Mathematica Slovaca

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Mathematica Slovaca, Vol. 49 (1999), No. 1, 27--33

Persistent URL: http://dml.cz/dmlcz/130223

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ON d^*-SUBALGEBRAS OF d-TRANSITIVE d^*-ALGEBRAS

YOUNG CHAN LEE* — HEE SIK KIM**

(*Communicated by Anatolij Dvurečenskij*)

ABSTRACT. In this paper we estimate the number of d^*-subalgebras of order i in a d-transitive d^*-algebra which is a generalization of BCK-algebras by using Hao's method.

1. Introduction

Y. Imai and K. Iséki [II] and K. Iséki [Is1] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [HL1], [HL2] Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. J. Neggers and H. S. Kim [NK] introduced the notion of d-algebras which is another generalization of BCK-algebras, and investigated relations between d-algebras and BCK-algebras. J. Neggers, Y. B. Jun and H. S. Kim [NJK] discussed ideal theory in d-algebras, and introduced the notions of d-subalgebra, d-ideal, $d^\sharp$-ideal and d^*-ideal, and investigated some relations among them. J. Hao [Ha] estimated the number of subalgebras of order i in a finite BCK-algebra X. In this paper we estimate the number of d^*-subalgebras of order i in a d-transitive d^*-algebra which is a generalization of BCK-algebras, by using Hao's method.

AMS Subject Classification (1991): Primary 06F35, 06A06.
Key words: d-subalgebra, d- (d^*-)algebras, adjoint matrix, d-transitive.

2. Preliminaries

A *d-algebra* is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

(1) $x * x = 0$,
(2) $0 * x = 0$,
(3) $x * y = 0$ and $y * x = 0$ imply $x = y$ for all x, y in X.

A BCK-algebra is a d-algebra $(X; *, 0)$ satisfying the following additional axioms:

(4) $\big((x * y) * (x * z)\big) * (z * y) = 0$,
(5) $\big(x * (x * y)\big) * y = 0$ for all x, y, z in X.

EXAMPLE 2.1. ([NK])

(a) Every BCK-algebra is a d-algebra.

(b) Let $X := \{0, 1, 2\}$ be a set with the Table 1.

$*$	0	1	2
0	0	0	0
1	2	0	2
2	1	1	0

Table 1.

Then $(X; *, 0)$ is a d-algebra, but not a BCK-algebra, since $\big(2 * (2 * 2)\big) * 2 = (2 * 0) * 2 = 1 * 2 = 2 \neq 0$.

(c) Let $\mathbb{R}$ be the set of all real numbers and define $x * y := x \cdot (x - y)$, $x, y \in \mathbb{R}$, where $\cdot$ and $-$ are the ordinary product and subtraction of real numbers. Then $x * x = 0$, $0 * x = 0$, $x * 0 = x^2$. If $x * y = y * x = 0$, then $x(x - y) = 0$ and $x^2 = xy$, $y(y - x) = 0$, $y^2 = xy$. Thus if $x = 0$, $y^2 = 0$, $y = 0$; if $y = 0$, $x^2 = 0$, $x = 0$ and if $xy \neq 0$, then $x = y$. Hence $(\mathbb{R}; *, 0)$ is a d-algebra, but not a BCK-algebra, since $(2 * 0) * 2 \neq 0$.

DEFINITION 2.2. ([NJK]) A d-algebra X is called a *d^*-algebra* if it satisfies the identity $(x * y) * x = 0$ for all $x, y \in X$.

Clearly, a BCK-algebra is a d^*-algebra, but the converse need not be true.

EXAMPLE 2.3. ([NJK]) Let $X := \{0, 1, 2, \dots\}$ and let the binary operation $*$ be defined as follows:

$$x * y := \begin{cases} 0 & \text{if } x \leq y, \\ 1 & \text{otherwise.} \end{cases}$$

Then $(X,*,0)$ is a d-algebra which is not a BCK-algebra (see [NK; Example 2.8]). We can easily see that $(X,*,0)$ is a d^*-algebra.

3. Main results

J. Neggers, Y. B. Jun and H. S. Kim [NJK] introduced the notion of d-algebras and investigated their properties related to the concepts of d- (d^*-)ideals. With this concept we obtain a generalization of J. Hao's results [Ha] in d-transitive d^*-algebras.

DEFINITION 3.1. ([NJK]) Let $(X;*,0)$ be a d- (d^*-)algebra and $\emptyset \neq I \subseteq X$. I is called a d- (d^*-)*subalgebra* of X if $x*y \in I$ whenever $x \in I$ and $y \in I$.

PROPOSITION 3.2. *Let $(X;*,0)$ be a d- (d^*-)algebra and let X_0 be a d- (d^*-)subalgebra of X. Then we have:*

(a) $0 \in X_0$,
(b) $(X_0;*,0)$ *is also a d- (d^*-)algebra of X,*
(c) X *is a d- (d^*-)subalgebra of X,*
(d) $\{0\}$ *is a d- (d^*-)subalgebra of X.*

Proof. Routine. □

Note that if $(X;*,0)$ is a BCK-algebra and $0 \neq x_0 \in X$, then $(\{0,x_0\};*,0)$ is a subalgebra of X. But this does not hold in the case of d- (d^*-) algebra.

EXAMPLE 3.3. Consider Example 2.1(b). We can easily see that $(\{0,1\};*,0)$ is not a d-subalgebra of X.

LEMMA 3.4. ([NJK]) *Let $(X;*,0)$ be a d-algebra. If $x \neq y$ and $x*y=0$, then $y*x \neq 0$.*

LEMMA 3.5. *Let $(X;*,0)$ be a d^*-algebra. If $x*y=z$, then $z*x=0$.*

Proof. Let $z := x*y$. Then $z*x = (x*y)*x = 0$, since X is a d^*-algebra. □

Remark. In the above Lemma 3.5, the *d^*-algebra* condition is necessary. Consider Example 2.1(b). We can see that $1*2=2$, but $2*1=1\neq 0$, and hence Lemma 3.5 does not hold.

J. Neggers and H. S. Kim [NK] introduced the notion of d-transitivity in a d-algebra.

DEFINITION 3.6. ([NK]) A d-algebra $(X;*,0)$ is said to be *d-transitive* if $x*z=0$ and $z*y=0$ imply $x*y=0$.

EXAMPLE 3.7. Consider the following d-algebra X with the Table 2.

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	2	0	0
3	3	3	3	0

Table 2.

We can easily see that $1*2=0$, $2*3=0$, but $1*3=1$, and hence $(X;*,0)$ is non-d-transitive d-algebra. Moreover, since $\{(1*3)*(1*2)\}*(2*3)=1\neq 0$, $(X;*,0)$ is not a BCK-algebra.

EXAMPLE 3.8. The d^*-algebra in Example 2.3 is a d-transitive.

DEFINITION 3.9. An ordered n-tuple $a_1, a_2, \dots, a_n$ of elements in a d-algebra X is called an *n-sequence.*

DEFINITION 3.10. Given an n-sequence $a_1, a_2, \dots, a_n$ of a d-algebra X, we construct a $(n-1)\times n$ matrix $\mathbf{A}$ as follows:

$$\mathbf{A}=\begin{pmatrix} a_1*a_2 & a_2*a_1 & \dots & a_n*a_1 \\ a_1*a_3 & a_2*a_3 & \dots & a_n*a_2 \\ \dots & \dots & \dots & \dots \\ a_1*a_n & a_2*a_n & \dots & a_n*a_{n-1} \end{pmatrix}.$$

$\mathbf{A}$ is called the *adjoint matrix* relative to the n-sequence $a_1, a_2, \dots, a_n$.

PROPOSITION 3.11. *Given a distinct n-sequence $a_1, a_2, \dots, a_n$ $(n\geq 2)$ of elements of a d-transitive d-algebra X, let $\mathbf{A}$ be the adjoint matrix relative to this sequence. Then there exists a column in $\mathbf{A}$ which is composed of non-zero elements.*

Proof. The proof is by induction on n. When $n=2$, let a_1, a_2 be a 2-sequence, where $a_1\neq a_2$, then its adjoint matrix is

$$\mathbf{A}=(a_1*a_2 \quad a_2*a_1).$$

If $a_1*a_2=a_2*a_1=0$, then by (3) we have $a_1=a_2$, a contradiction. So the proposition is true for the case $n=2$.

Now assume that the proposition is true for $n-1$.

Let $a_1, a_2, \dots, a_n$ be a distinct n-sequence. Then the adjoint matrix relative to this n-sequence is

$$\mathbf{A}_n = \begin{pmatrix} a_1 * a_2 & a_2 * a_1 & \cdots & a_{n-1} * a_1 & a_n * a_1 \\ a_1 * a_3 & a_2 * a_3 & \cdots & a_{n-1} * a_2 & a_n * a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_1 * a_{n-1} & a_2 * a_{n-1} & \cdots & a_{n-1} * a_{n-2} & a_n * a_{n-2} \\ a_1 * a_n & a_2 * a_n & \cdots & a_{n-1} * a_n & a_n * a_{n-1} \end{pmatrix}.$$

Set

$$\mathbf{A}_{n-1} = \begin{pmatrix} a_1 * a_2 & a_2 * a_1 & \cdots & a_{n-1} * a_1 \\ a_1 * a_3 & a_2 * a_3 & \cdots & a_{n-1} * a_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1 * a_{n-1} & a_2 * a_{n-1} & \cdots & a_{n-1} * a_{n-2} \end{pmatrix}.$$

It is obvious that $\mathbf{A}_{n-1}$ is the adjoint matrix relative to the $(n-1)$-sequence $a_1, a_2, \dots, a_{n-1}$. For this $(n-1)$-sequence we certainly have $a_i \neq a_j$ whenever $i \neq j$. Then, by the induction hypothesis, we know that there exists in $\mathbf{A}_{n-1}$ a column which is composed of non-zero elements. Without loss of generality, we can assume that the first column of $\mathbf{A}_{n-1}$ is composed of non-zero elements, i.e.,

$$\begin{cases} a_1 * a_2 \neq 0, \\ a_1 * a_3 \neq 0, \\ \quad \vdots \\ a_1 * a_{n-1} \neq 0. \end{cases} \tag{a}$$

Now, if $a_1 * a_n \neq 0$, then the elements in the first column of $\mathbf{A}_n$ are all non-zero, so we are done.

If $a_1 * a_n = 0$, then since $a_1 \neq a_n$, by Lemma 3.4, we have

$$a_n * a_1 \neq 0. \tag{b}$$

For $2 \leq i \leq n-1$, we shall show that we also have

$$a_n * a_i \neq 0. \tag{c}$$

In fact, if $a_n * a_i = 0$, then since $a_1 * a_n = 0$, we have

$$a_1 * a_i = 0 \qquad (2 \leq i \leq n-1). \tag{d}$$

But (d) contradicts (a). By (b) and (c) we know that the n-th column of $\mathbf{A}_n$ is composed of non-zero elements. Therefore the conclusion is also true for n. The proposition is proved by induction. □

PROPOSITION 3.12. *Every d-transitive d^*-algebra X of order $n+1$ contains a d^*-algebra of order n $(n \geq 1)$.*

Proof. Let $X = \{0, a_1, a_2, \dots, a_n\}$ be a d-transitive d^*-algebra of order $n+1$, where $a_1, a_2, a_3, \dots, a_n$ are distinct non-zero elements of X. We construct the adjoint matrix $\mathbf{A}_n$ relative to $a_1, a_2, a_3, \dots, a_n$ as follows:

$$\mathbf{A}_n = \begin{pmatrix} a_1 * a_2 & a_2 * a_1 & \cdots & a_{n-1} * a_1 & a_n * a_1 \\ a_1 * a_3 & a_2 * a_3 & \cdots & a_{n-1} * a_2 & a_n * a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_1 * a_{n-1} & a_2 * a_{n-1} & \cdots & a_{n-1} * a_{n-2} & a_n * a_{n-2} \\ a_1 * a_n & a_2 * a_n & \cdots & a_{n-1} * a_n & a_n * a_{n-1} \end{pmatrix}.$$

By Proposition 3.11 there exists in $\mathbf{A}_n$ a column which is composed of non-zero elements. Without loss of generality, we can assume that the elements in the n-th column of $\mathbf{A}_n$ are all non-zero, i.e.,

$$a_n * a_i \neq 0, \qquad i = 1, \dots, n-1. \tag{e}$$

Now we shall show that $T = \{0, a_1, a_2, \dots, a_{n-1}\}$ is a subalgebra of order n in X. In fact, if T is not a subalgebra of X, then there exist i, j $(1 \leq i, j \leq n-1)$ such that $i \neq j$ and $a_i * a_j = a_n$. Since X is a d^*-algebra, by Lemma 3.5, we have

$$a_n * a_i = 0 \tag{f}$$

which contradicts (e). This completes the proof. □

As a consequence of Proposition 3.12 we may estimate the number of d^*-subalgebras of order i in a d-transitive d^*-algebra.

THEOREM 3.13. *Let X be a d-transitive d^*-algebra of order n. Then*

$$1 \leq N(i) \leq \binom{n-1}{i-1} \qquad (i = 1, 2, \dots, n)$$

where $N(i)$ denotes the number of d^*-subalgebras of order i in X.

Proof. This is a direct consequence of Proposition 3.12. □

Acknowledgement

The authors are deeply grateful to the referee for some valuable suggestions.

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Received December 4, 1996
Revised February 22, 1997

* *Department of Mathematics*
Hannam University
Taejon 300-791
KOREA

** *Department of Mathematics*
Hanyang University
Seoul 133-791
KOREA

E-mail: heekim@email.hanyang.ac.kr