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THEOREMS OF THE NORDHAUS-GADDUM TYPE FOR k-UNIFORM HYPERGRAPHS

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In 1956 E. A. Nordhaus and J. W. Gaddum [5] proved the assertion

$$2\cdot\sqrt{n}\leqq\chi(G)+\chi(\bar{G})\leqq n+1, \tag{1}$$

where G is a finite undirected graph without loops and multiple edges, $\bar{G}$ is its complement, n is the number of vertices of the graph G and $\chi(G)$ or $\chi(\bar{G})$ is the chromatic number of the graph G or $\bar{G}$, respectively. Since then several theorems dealing with the characteristics of the graph G and its complement have been published [2—4]. These theorems are called theorems of the Nordhaus—Gaddum type.

In the presented paper a theorem of the Nordhaus—Gaddum type for the chromatic and achromatic number of k-uniform hypergraphs is proved. The method used in proving Theorem 1 may be applied for proving the known Nordhaus—Gaddum theorem (1) as well. By means of the method used in this proof one can obtain the description of the structures of all graphs for which the relation $\chi(G)+\chi(\bar{G}) = n+1$ is valid.

1. Basic Notions

(Cf. Berge [1].) The usual terminology for graphs and hypergraphs will be applied. Let us recall the following notions.

By a hypergraph H is meant a couple $\langle X, \mathscr{E}\rangle$, where X is a finite set of elements called vertices and $\mathscr{E}=\{E_1, \ldots, E_m\}$ is a finite system of non-empty subsets of X called edges, where $E_i\neq E_j$ for $i, j\in\{1, \ldots, m\}$, $i\neq j$.

A hypergraph is k-uniform, $k>1$, if all edges have cardinality k. A k-uniform hypergraph with $n\geqq k$ vertices is complete if its set of edges has the cardinality $\binom{n}{k}$.

The complement of a k-uniform hypergraph $H=\langle X, \mathscr{E}\rangle$ its the hypergraph

$\bar{H}=\langle X, \bar{\mathscr{E}}\rangle$ if $|\mathscr{E}\cup\bar{\mathscr{E}}|=\binom{n}{k}$ and $\mathscr{E}\cap\bar{\mathscr{E}}=\emptyset$. (By $|\mathscr{E}\cup\bar{\mathscr{E}}|$ the cardinality of the set $\mathscr{E}\cup\bar{\mathscr{E}}$ is denoted.)

A hypergraph $H\langle N\rangle=\langle X, \mathscr{E}_N\rangle$ is said to be the k-uniform subhypergraph of a k-uniform hypergraph $H=\langle X, \mathscr{E}\rangle$ induced by a set N if $N\subseteq X$ and $\mathscr{E}_N$ is the system of all edges $E_i\in\mathscr{E}$ such that $E_i\subseteq N$.

A set $S\subseteq X$ of vertices of H is called stable if for all edges $E_i\in\mathscr{E}$ we have $E_i\cap(X-S)\neq\emptyset$.

A stable set $S\subseteq X$ of $H=\langle X, \mathscr{E}\rangle$ is said to be maximal if for each vertex $x\in(X-S)$ the set $S\cup\{x\}$ fails to be a stable set of H.

A partition of the vertex set X of $H=\langle X, \mathscr{E}\rangle$ into disjoint stable subsets is called a colouring of H, whereby the vertices belonging to the same stable subsets are given the same colour and the vertices belonging to distinct stable subsets are given different colours. Two colours are adjacent if there exists an edge containing vertices to which these two distinct colours are given. A colouring of H is complete if all pairs of the used colours are mutually adjacent.

The chromatic number $\chi(H)$ or the achromatic number $\psi(H)$ of a hypergraph H is the least or greatest number, respectively, of colours used in a complete colouring of H.

2. Chromatic number

Lemma 1. *In a k-uniform hypergraph $H=\langle X, \mathscr{E}\rangle$ with $\chi(H)=q$ there exists a colouring $\{S_1, \ldots, S_q\}$ having the following properties:*

1° $|S_q|\geqq|S_{q-1}|\geqq\ldots\geqq|S_{r+1}|\geqq k$
$|S_r|=\ldots=|S_2|=k-1\geqq|S_1|$.

2° *For each $i=1, 2, \ldots, q$, the S_i is a maximal stable set in $H\langle S_i\cup\ldots\cup S_1\rangle$.*

3° *If $|S_r\cup\ldots\cup S_1|\geqq k$, then $H\langle S_r\cup\ldots\cup S_1\rangle$ is a complete k-uniform subhypergraph of H.*

Proof. From the assumption of the lemma it follows that there exists a partition of the set of vertices X in q disjoint stable sets $S_1^1, \ldots, S_q^1$ with the property $|S_q^1|\geqq\ldots\geqq|S_1^1|$. Let S_q^2 be a maximal stable set in $H=\langle X, \mathscr{E}\rangle$ such that $S_q^1\subseteq S_q^2$. Gradually for each $i=q-1, \ldots, 2$ let S_i^2 be a maximal stable set in $H\langle X-(S_q^2\cup\ldots\cup S_{i+1}^2)\rangle$ such that $S_i^1-(S_q^2\cup\ldots\cup S_{i+1}^2)\subseteq S_i^2$. Let us arrange the sets $S_q^2, \ldots, S_1^2$ according to the cardinality and let us denote them $S_q^3, \ldots, S_1^3$, thus $|S_q^3|\geqq\ldots\geqq|S_1^3|$. The method according to which we have obtained from the colouring $\{S_q^1, \ldots, S_1^1\}$ the colouring $\{S_q^3, \ldots, S_1^3\}$, will be called the n-process. By a finite number of the n-process application we have a colouring $\{S_q^0, \ldots, S_1^0\}$ which has the properties 1° and 2°.

If this colouring has not the property 3°, i. e.

$$|S_q^0|\geqq\ldots\geqq|S_{r+1}^0|\geqq k,\quad |S_r^0|=\ldots=|S_2^0|=k-1\geqq|S_1^0|$$

and $H\langle S_r^0\cup\ldots\cup S_1^0\rangle$ is not a complete k-uniform subhypergraph, we shall recolour the vertices of this subhypergraph with r colours so that the k vertices which do not form an edge will be given one colour and the other vertices will be coloured with $(r-1)$ colours. We shall arrange these colour classes of H according to the cardinality and we use the n-process. Since the set of vertices X is finite, after a finite number of applications of this method we get a colouring that has the properties 1°, 2° and 3° from Lemma 1.

Lemma 2. *For a* k-uniform hypergraph $H=\langle X, \mathscr{E}\rangle$ *with* n *vertices,*

$$\chi(H)\leqq\left]\frac{n}{k-1}\right[$$

holds.

($]a[$ denotes the smallest integer $\geqq a$.)

Lemma 3. *For a k-uniform hypergraph* $H=\langle X, \mathscr{E}\rangle$, $X=S_1\cup\ldots\cup S_m$, *the inequality*

$$\chi(H)\leqq\chi(H\langle S_1\rangle)+\ldots+\chi(H\langle S_m\rangle)$$

holds.

The proof of Lemma 2 and Lemma 3 follows immediately from the definition of the chromatic number and subhypergraph.

Lemma 4. *For a k-uniform hypergraph* $H=\langle X, \mathscr{E}\rangle$ *and its complement* $\bar{H}=\langle X, \bar{\mathscr{E}}\rangle$

$$\chi(\bar{H})\leqq\left]\frac{n-\chi(H)+2}{k-1}\right[$$

holds, where n *is the number of vertices of the hypergraph H and* $k\geqq 3$.

Proof. Let $\chi(H)=q$. If $q=1$ or $q=2$, then the assertion of the lemma is valid. Let $q\geqq 3$. According to Lemma 1 we can assume that the colouring $\{S_1, \ldots, S_q\}$ of H has the properties 1°, 2° and 3°. We shall define a colouring of the hypergraph $\bar{H}$ by the means of the properties of the colouring of the hypergraph H.

Let $x_{r+1}^k\in S_{r+1}$; then there exist vertices $x_{r+2}^1, \ldots, x_{r+2}^{k-1}\in S_{r+2}$ such that $\{x_{r+1}^k, x_{r+2}^1, \ldots, x_{r+2}^{k-1}\}\in\mathscr{E}$.

Let $x_{r+2}^k\in S_{r+2}-\{x_{r+2}^1, \ldots, x_{r+2}^{k-1}\}$; then there exist vertices $x_{r+3}^1, \ldots, x_{r+3}^{k-1}\in S_{r+3}$ such that $\{x_{r+2}^k, x_{r+3}^1, \ldots, x_{r+3}^{k-1}\}\in\mathscr{E}$.

Proceeding analogously in the next steps, we get finally:

If $x_{q-1}^k \in S_{q-1} - \{x_{q-1}^1, \ldots, x_{q-1}^{k-1}\}$, then there exist vertices $x_q^1, \ldots, x_q^{k-1} \in S_q$ such that $\{x_{q-1}^k, x_q^1, \ldots, x_q^{k-1}\} \in \mathscr{E}$.

It means that for each $i = 1, 2, \ldots, q-1-r$ the vertices of the set $\{x_{r+i}^k, x_{r+i+1}^1, \ldots, x_{r+i+1}^{k-1}\}$ do not form an edge in $\bar{H}$, i. e. we can colour them with one colour.

Let us analyse some examples.

1. If $r=0$ or $r=1$, then from the preceding consideration it follows that in $\bar{H}$ there exist at least $(q-1)$ k-element subsets of the set X that do not form the edge in $\bar{H}$, thus to each of them we can coordinate just one colour. By using Lemma 2 and Lemma 3 we get

$$\chi(\bar{H}) \leqq \left]\frac{n-(q-1)k}{k-1}\right[+ q - 1 = \left]\frac{n-q+1}{k-1}\right[.$$

2. If $q-r=0$. then H is a complete hypergraph and thus

$$\chi(\bar{H}) = 1.$$

3. If $q-r=1$, then the subhypergraph $H\langle S_{q-1} \cup \ldots \cup S_1 \rangle$ of the hypergraph H is complete and thus $S_{q-1} \cup \ldots \cup S_1$ is a stable set in $\bar{H}$. Then

$$\chi(\bar{H}) \leqq \left]\frac{|S_q|}{k-1}\right[+ 1 = \left]\frac{n - |S_{q-1} \cup \ldots \cup S_1|}{k-1}\right[+ 1 =$$

$$= \left]\frac{n-(q-2)(k-1)-|S_1|}{k-1}\right[+ 1 \leqq \left]\frac{n-q+2-(k-2)(q-3)}{k-1}\right[\leqq$$

$$\leqq \left]\frac{n-q+2}{k-1}\right[.$$

4. If $2 \leqq q-r \leqq q-2$, then in the subhypergraph $\bar{H}\langle X - (S_r \cup \ldots \cup S_1)\rangle$ there exist at least $(q-r-1)$ k-element subsets of the set X which do not form an edge in $\bar{H}$, thus to each of them we can coordinate just one colour. Thus

$$\chi(\bar{H}\langle X - (S_r \cup \ldots \cup S_1)\rangle) \leqq \left]\frac{\sum_{i=r+1}^{q} |S_i| - (q-r-1)k}{k-1}\right[+$$

$$+ q - r - 1 = \left]\frac{\sum_{i=r+1}^{q} |S_i| - q + r + 1}{k-1}\right[$$

but

$$\sum_{i=r+1}^{q} |S_i| = n - (r-1)(k-1) - |S_1|,$$

hence

$$\chi(\bar{H}\langle X-(S_r\cup\ldots\cup S_1)\rangle)\leqq\left]\frac{n-(r-1)(k-1)-|S_1|-q+r+1}{k-1}\right[.$$

Since

$$\chi(\bar{H}\langle S_r\cup\ldots\cup S_1\rangle)=1,$$

we have

$$\chi(\bar{H})\leqq\left]\frac{n-q+r+1-|S_1|}{k-1}\right[-r+2\leqq\left]\frac{n-q+2}{k-1}\right[.$$

The proof of Lemma 4 is complete.

Theorem 1. *For a k-uniform hypergraph $H=\langle X, \mathscr{E}\rangle$, $k\geqq 3$, with n vertices*

$$2\sqrt{\frac{n}{k-1}}\leqq\chi(H)+\chi(\bar{H})\leqq\left]\frac{2(n+1)}{k}\right[+1$$

holds.

Proof. a) Let $\chi(H)=q$. In H there exists at least one colour set S_i, for which $|S_i|\geqq\frac{n}{q}$. Then in $\bar{H}$ the subhypergraph induced by the set of vertices S_i is complete, thus

$$\chi(\bar{H})\geqq\chi(\bar{H}\langle S_i\rangle)\geqq\frac{\frac{n}{q}}{k-1}=\frac{n}{q(k-1)}$$

and

$$\chi(H)+\chi(\bar{H})\geqq q+\frac{n}{q(k-1)}.$$

We shall find out for which q the expression $q+\frac{n}{q(k-1)}$ has the minimum value. We shall find the local minimum of the continuous function $f(x)=x+\frac{n}{x(k-1)}$ in the interval $\left\langle 1, \frac{n}{k-1}\right\rangle$. The local minimum is for $x=\sqrt{\frac{n}{k-1}}$. It means that

$$\chi(H)+\chi(\bar{H})\geqq\sqrt{\frac{n}{k-1}}+\frac{n}{\sqrt{\frac{n}{k-1}}(k-1)},$$

and after a modification we get

$$\chi(H)+\chi(\bar{H})\geqq 2\sqrt{\frac{n}{k-1}}.$$

b) From Lemma 4 it follows that

$$\chi(\bar{H})\leqq \left]\frac{n-\chi(H)+2}{k-1}\right[, \quad \chi(H)\leqq \left]\frac{n-\chi(\bar{H})+2}{k-1}\right[.$$

These two inequalities can be modified:

$$-(k-1)<n-\chi(H)+2-\chi(\bar{H})\,(k-1)$$
$$-(k-1)<n-\chi(\bar{H})+2-\chi(H)\,(k-1)\,.$$

After the addition we get

$$-2(k-1)<2n+4-k(\chi(H)+\chi(\bar{H})),$$

hence we have

$$\chi(H)+\chi(\bar{H})\leqq \left]\frac{2(n+1)}{k}\right[+1.$$

3. Achromatic number

Theorem 2. *For a k-uniform hypergraph $H=\langle X, \mathscr{E}\rangle$, $k\geqq 3$, with n vertices,*

$$n+1\leqq \psi(H)+\psi(\bar{H})\leqq 2n$$

holds.

Proof. The validity of the upper bound is obvious. Now we shall prove the validity of the lower bound.

1. Let $k\geqq 4$. If $\psi(H)=n$ or $\psi(H)=1$, then the assertion of Theorem 2 is valid.

If $1<\psi(H)<n$, then in H there exists a couple of non-adjacent vertices, i. e. in $\bar{H}$ there are all edges containing this couple of vertices. From this it follows that $\psi(\bar{H})=n$, i. e. the assertion is valid.

2. Let $k=3$. If $\psi(H)=n$ or $\psi(H)=1$, then the assertion is valid.

Let $1<\psi(H)<n$. The set of vertices X of the hypergraph H may be decomposed into $\psi(H)$ disjoint coloured subsets as follows:

$$X=M_1\cup\ldots\cup M_m\cup N_1\cup\ldots\cup N_s\cup R_1\cup\ldots\cup R_r\cup L_1\cup\ldots\cup L_p\,,$$

whereby we have

$$\text{(a)}\qquad |M_1|=\ldots=|M_m|=|N_1|=\ldots=|N_s|=1\,,$$
$$|R_1|=\ldots=|R_r|=2,$$
$$|L_i|\geqslant 3 \text{ for each } i=1,2,\ldots,p\,.$$

(b) Each vertex belonging to the set $M_1 \cup \ldots \cup M_m$ is adjacent to all vertices of H. The vertices belonging to the set $N_1 \cup \ldots \cup N_s$ are not adjacent to all the vertices of H.

Each vertex belonging to $N_1 \cup \ldots \cup N_s$ is adjacent to all the vertices of $\bar{H}$, because of the 3-uniformity of the hypergraph.

For each $i = 1, 2, \ldots, p$ there exists in the set L_i at most one vertex adjacent to all the vertices of H, i. e. at least $|L_i| - 1$ vertices belonging to L_i are adjacent to all the vertices of $\bar{H}$.

For each $i = 1, 2, \ldots$, r there exists at least one vertex $x_i \in R_i$ which is adjacent to all the vertices of $\bar{H}$, which follows from the following consideration:

If the vertices x_{i1}, x_{i2} belonging to R_i are adjacent to all the vertices of the set $X - \{x_{i1}, x_{i2}\}$ in the hypergraph H, then they are non-adjacent to each other, i. e. in $\bar{H}$ they are adjacent to all the vertices. If some of the vertices $x_{i1}, x_{i2} \in R_i$ is not adjacent to all the vertices of the set $X - \{x_{i1}, x_{i2}\}$ in the hypergraph H, then it is adjacent to all the vertices of $\bar{H}$.

Then for the achromatic number $\psi(\bar{H})$ of $\bar{H}$

$$\psi(\bar{H}) \geqq s + r + \sum_{i=1}^{p} (|L_i| - 1) + 1$$

holds. Then

$$\psi(H) + \psi(\bar{H}) \geqq m + s + r + p + s + r + \sum_{i=1}^{p} (|L_i| - 1) + 1,$$

but

$$p + m + s + 2r + \sum_{i=1}^{p} (|L_i| - 1) = n,$$

then

$$\psi(H) + \psi(\bar{H}) \geqq n + s + 1 \geqq n + 1,$$

and the theorem is proved.

Remark. It is easy to verify that for each k and each $n > k + 1$ there exist k-uniform hypergraphs for which the equality in the upper or the lower bound from Theorem 2 is fulfilled.

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ТЕОРЕМЫ ТИПА НОРДХАУСА—ГАДДУМА ДЛЯ k-УНИФОРМНЫХ ГИПЕРГРАФОВ

Франтишек Олейник

Резюме

В этой работе приведены результаты, которие принадлежат к так называемому классу Нордхауса—Гаддума, для k-униформных гиперграфов.

Хроматические числа $\chi(H)$ и $\chi(\bar{H})$ k-униформного гиперграфа H с n вершинами и его дополнения $\bar{H}$ удовлетворяют неравенствам

$$2\sqrt{\frac{n}{k-1}} \leqq \chi(H) + \chi(\bar{H}) \leqq \left]\frac{2(n+1)}{k}\right[+ 1$$

Ахроматические числа $\psi(H)$ и $\psi(\bar{H})$ k-униформного гиперграфа H с n вершинами и его дополнения $\bar{H}$ удовлетворяют неравенствам

$$n + 1 \leqq \psi(H) + \psi(\bar{H}) \leqq 2n.$$