Snežana Matić-Kekić; Dragan M. Acketa

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ON MINIMAL AREA AUGMENTATION OF DIGITAL CONVEX n-GONS

SNEŽANA MATIĆ-KEKIĆ — DRAGAN M. ACKETA

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ABSTRACT. Let there be given a digital convex $2n$-gon P with parallel edges (a centrally symmetric one). Let s denote the minimal edge size of a digital square in which P can be inscribed. An $O(ns)$ algorithm is presented, which determines a pair of parallel edges, the insertion of which leads to the minimal area increase; in other words, the area difference between the centrally symmetric digital convex $(2n + 2)$-gon obtained and P is the minimal possible. It is shown that this algorithm can be generalized, while preserving the same complexity, to the minimal area augmentation with a pair of congruent parallel edges of a non-symmetric digital convex polygon with an arbitrary number of edges.

1. Introduction

A *digital convex polygon* (a *d.c. polygon* or *d.c.p.* for short) is a polygon with all the vertices belonging to the integer grid and with all the angles strictly less than π radians. Let the *diameter* of a d.c.p. denote the minimal edge size of a digital square in which this d.c.p. can be inscribed.

The following optimization problem is considered:

Given a digital convex n-gon, add two congruent parallel edges to it in such a way that a digital convex $(n+2)$-gon is produced and that the area difference between these two polygons is the minimal possible.

An algorithm for solving a simpler subproblem will be primarily given in Section 2, while Section 3 contains the generalization of this solution to the problem itself.

The subproblem reads:

Given a centrally symmetric (parallel-sided) digital convex $2n$-gon ($n > 1$), construct a centrally symmetric digital convex $(2n+2)$-gon from it by adding

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two (congruent parallel) edges in such a manner that the minimal possible area increase is produced.

An $O(ns)$ construction for solving both the subproblem and the general problem itself will be given, where s denotes the diameter of the initial d.c. polygon. More precisely, $O(s)$ elementary steps will be expended for finding the locally optimal insertion at a given position. The number of positions (insertions) to be considered is $O(n)$. Finally, the adjustments necessary for transition from one position to the next can be performed in constant time.

A motivation for considering centrally symmetric digital convex $2n$-gons can be found in Theorem 2 of [6], which says that the search for d.c. $2n$-gons with minimal area can be restricted without loss of generality to the centrally symmetric case.

Starting from the unit square, an iterative application of a minimal area augmentation similar to those treated by the subproblem can be applied to a greedy search for a digital convex $2n$-gon with minimal area. Some hints allow the complexity of this search to be only $O(n^2)$ ([1]), instead of $O(n^2s)$. This greedy search gives optimal solutions for all the values of n for which such solutions are known ([6]) (that is, for $n \leq 11$).

A related optimization problem of finding a d.c. n-gon with minimal diameter has been formulated in [7]. Asymptotic and algorithmic aspects of the problem have been considered in more detail, in., e.g., [2], [3] respectively, while an exact solution for a given n has been given in [5].

Let $g(v)$ and $a(v)$ respectively denote the number of *interior* points (*G-area*) and the area in the Euclidean sense (*A-area*) of a d.c. v-gon without internal grid points on its edges. These two functions are related by $a(v) = g(v) + v/2 - 1$ for $v \geq 3$, which follows from the Pick's theorem ([4]).

Let $y_{\min}$ and $x_{\max}$ respectively denote the minimal y-coordinate and the maximal x-coordinate of the d.c.p. P considered. Generally, the SE-arc (*south-east arc*) of P is the sequence of consecutive edges (V_i, V_{i+1}), $1 \leq i \leq c-1$, where V_i denotes a vertex (x_i, y_i) of P; $x_1 < x_2 < x_3 < \cdots < x_{c-1} < x_c = x_{\max}$; $y_{\min} = y_1 \leq y_2 < y_3 < \cdots < y_{c-1} < y_c$. The NE-arc, the NW-arc and the SW-arc of a d.c.p. are defined similarly.

Given an edge $e = \big((x_1, y_1), (x_2, y_2)\big)$ of a d.c.p., the *edge slope* of e denotes the fraction:

$$\begin{cases} \frac{|x_1-x_2|}{|y_1-y_2|} & \text{if } e \in \text{NE- or SW-arc;} \\ \\ \frac{|y_1-y_2|}{|x_1-x_2|} & \text{if } e \in \text{SE- or NW-arc.} \end{cases}$$

The smallest integer not smaller than x is denoted by $\lceil x \rceil$, while the greatest integer not greater than x is denoted by $\lfloor x \rfloor$. The greatest common divisor of integers X and Y is denoted by $\gcd(X, Y)$.

2. An algorithm

Let there be given a centrally symmetric d.c. $2n$-gon (Fig. 1) with diameter s which has k sorted edge slopes $\frac{b_1}{a_1} < \frac{b_2}{a_2} < \cdots < \frac{b_k}{a_k}$ in the NW-arc, so that the natural numbers b_i and a_i are relatively prime for $1 \le i \le k$, and which has $n-k$ sorted edge slopes $\frac{d_1}{c_1} < \frac{d_2}{c_2} < \cdots < \frac{d_{n-k}}{c_{n-k}}$ in the SW-arc, so that the natural numbers d_i and c_i are relatively prime for $1 \le i \le n-k$.

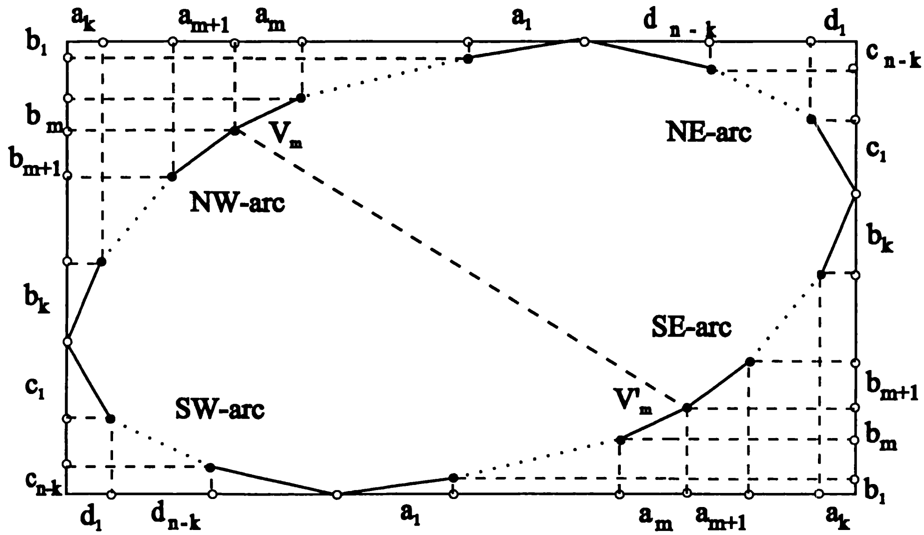

FIGURE 1. A centrally symmetric digital convex $2n$-gon.

The following algorithm is proposed:

Input. The sequence $S(n)$ which represents the initial $2n$-gon and consists of edge slopes belonging to the NW- or to the SW-arc (both sequences of edge slopes are sorted in increasing order):

$$S(n) = \left\{ \frac{b_1}{a_1}, \frac{b_2}{a_2}, \ldots, \frac{b_k}{a_k}; \ \frac{d_1}{c_1}, \frac{d_2}{c_2}, \ldots, \frac{d_{n-k}}{c_{n-k}} \right\}.$$

Output. The position m^* of the optimal edge insertion and the corresponding edge slope q_{m^*}/p_{m^*}.

A centrally symmetric digital convex $(2n+2)$-gon with a minimal area augmentation w.r.t. the initial centrally symmetric d.c. $2n$-gon is constructed by adding the edge slope q_{m^*}/p_{m^*} to the position between the m^*th and (m^*+1)th edge slope of the NW-arc for $0 \leq m^* \leq k$ or to the position between the $(m^* - k - 1)$th and $(m^* - k)$th edge slope of the SW-arc for $k+1 \leq m^* \leq n+1$, as well as to the corresponding position in the SE-arc, respectively in the NE-arc.

The algorithm has the following outlook:

Start with two variables X and Y (their geometric sense will be illustrated by Fig. 2, Section 2.1) by:

$$X = -\sum_{i=1}^{k} a_i + \sum_{i=1}^{n-k} d_i \quad \text{and} \quad Y = \sum_{i=1}^{k} b_i + \sum_{i=1}^{n-k} c_i .$$

Edge slope q_{m^*}/p_{m^*} is determined as follows:

Let $\Delta\left(S(n), \frac{q}{p}\right)$ denote the G-area increase of a d.c. $2n$-gon represented by the sequence $S(n)$, after two edges with the edge slope $\dfrac{q}{p}$ are inserted in two opposite positions within given two opposite arcs, so that the convexity (and consequently the central symmetry) is preserved. A necessary condition for this increase to be minimal is that the natural numbers q and p are relatively prime, so the assumption that this condition is satisfied will be made in the rest of the paper.

Let a sequence $\dfrac{q_m}{p_m}$ (for $0 \leq m \leq n+1$) be defined by the property:

$$\Delta\left(S(n), \frac{q_m}{p_m}\right) = \begin{cases} \displaystyle\min_{\frac{b_m}{a_m} < \frac{q}{p} < \frac{b_{m+1}}{a_{m+1}}} \Delta\left(S(n), \frac{q}{p}\right), & 0 \leq m \leq k\,, \\ \displaystyle\min_{\frac{d_{m-(k+1)}}{c_{m-(k+1)}} < \frac{q}{p} < \frac{d_{m-k}}{c_{m-k}}} \Delta\left(S(n), \frac{q}{p}\right), & k+1 \leq m \leq n+1\,, \end{cases}$$

where $b_0/a_0 = 0$, $b_{k+1}/a_{k+1} = 5s$, $d_0/c_0 = 0$ and $d_{n-k+1}/c_{n-k+1} = 5s$.

In other words, $\dfrac{q_m}{p_m}$ $(0 \leq m \leq n+1)$ is the optimal solution of a local insertion. It denotes the edge slope of an edge the insertion of which between the mth and the $(m+1)$th edge slope of $S(n)$ gives the representation of a centrally symmetric d.c. $(2n+2)$-gon with the minimal area augmentation w.r.t. the initial centrally symmetric d.c. $2n$-gon, when only the insertions on that fixed position are considered.

Then $\dfrac{q_{m^*}}{p_{m^*}}$ is the edge slope which satisfies:

$$\Delta\left(S(n), \frac{q_{m^*}}{p_{m^*}}\right) = \min_{0 \leq m \leq n+1} \Delta\left(S(n), \frac{q_m}{p_m}\right).$$

Remark. The edge slopes b_0/a_0, b_{k+1}/a_{k+1}, d_0/c_0 and d_{n-k+1}/c_{n-k+1} do not exist in the initial d.c. $2n$-gon. They are introduced in order to unify the conditions concerned with the edge insertions: the same conditions cover the insertions before the first edge of an arc, after the last one, and between them. The choice of the constant $5s$ is in accordance with Lemma 5.

2.1. Calculation of $\Delta\left(S(n), \frac{q}{p}\right)$.

The next lemma says that $\Delta\left(S(n), \frac{q}{p}\right)$ can be calculated in a constant time. The proof will be given for insertions into the NW-arc and the SE-arc, the proof for insertions into the SW-arc and the NE-arc being quite similar:

LEMMA 1. *If two parallel edges with edge slope q/p, with q and p relatively prime, are inserted between some two consecutive edges belonging to the NW-arc and between the two opposite edges of the SE-arc of a centrally symmetric d.c. $2n$-gon with the edge slopes b_m/a_m and b_{m+1}/a_{m+1}, $(\frac{b_m}{a_m} < \frac{q}{p} < \frac{b_{m+1}}{a_{m+1}})$, then the G-area increase produced by this insertions is equal to*

$$\Delta\left(S(n), \frac{q}{p}\right) = Xq + Yp - 1\,,$$

where

$$X = \sum_{i=1}^{m} a_i - \sum_{i=m+1}^{k} a_i + \sum_{i=1}^{n-k} d_i \quad \textit{and} \quad Y = \sum_{i=m+1}^{k} b_i - \sum_{i=1}^{m} b_i + \sum_{i=1}^{n-k} c_i\,.$$

Proof. Let V_m and V'_m be the common vertices (see Fig. 1) of the edges with the edge slopes b_m/a_m and b_{m+1}/a_{m+1} in the NW- and the SE-arc respectively. Three situations will be distinguished, depending on the location of the point V'_m after the vector $\overrightarrow{V_mV'_m}$ is translated so that the point V_m coincides with $(0,0)$; the cases a), b) and c) in Figures 2–4 correspond to the situations when the translated V'_m belongs to the SW-, SE- or NE-quadrant respectively. These three situations can be also characterized by the pairs of inequalities ($X \le 0$ and $Y > 0$), ($X > 0$ and $Y > 0$), ($X > 0$ and $Y \le 0$) respectively (Fig. 2).

Figure 2 also explains the geometric sense of the expressions X and Y in each of the three cases; the abbreviations

$$A_{1m} = \sum_{i=1}^{m} a_i\,, \qquad A_{mk} = \sum_{i=m+1}^{k} a_i\,, \qquad B_{1m} = \sum_{i=1}^{m} b_i\,, \qquad B_{mk} = \sum_{i=m+1}^{k} b_i\,,$$

$$C = \sum_{i=1}^{n-k} c_i\,, \qquad D = \sum_{i=1}^{n-k} d_i$$

are used in it.

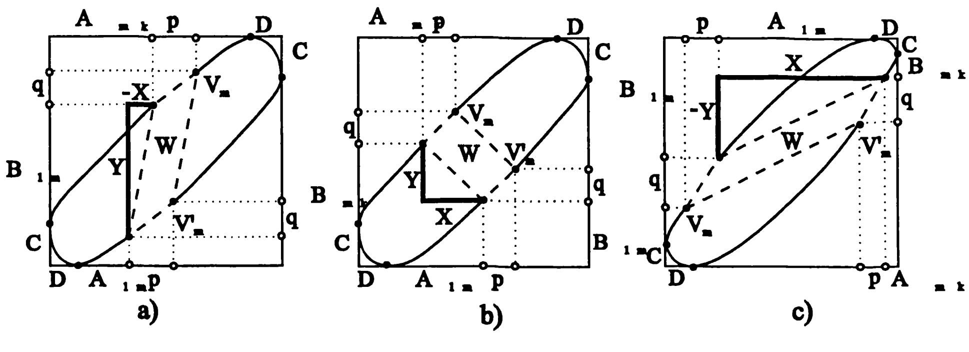

FIGURE 2. Geometric sense of X and Y.

The additional G-area $\Delta\Big(S(n), \dfrac{q}{p}\Big)$, after inserting the edge with edge slope $\dfrac{q}{p}$, is equal to the G-area of the parallelogram W inscribed into the rectangle with the side lengths $p + |X|$ and $q + |Y|$ in accordance with Figures 2 and 3, augmented for the number of interior grid points belonging to the side of W parallel with $V_m V_m'$ (note that the remaining two sides of W do not have interior grid points since the numbers q and p are relatively prime).

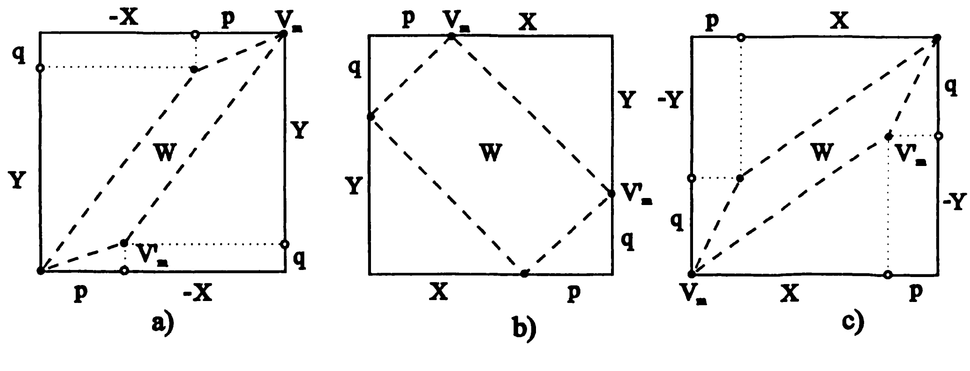

FIGURE 3. The A-area increase parallelogram.

Figure 3b) gives that the A-area of W is equal to $(X+p)(Y+q) - XY - pq = Xq + Yp$. Similarly, the A-area of W in Figure 3c) is equal to $(X+p)(-Y+q) - X(-Y) - 2(-Y)p - pq = Xq + Yp$, and the same result is obtained for Figure 3a).

Observe that the number v of grid points on the edges of the parallelogram W is equal to $4+2\cdot\big(\gcd(X,Y)-1\big)$. The Pick's theorem gives that the G-area of W is equal to (the A-area of W) $-\frac{v}{2}+1$. Thus the G-area of W is equal to $Xq+Yp-\gcd(X,Y)$ in each one of the three cases.

Since the number of interior grid points belonging to the side of W parallel with $V_mV'_m$ is equal to $\gcd(X,Y)-1$, it follows that the G-area increase $\Delta\Big(S(n),\frac{q}{p}\Big)$ is equal to $Xq+Yp-1$. □

2.2. Determination of the sequence $\frac{q_m}{p_m}$.

The problem of determining the member $\frac{q_m}{p_m}$ of the sequence introduced at the beginning of Section 2 can be reduced to a problem of linear integer programming. This problem will be considered only for the insertion into the NW-arc (and the SE-arc). This branch of the problem will be denoted by *LIP*, the other branch is quite analogous:

$$\min(p\cdot Y+q\cdot X)\,,$$

under the constraints:

$$\textbf{I.}\quad \frac{b_m}{a_m}<\frac{q}{p}<\frac{b_{m+1}}{a_{m+1}}\qquad\qquad \textbf{II.}\quad p\cdot Y+q\cdot X\geq 1\qquad\qquad (\text{LIP})$$

III. p and q are relatively prime natural numbers,
where X and Y are given integer constants.

The constraint II follows from the fact that the minimal possible G-area increase is equal to 0.

Three cases (denoted by a), b), c)) are shown on the Figures 4a)–4c), in depending on the sign of the constants X and Y:

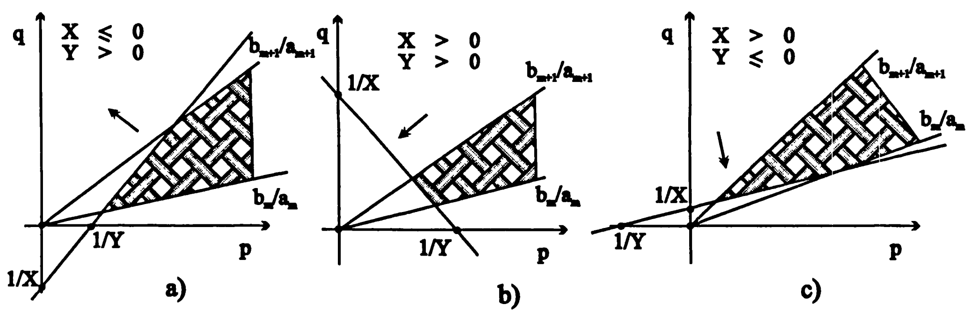

FIGURE 4. The feasible set of LIP.

LEMMA 2. *The inequalities* $\frac{Y}{-X} > \frac{b_{m+1}}{a_{m+1}}$ *and* $\frac{-Y}{X} < \frac{b_m}{a_m}$ *are satisfied in cases* a) *and* c) *respectively.*

P r o o f. The proof for case a) will be given, the other being completely similar.

Since $\frac{b_{m+1}}{a_{m+1}} < \frac{b_i}{a_i} \implies b_{m+1} \cdot a_i < a_{m+1} \cdot b_i$ for $m+2 \le i \le k$, and $\frac{b_{m+1}}{a_{m+1}} > \frac{b_i}{a_i} \implies b_{m+1} \cdot a_i > a_{m+1} \cdot b_i$ for $1 \le i \le m$, it follows that

$$\begin{aligned} &b_{m+1} \cdot a_{m+2} + \cdots + b_{m+1} \cdot a_k - b_{m+1} \cdot a_1 - \cdots - b_{m+1} \cdot a_m \\ &< a_{m+1} \cdot b_{m+2} + \cdots + a_{m+1} \cdot b_k - a_{m+1} \cdot b_1 - \cdots - a_{m+1} \cdot b_m . \end{aligned}$$

Thus

$$\begin{aligned} \frac{b_{m+1}}{a_{m+1}} &< \frac{(b_{m+1} + \cdots + b_k) - (b_1 + \cdots + b_m)}{(a_{m+1} + \cdots + a_k) - (a_1 + \cdots + a_m)} \\ &< \frac{(b_{m+1} + \cdots + b_k) - (b_1 + \cdots + b_m) + (c_1 + \cdots + c_{n-k})}{(a_{m+1} + \cdots + a_k) - (a_1 + \cdots + a_m) - (d_1 + \cdots + d_{n-k})} \\ &= \frac{Y}{-X} . \end{aligned}$$

□

LEMMA 3. *The problem LIP has always an optimal solution.*

P r o o f. Constraint I says that the feasible set is a subset of the interior of the angle determined by the half-lines $q = \frac{b_{m+1}}{a_{m+1}} \cdot p$, $p > 0$, and $q = \frac{b_m}{a_m} \cdot p$, $p > 0$. The line $L_1 \colon p \cdot Y + q \cdot X = 1$ intersects both sides of the angle; this follows from $X > 0$, $Y > 0$ in case b) and from Lemma 2 in cases a) and c). Constraint II implies that the whole feasible set lies outside the triangle which is cut from the angle by L_1. That the feasible set is non-empty follows from the fact that infinitely many points (p, q) satisfying the constraint III lie within the angle, but only finitely many of them lie within the triangle.

Since the optimization direction is orthogonal w.r.t. to the line L_1, the optimal solution(s) can be found on the line $L_t \colon p \cdot Y + q \cdot X = t$ for some natural number t. More precisely, the smallest possible value of t should be found such that the intersection of L_t and the feasible set of LIP is non-empty. □

2.3. Complexity.

The following two lemmas will be used for proving the complexity of the proposed algorithm:

LEMMA 4. *The difference of two consecutive edge slopes of a digital convex polygon with diameter s is greater than $\frac{1}{2s}$.*

Proof. The minimal edge slope of an edge of the d.c.p. considered is equal to $1/s$. It is easy to show that the absolute value of the tangens of the angle determined by three of its vertices is bounded below by $1/(2s-1)$.

Consider two consecutive edge slopes $\frac{b_m}{a_m}$ and $\frac{b_{m+1}}{a_{m+1}}$ of this d.c.p. (where $\frac{b_m}{a_m} < \frac{b_{m+1}}{a_{m+1}}$). The following three angles are introduced:

$$\gamma_m = \operatorname{arctg}\left(\frac{b_m}{a_m}\right), \qquad \gamma_{m+1} = \operatorname{arctg}\left(\frac{b_{m+1}}{a_{m+1}}\right) \qquad \text{and} \qquad \delta_m = \gamma_{m+1} - \gamma_m .$$

Then

$$\begin{aligned}\frac{b_{m+1}}{a_{m+1}} - \frac{b_m}{a_m} &= \operatorname{tg}(\gamma_{m+1}) - \operatorname{tg}(\gamma_m)\\ &= \operatorname{tg}(\delta_m)\cdot\left(1 + \operatorname{tg}(\gamma_{m+1})\cdot\operatorname{tg}(\gamma_m)\right)\\ &\geq \frac{1}{2s-1}\cdot\left(1+\frac{1}{s^2}\right) > \frac{1}{2s}.\end{aligned}$$

□

LEMMA 5. *If $\frac{q_m}{p_m}$ is an optimal solution of LIP, then*

$$q_m \leq 4s+1 \qquad \textit{and} \qquad p_m \leq 4s+1 .$$

Proof. Constraint I of LIP implies that each feasible point (q,p) satisfies

$$q \in \left(\frac{b_m}{a_m}\cdot p,\ \frac{b_{m+1}}{a_{m+1}}\cdot p\right).$$

A sufficient condition for this interval to contain an integer is that the width of the interval belongs to the open interval $(1,2)$. Since an integer may exist inside a smaller interval, it follows that

$$p_m \leq \left\lceil \frac{2}{\frac{b_{m+1}}{a_{m+1}} - \frac{b_m}{a_m}} \right\rceil \leq \frac{2}{\frac{b_{m+1}}{a_{m+1}} - \frac{b_m}{a_m}} + 1 \overset{\text{(Lemma 4)}}{<} 4s+1 .$$

Similarly,

$$q_m \leq \frac{2}{\frac{a_m}{b_m} - \frac{a_{m+1}}{b_{m+1}}} + 1 < 4s+1 .$$

□

The complexity of the proposed algorithm is given by the following theorem:

THEOREM 1. *The complexity of finding two congruent parallel edges, which can be added to a centrally symmetric digital convex $2n$-gon with diameter s in such a way that a digital convex $(2n+2)$-gon is produced, and that the area difference between the two polygons is the minimal possible — is equal to $O(ns)$.*

Proof. Given the input sequence $S(n)$, an attempt is made to insert an edge at each of the $n+2$ positions determined by the present n edges. These positions are passed only once. After an $O(n)$ initialization, the values of X and Y are adjusted in constant time for each insertion, (their initial values are defined in Section 2.1):

The adjustments $X \leftarrow X + 2 \cdot a_{m+1}$; $Y \leftarrow Y - 2 \cdot b_{m+1}$ are used after the mth insertion $(0 \le m \le k-1)$.

The transition from the NW-arc to the SW-arc is performed after the kth insertion by using the adjustments: $X \leftarrow Y$; $Y \leftarrow -X$; note that $X = A + D$; $Y = -B + C$ before the transition, and $X = B - C$; $Y = A + D$ after the transition, where A and B respectively denote the sums: $A = \sum_{i=1}^{k} a_i$ and $B = \sum_{i=1}^{k} b_i$.

The adjustments $X \leftarrow X + 2 \cdot c_{m-k}$; $Y \leftarrow Y - 2 \cdot d_{m-k}$ are used after the mth insertion $(k+1 \le m \le n)$.

The current values m^*, q_{m^*}, p_{m^*} and $\Delta\Big(S(n), \dfrac{q_{m^*}}{p_{m^*}}\Big)$ (see the introductory part of Section 2) are kept in memory.

It follows from Lemma 5 that the edge slope $\dfrac{q_m}{p_m}$, which is to be inserted at each particular position m, $0 \le m \le n+1$, can be determined by at most $O(s)$ elementary tests. The value of p_m is equal to the smallest natural number i, for which the following $O(1)$ test is satisfied:

IF none of the fractions $\dfrac{b_m}{a_m} \cdot i$ and $\dfrac{b_{m+1}}{a_{m+1}} \cdot i$ is an integer

THEN $\left\lceil \dfrac{b_m}{a_m} \cdot i \right\rceil \le \left\lfloor \dfrac{b_{m+1}}{a_{m+1}} \cdot i \right\rfloor$ ELSE $\left\lceil \dfrac{b_m}{a_m} \cdot i \right\rceil < \left\lfloor \dfrac{b_{m+1}}{a_{m+1}} \cdot i \right\rfloor$.

Lemma 5 gives that $O(s)$ such tests are required.

The value of q_m is then determined as $\left\lceil \dfrac{b_m}{a_m} \cdot p_m \right\rceil$ in cases b) and c), while $q_m = \left\lfloor \dfrac{b_{m+1}}{a_{m+1}} \cdot p_m \right\rfloor$ in case a).

Since $O(n)$ possibilities for edge insertion are considered, and the proposed way of determining $\dfrac{q_m}{p_m}$ is of complexity $O(s)$, it follows that the total complexity of the proposed algorithm is $O(ns)$. □

Remarks. Iterative applications (with more than one iterative step) of the proposed algorithm need not result in a d.c. $(2n+2i)$-gon $(i \geq 2)$ with the minimal possible area difference w.r.t. the initial d.c. $2n$-gon. For example, some counterexamples were provided for the $O(n^2)$ algorithm ([1]) for determining a digital convex $2n$-gon with minimal possible area, which show that there exist non-greedy solutions with a smaller area than the area of the greedy optimal solution. However, it should be noted that the algorithm given in [6] for solving the same problem, which necessarily reaches an optimal solution, has an exponential complexity.

The following hint reduces the complexity of the algorithm from [1] to only $O(n^2)$, in comparison with $O(n^2 s)$, which would follow from iterative applications of the algorithm proposed here:

When looking for a digital convex $2n$-gon with minimal possible area, it follows from the results of [6] that $\frac{q_m}{p_m}$ can be determined in constant time by $\frac{q_m}{p_m} = \frac{b_m + b_{m+1}}{a_m + a_{m+1}}$.

3. General case

Let a general d.c. n-gon be represented by the sequence

$$S(n) = \left\{ \frac{b_1}{a_1}, \ldots, \frac{b_k}{a_k};\ \frac{d_1}{c_1}, \ldots, \frac{d_l}{c_l};\ \frac{f_1}{e_1}, \ldots, \frac{f_{k'}}{e_{k'}};\ \frac{h_1}{g_1}, \ldots, \frac{h_{l'}}{g_{l'}} \right\},$$

where $k + l + k' + l' = n$, and the subsequences of sorted edge slopes belonging to NW-, SW-, SE- and NE-arc respectively are listed in increasing order and separated by the ";" sign.

When the edge slope and the positions of two additional edges are fixed, the following analogue of Lemma 1 provides the constant time calculation of the G-area increase with the generalized problem:

LEMMA 6. *If two parallel edges with edge slope q/p, with q and p relatively prime, are inserted between the edges of the NW-arc with the edge slopes b_m/a_m and b_{m+1}/a_{m+1}, respectively between the edges of the SE-arc with the edge slopes $f_{m'}/e_{m'}$ and $f_{m'+1}/e_{m'+1}$, $(\max\{\frac{b_m}{a_m}, \frac{f_{m'}}{e_{m'}}\} < \frac{q}{p} < \min\{\frac{b_{m+1}}{a_{m+1}}, \frac{f_{m'+1}}{e_{m'+1}}\})$, then the G-area increase produced by this insertions is equal to $Xq + Yp - 1$, where*

$$X = \sum_{i=1}^{l} d_i + \sum_{i=1}^{m'} e_i - \sum_{i=m+1}^{k} a_i \quad \textit{and} \quad Y = \sum_{i=1}^{l} c_i - \sum_{i=1}^{m'} f_i + \sum_{i=m+1}^{k} b_i .$$

The proof is also analogous to the proof of Lemma 1. The initial values of the new variables X and Y are obtained from the above expressions by putting $(m, m') = (0, 0)$.

Remark. The equalities $X = X'$ and $Y = Y'$ are satisfied, where the variables X' and Y' are defined by

$$X' = \sum_{i=1}^{l'} h_i + \sum_{i=1}^{m} a_i - \sum_{i=m'+1}^{k'} e_i, \qquad Y' = \sum_{i=1}^{l'} g_i - \sum_{i=1}^{m} b_i + \sum_{i=m'+1}^{k'} f_i.$$

The sums obtained on the both sides of these equalities, after the summands with negative sign are interchanged, are equal to the lengths within pairs of opposite sides of the minimal rectangle inscribed around the initial d.c.p.

The eight auxiliary edge slopes are introduced for unifying reasons, analogously to those with the centrally symmetric case subproblem:

$$b_0/a_0 = d_0/c_0 = f_0/e_0 = h_0/g_0 = 0 \qquad \text{and}$$
$$b_{k+1}/a_{k+1} = d_{l+1}/c_{l+1} = f_{k'+1}/e_{k'+1} = h_{l'+1}/g_{l'+1} = 5s.$$

It is easy to show that the statement of Lemma 2 remains valid after the constants X and Y are replaced by their generalized analogues (introduced with Lemma 6). Similarly, it is easy to conclude that the generalization preserves the statements of Lemmas 3, 4 and 5; Lemma 4 is already formulated for a general d.c.p.

The following theorem claims that the generalization does not increase the complexity:

THEOREM 2. *The complexity of finding two congruent parallel edges which can be added to a digital convex n-gon with diameter s in such a way that a digital convex $(n+2)$-gon is produced, and that the area difference between the two polygons is minimal possible — is equal to $O(ns)$.*

The local optimization problem (finding the edge slope which gives the minimal G-area increase caused by the insertion on fixed positions, determined by a pair of indices (m, m') within some two opposite arcs) is now solved by a linear integer programming problem completely analogous to LIP considered in Section 2.2; the only differences between the two problems are related to the fact that the new definitions of constants X and Y are used, and that the constraint I is now replaced by:

$$\max\left\{\frac{b_m}{a_m}, \frac{f_{m'}}{e_{m'}}\right\} < \frac{q}{p} < \min\left\{\frac{b_{m+1}}{a_{m+1}}, \frac{f_{m'+1}}{e_{m'+1}}\right\}.$$

Similarly as with the last part of proof of Theorem 1, it can be proved that the complexity of the generalized version of LIP remains $O(s)$.

The final step for proving Theorem 2 should provide that the number of positions for insertion is $O(n)$ again:

LEMMA 7. *The number of positions for insertion of a pair of parallel congruent edges is upperbounded by* $n+2$.

P r o o f . In accordance with the notations at the beginning of this section, it suffices to show that the number of positions for the insertion of a pair of parallel congruent edges into the NW-arc and the SE-arc is upperbounded by $k+k'+1$.

Finding the next position in the sorted (merged) sequences of edge slopes within the both arcs can be made in constant time by the following branching:

IF the edge slopes on the $(m+1)$th position of the NW-arc and the $(m'+1)$th position in the SE-arc are equal
THEN raise both m and m' by 1
ELSE IF the $(m+1)$th edge slope in the NW-arc is smaller than the $(m'+1)$th edge slope in the SE-arc
THEN raise m by 1 ELSE raise m' by 1.

After the initialization with $m = m' = 0$, the pairs of positions (m, m') for insertion, determined by this procedure, necessarily satisfy the generalized version of the constraint I of the generalized linear integer programming problem.

The upper bound of the lemma is reached when there are no two common edge slopes in any of the two pairs of opposite arcs. □

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Institute of Mathematics
Trg Dositeja Obradovića 4
YU-21000 Novi Sad
YUGOSLAVIA