \wedge \overline{\theta}_i = \overline{\theta}_i \wedge \bigvee\big(\alpha_j :\ j \in I - \{i\}\big) = \bigvee\big(\overline{\theta}_i \wedge \alpha_j :\ j \in I - \{i\}\big) = 0_A\,.$$

because $\alpha_j \wedge \overline{\theta}_i = 0_A$ for all $j \neq i$. Therefore, (8) holds.

To prove (c), first we observe that $\alpha_i \leq \overline{\theta}_i$ for each $i \in I$. Hence $1_A = \alpha_i \vee \theta_i \leq \overline{\theta}_i \vee \theta_i$. Moreover, θ_i and $\overline{\theta}_i$ are permutable (because $\theta_i \in \mathrm{DCon}(A)$), and then $1_A = \theta_i \circ \overline{\theta}_i$.

Finally, we have to show that (b) is satisfied. Since $\theta_i = \bigvee(\alpha_j :\ j \neq i) \leq \bigvee(\overline{\theta}_j :\ j \neq i)$, we obtain $1_A = \theta_i \vee \overline{\theta}_i \leq \bigvee(\overline{\theta}_i :\ i \in I) = \bigvee(\theta(\{i\}) :\ i \in I) \leq \bigvee(\theta(M) :\ M \in \mathcal{L})$. Hence, $1_A = \bigvee(\theta(M) :\ M \in \mathcal{L})$. Thus the family θ_i $(i \in I)$ of congruences on A satisfies the conditions (8), (b), and (c). Therefore, $0_A = \prod_{\mathcal{L}}(\theta_i :\ i \in I)$, and hence by Theorem 1 we conclude that $A \cong \prod_{\mathcal{L}}(A_i :\ i \in I)$, where $A_i = A/\theta_i$.

From Lemma 4, it follows that every A_i is directly indecomposable, because θ_i is a coatom of $\mathrm{DCon}(A)$. This ends the proof of Theorem 4.

Now we obtain:

THEOREM 5. *Let A be an algebra whose congruence lattice is distributive and let $\mathrm{DCon}(A)$ be a $\vee$-closed sublattice in $\mathrm{Con}(A)$. Then any full subdirect decomposition of A into directly indecomposable factors is a weak direct product decomposition of A.*

P r o o f . Let A be a full subdirect product of directly indecomposable algebras A_i $(i \in I)$, i.e.

$$A = \prod\nolimits_{\mathcal{P}(I)}(A_i :\ i \in I)\,.$$

By Theorem 4, A is isomorphic to a weak direct product of directly indecomposable algebras B_j, $j \in J$. Let

$$f\colon A \cong \prod\nolimits_{\mathcal{F}(I)}(B_j :\ j \in J)\,.$$

Using Theorem 3, we obtain that there exists a bijection $\sigma\colon I \to J$ such that $\sigma\big(I(x,y)\big) = J\big(f(x), f(y)\big)$ for all $x, y \in A$. From the fact that the set $J\big(f(x), f(y)\big)$ is finite, we deduce that $I(x,y)$ is finite. Therefore, A is a weak direct product of algebras A_i, $i \in I$.

The following lemma can be deduced from the proof of [1; Lemma 1.4].

LEMMA 5. *If A is an algebra whose congruence lattice is completely distributive, then* $\mathrm{DCon}(A)$ *is a* $\vee$*-closed sublattice of* $\mathrm{Con}(A)$.

R e m a r k 1. By this lemma, Theorem 4 implies [1; Theorems 1.6 and 1.7].

R e m a r k 2. By Lemma 5 and Theorem 5 we obtain [1; Theorem 1.8].

Let L be a lattice. We say that L satisfies the *restricted chain condition* if every interval of L satisfies the ascending or the descending chain condition (cf. [2]).

The lattice L is called *discrete* if all bounded chains in L are finite (cf. [3]) and L is *weakly discrete* if there exists a maximal finite chain between any comparable elements (cf. [1]).

Each discrete lattice is weakly discrete and it satisfies the restricted chain condition. If a lattice L satisfies the restricted chain condition, then we conclude from the proof of Theorem 6.3 (see [2; p. 106]) that $\mathrm{DCon}(L)$ is $\vee$-closed in $\mathrm{Con}(L)$. If L is a weakly discrete lattice, then by [1; Lemma 1.9] we get that $\mathrm{Con}(L)$ is completely distributive, and hence $\mathrm{DCon}(L)$ is a $\vee$-closed sublattice of $\mathrm{Con}(L)$.

From this and Theorem 4 we obtain:

THEOREM 6. (see H a s h i m o t o [2; Theorem 6.3] and D r a š k o v i č o v á [1; Corollary 1.12]). *If a lattice L is weakly discrete or if L satisfies the restricted chain condition, then L is isomorphic to a weak direct product of directly indecomposable lattices.*

COROLLARY. (cf. [3; Theorem 2.16]). *Any discrete lattice is isomorphic to a weak direct product of directly indecomposable lattices.*

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