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ASYMPTOTIC BEHAVIOUR AND OSCILLATION OF SOLUTIONS OF NEUTRAL DELAY DIFFERENCE EQUATIONS OF ARBITRARY ORDER

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ABSTRACT. The authors obtain results on the asymptotic properties of solutions of a higher order nonlinear neutral delay difference equation. Examples illustrating the results are included and some suggestions for further research are indicated.

1. Introduction

Consider the neutral difference equation

$$\Delta^m[y_{n-m+1} + p_{n-m+1}y_{n-m+1-k}] + \delta F(n, y_{n-l}) = 0\,, \tag{E}$$

where $m \geq 1$, $\delta = \pm 1$, Δ denotes the forward difference operator $\Delta y_n = y_{n+1} - y_n$, $\Delta^i y_n = \Delta(\Delta^{i-1}y_n)$, $1 \leq i \leq m$, $k, l \in \mathbb{N} = \{0, 1, 2, \dots\}$, $\{p_n\}$ is a sequence of real numbers, $F\colon \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ is continuous with $uF(n,u) \geq 0$ for $u \neq 0$ and $n \geq N_0$, and $F(n,u) \not\equiv 0$ for $u \in \mathbb{R} \setminus \{0\}$ and $n \geq N_1$ for every $N_1 \geq N_0$. By a solution of (E), we mean a sequence $\{y_n\}$ of real numbers which is defined for $n \geq N_0 - M$ where $M = \max\{k, l\} + m - 1$ and which satisfies (E) for $n \geq N_0$. A solution $\{y_n\}$ of (E) is said to be *nonoscillatory* if the terms y_n are either eventually all positive or eventually all negative. Otherwise, the solution is called *oscillatory.*

Here we examine the oscillatory and asymptotic behavior of solutions of (E). If $p_n \equiv 0$, equation (E) becomes an mth order difference equation with a delay, and a good deal is known about the asymptotic and oscillatory properties of solutions of equations of this type especially when $m = 1$ or 2 and the equation is linear; for recent contributions see, for example, the papers of Cheng et

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al. [5], Erbe and Zhang [7], Jaroš and Stavroulakis [13], Ladas et al. [6], [14]–[16], and Patula [21]–[22] for linear equations, and Bykov et al. [2]–[4], Erbe and Zhang [8], He [11], Hooker and Patula [12], Szmanda [23], and Wang and Yu [29] for nonlinear equations, as well as the monographs by Agarwal [1] and Lakshmikantham and Trigiante [17]. When $p_n \neq 0$, considerably fewer results of this type are known, and many of these are for linear equations with $m = 1$ or 2 (see Georgiou et al. [9]–[10], Lalli et al. [18]–[20], Thandapani et al. [24]–[28], Zafer and Dahiya [30], and the references contained therein). The results here are partially motivated by the papers [9], [10], [18], [20], [24], and [27], and in fact we generalize some of the results in those papers. Examples illustrating our results are included. The final section of the paper contains some suggestions for further research.

2. Oscillatory and asymptotic behavior

Many of our results will require the condition that if $\{u_n\}$ is a sequence with $u_n > 0$ ($u_n < 0$) and $\liminf\limits_{n\to\infty} |u_n| > 0$, then

$$\sum_{i=N_0}^{\infty} F(i, u_i) = \infty \ (-\infty). \tag{1}$$

We will begin with a lemma that will facilitate proving the main results in this paper. In some parts of the lemma, we assume that there exist constants P_1 and P_2 such that either

$$P_1 \leq p_n \leq 0, \tag{2}$$

$$-1 \leq p_n \leq 0, \tag{3}$$

or

$$P_2 \leq p_n \leq -1. \tag{4}$$

For notational purposes, we let

$$z_n = y_n + p_n y_{n-k}.$$

We give proofs only for the case when a nonoscillatory solution is eventually positive since the proofs for an eventually negative solution are similar. In addition, when the proof for $\delta = -1$ is completely analogous to the proof for $\delta = +1$, only the latter will be given.

LEMMA 1. *Suppose that condition* (1) *holds and* $\{y_n\}$ *is an eventually positive* (*negative*) *solution of* (E) *with* $\delta = +1$ *[*$\delta = -1$*]. Then:*

(a) $\{\Delta^{m-1} z_n\}$ *is an eventually decreasing* (*increasing*) *[increasing* (*decreasing*)*] sequence and*

$$\Delta^{m-1} z_n \to \delta L < \infty \ (> -\infty) \qquad \text{as} \quad n \to \infty .$$

(b) *If* $\delta L > -\infty$ $(< \infty)$, *then* $\liminf_{n\to\infty} |y_n| = 0$.

(c) *If* $z_n \to 0$ *as* $n \to \infty$, *then* $\{\Delta^i z_n\}$ *is monotonic and*

$$\Delta^i z_n \to 0 \qquad \text{as} \quad n \to \infty \text{ and } \Delta^i z_n \Delta^{i+1} z_n < 0 \tag{5}$$

for $i = 0, 1, \ldots, m-1$.

(d) *Let* $z_n \to 0$ *as* $n \to \infty$. *If* m *is even [odd], then* $z_n < 0$ $(z_n > 0)$. *If* m *is odd [even], then* $z_n > 0$ $(z_n < 0)$.

(e) *If* (2) *holds, then either* $\{\Delta^i z_n\}$ *is decreasing* (*increasing*) *[increasing* (*decreasing*)*] with*

$$\Delta^i z_n \to -\infty \ (\infty) \ [\infty \ (-\infty)] \qquad \text{as} \quad n \to \infty \tag{6}$$

for $i = 0, 1, \ldots, m-1$, *or* $\{\Delta^i z_n\}$ *is monotonic and* (5) *holds.*

(f) *If* (2) *holds and* m *is even, then* $z_n < 0$ $(z_n > 0)$ $[> 0 \ (< 0)]$. *If* (5) *holds and* m *is odd, then* $z_n > 0$ $(z_n < 0)$ $[< 0 \ (> 0)]$.

(g) *If* (3) *[*(2)*] holds, then* (5) *holds [either* (5) *holds or* $|y_n| \to \infty$ *as* $n \to \infty$*].*

(h) *If* (4) *holds and* m *is odd [even], then* (6) *holds.*

Proof. Suppose that $\{y_n\}$ is an eventually positive solution of (E). Then there exists an integer $N_1 \ge N_0$ such that $y_{n-m+1-k} > 0$ and $y_{n-l} > 0$ for $n \ge N_1$. From (E), we have $\delta \Delta^m z_{n-m+1} = -F(n, y_{n-l}) \le 0$, so clearly part (a) holds. Summing (E) from N_1 to n and then letting $n \to \infty$, we have

$$\sum_{i=N_1}^{\infty} F(i, y_{i-l}) = \delta \Delta^{m-1} z_{N_1-m+1} - \delta L .$$

From condition (1), we see that (b) holds.

In order to prove (c), suppose that $\delta = +1$ and $z_n \to 0$ as $n \to \infty$. By (a), $\{\Delta^{m-1} z_n\}$ is eventually decreasing. If $\Delta^{m-1} z_n \to L < 0$ as $n \to \infty$, then there exists $L_1 < 0$ and an integer $N_2 \ge N_1$ such that $\Delta^{m-1} z_n \le L_1$ for $n \ge N_2$. This contradicts $z_n \to 0$ as $n \to \infty$. If $\Delta^{m-1} z_n \to L > 0$ as $n \to \infty$, then $\Delta^{m-1} z_n \ge L$ for $n \ge N_1$ again contradicting $z_n \to 0$ as $n \to \infty$. Therefore, $\Delta^{m-1} z_n \to 0$ as $n \to \infty$. Since $\{\Delta^{m-1} z_n\}$ is decreasing and $F(n, y_{n-l}) \not\equiv 0$, we have $\Delta^{m-1} z_n > 0$ for $n \ge N_1$. Hence, if $m \ge 2$, then $\{\Delta^{m-2} z_n\}$ is increasing, and so $\Delta^{m-2} z_n \to L_2 > -\infty$ as $n \to \infty$. If $L_2 < 0$, then $\Delta^{m-2} z_n \le L_2$ for

$n \geq N_1$, which contradicts $z_n \to 0$ as $n \to \infty$. Now assume $L_2 > 0$; then there is an $L_3 > 0$ and an integer $N_3 \geq N_1$ such that $\Delta^{m-2} z_n \geq L_2$ for $n \geq N_3$. Again this contradicts $z_n \to 0$ as $n \to \infty$. Thus, $\Delta^{m-2} z_n \to 0$ as $n \to \infty$, and since $\{\Delta^{m-2} z_n\}$ is increasing, we have $\Delta^{m-2} z_n < 0$ for $n \geq n_1$. Continuing in this fashion we see that (5) holds.

Part (d) follows immediately from (5) since $\delta \Delta^m z_{n-m+1} \leq 0$. To prove (e) for $\delta = +1$, first note that from (a) and (b), we have $\{\Delta^{m-1} z_n\}$ is decreasing, $\Delta^{m-1} z_n \to L \geq -\infty$ as $n \to \infty$, and $\liminf_{n\to\infty} y_n = 0$ if $L > -\infty$. If $L = -\infty$, then clearly (6) holds.

If $-\infty < L < 0$, then eventually $z_n \leq L_1$ for some $L_1 < 0$, and so $P_1 y_{n-k} \leq p_n y_{n-k} < z_n$ contradicting $\liminf_{n\to\infty} y_n = 0$. Hence, $L \geq 0$. If $L > 0$, then eventually $y_n \geq z_n \geq L_2 > 0$, which contradicts $\liminf_{n\to\infty} y_n = 0$. Thus, we have $\Delta^{m-1} z_n \to 0$ as $n \to \infty$. Moreover, $\Delta^{m-1} z_n > 0$ since $\{\Delta^{m-1} z_n\}$ is decreasing and $F(n, y_{n-l}) \not\equiv 0$. Hence, $\{\Delta^{m-2} z_n\}$ is increasing. In addition, $\Delta^{m-2} z_n < 0$ for otherwise $\{\Delta^{m-2} z_n\}$ is eventually positive and increasing, which in turn implies $\{z_n\}$ has a positive lower bound contradicting $\liminf_{n\to\infty} y_n = 0$. Now if $\Delta^{m-2} z_n \to L_3 < 0$ as $n \to \infty$, then it is easy to see that $z_n \leq L_4 < 0$ eventually. This again contradicts $\liminf_{n\to\infty} y_n = 0$. Thus, $\{\Delta^{m-2} z_n\}$ is increasing and tends to zero as $n \to \infty$. Continuing in this way we see that (6) holds.

The proof of (f) follows from the fact that either (5) or (6) implies $z_n < 0$ $[z_n > 0]$ if m is even, and (5) implies $z_n > 0$ $[z_n < 0]$ when m is odd.

To prove (g) when $\delta = +1$, suppose (5) does not hold. Then, by part (e), (6) holds, so $z_n < 0$ for $n \geq N_2$ for some $N_2 \geq N_1$. Since $p_n \geq -1$, we have

$$y_n \leq -p_n y_{n-k} \leq y_{n-k}.$$

This implies that $\{y_n\}$ is bounded contradicting (6). If $\delta = -1$, and (5) does not hold, then part (e) implies that (6) holds, and so $z_n \to \infty$ as $n \to \infty$. By (2), we have $z_n \leq y_n \to \infty$ as $n \to \infty$.

Finally, to prove (h), if (6) does not hold, then (5) holds. This implies that $\liminf_{n\to\infty} y_n = 0$. Part (f) implies $z_n > 0$ for $n \geq N_2$ for some $N_2 \geq N_1$. Hence, $y_n > -p_n y_{n-k} \geq y_{n-k}$, which contradicts $\liminf_{n\to\infty} y_n = 0$. □

Our first theorem places very mild restrictions on the sequence $\{p_n\}$, and as a consequence, the conclusions in the theorem are not very strong. However, it does give us the flavour of the results to be obtained in the subsequent theorems.

THEOREM 2. *Suppose that condition* (1) *holds, m is either even or odd, and $\{y_n\}$ is a nonoscillatory solution of* (E).

(i) *If $\delta = +1$ and there exists a constant P_3 such that*

$$P_3 \leq p_n,$$

then either $|y_n| \to \infty$ *as* $n \to \infty$ *or* $\liminf_{n\to\infty} |y_n| = 0$. *Moreover, if* $-1 \le P_3$, *then the second conclusion holds.*

(ii) *If* $\delta = -1$ *and there exists* P_4 *such that*

$$p_n \le P_4,$$

then either $\limsup_{n\to\infty} |y_n| = \infty$ *or* $\liminf_{n\to\infty} |y_n| = 0$. *In addition, if* $P_4 \le 0$, *then either* $|y_n| \to \infty$ *as* $n \to \infty$ *or* $\liminf_{n\to\infty} |y_n| = 0$.

Proof. Let $\{y_n\}$ be an eventually positive solution of (E), say, $y_{n-m+1-k} > 0$ and $y_{n-l} > 0$ for $n \ge N_1$ for some $N_1 \ge N_0$. Part (a) of Lemma 1 implies $\Delta^{m-1} z_n \to \delta L < \infty$ as $n \to \infty$, and part (b) of Lemma 1 implies $\liminf_{n\to\infty} y_n = 0$ if $\delta L > -\infty$. If $\delta L = -\infty$, then $\delta z_n \to -\infty$ as $n \to \infty$. If (i) holds, $z_n \to -\infty$ as $n \to \infty$, and so

$$P_3 y_{n-k} \le y_n + p_n y_{n-k} = z_n \to -\infty$$

as $n \to \infty$. Hence, $p_n < 0$ eventually and $y_n \to \infty$ as $n \to \infty$. If $P_3 \ge -1$, then either $\liminf_{n\to\infty} y_n = 0$ or $y_n + p_n y_{n-k} = z_n < 0$ for all large n. Thus, $y_n < -p_n y_{n-k} \le y_{n-k}$, which implies $\{y_n\}$ is bounded and this contradicts $L = -\infty$. If (ii) holds, $z_n \to \infty$ as $n \to \infty$, so we have $z_n \le y_n + P_4 y_{n-k} \to \infty$ as $n \to \infty$. This implies $\limsup_{n\to\infty} y_n = \infty$. If $P_4 \le 0$, then $z_n \le y_n \to \infty$ as $n \to \infty$. □

Remark. Theorem 2 generalizes Theorem 2.3 in [18].

For our next theorem, we ask that there exists a positive constant P_5 such that

$$0 \le p_n \le P_5 < 1. \tag{7}$$

THEOREM 3. *Suppose that conditions* (1) *and* (7) *hold.*

(i) *If* m *is even and* $\delta = +1$, *then all solutions of* (E) *are oscillatory, while if* $\delta = -1$, *any solution* $\{y_n\}$ *of* (E) *is either oscillatory,* $y_n \to 0$ *as* $n \to \infty$, *or* $|y_n| \to \infty$ *as* $n \to \infty$.

(ii) *If* m *is odd and* $\delta = +1$, *then either* $\{y_n\}$ *is oscillatory or* $y_n \to 0$ *as* $n \to \infty$, *while if* $\delta = -1$, *then either* $\{y_n\}$ *is oscillatory or* $|y_n| \to \infty$ *as* $n \to \infty$.

Proof. Let $\{y_n\}$ be an eventually positive solution of (E), say $y_{n-m+1-k} > 0$ and $y_{n-l} > 0$ for $n \ge N_1 \ge N_0$. By part (a) of Lemma 1, we have $\{\delta\Delta^{m-1} z_n\}$ is decreasing and $\{\delta\Delta^{m-1} z_n\}$ converges to $\delta L \ge -\infty$ as $n \to \infty$. If $\delta L = -\infty$, then z_n is eventually negative if $\delta = +1$, and $z_n \to \infty$ if $\delta = -1$. Moreover, since $p_n \ge 0$, the first possibility is excluded. If $z_n \to \infty$, then $\{z_n\}$ is increasing since $\{\delta\Delta^{m-1}z_n\}$ has fixed sign. Hence, we have $z_n = y_n + p_n y_{n-k} \le y_n + p_n z_{n-k} \le y_n + P_5 z_n$, so $z_n[1 - P_5] \le y_n \to \infty$ as $n \to \infty$.

If $\delta L > -\infty$, $\liminf_{n\to\infty} y_n = 0$. Since $\{z_n\}$ is monotonic, $z_n \to l$ as $n \to \infty$. Observe that $l \ge 0$ since $l < 0$ implies $y_n < 0$. Assume $l > 0$. If $\{z_n\}$ is increasing, we again obtain $z_n[1 - P_5] \le y_n$, which contradicts $\liminf_{n\to\infty} y_n = 0$. If $\{z_n\}$ is decreasing, let $1 - P_5 = \varepsilon > 0$. Then $z_n \le y_n + P_5 z_{n-k}$, and since l is finite,

$$\frac{z_n}{z_{n-k}} \le \frac{y_n}{z_{n-k}} + P_5 \le \frac{y_n}{l} + P_5 .$$

Since $P_5 + \frac{\varepsilon}{2} < 1$, there exists $N_2 > N_1$ such that $\frac{z_n}{z_{n-k}} \ge P_5 + \frac{\varepsilon}{2}$ for $n \ge N_2$. Hence, $y_n \ge \frac{l\varepsilon}{2}$ for $n \ge N_2$ contradicting $\liminf_{n\to\infty} y_n = 0$. Thus, $z_n \to 0$ as $n \to \infty$.

To complete the proof, just observe that part (d) of Lemma 1 implies that for m even $z_n < 0$ if $\delta = +1$, and $z_n > 0$ if $\delta = -1$. But $z_n < 0$ contradicts $y_n > 0$, and $z_n > 0$ implies $y_n \le z_n \to 0$ as $n \to \infty$. Hence (i) holds. If m is odd, Lemma 1(d) implies $z_n > 0$ if $\delta = +1$, and $z_n < 0$ if $\delta = -1$; part (ii) then follows. □

EXAMPLES. The equation

$$\Delta^m[y_{n-m+1} + p y_{n-m}] + \frac{(-1)^{m+1}(1+p\,\mathrm{e})(\mathrm{e}-1)^m\,\mathrm{e}^{(\gamma-1)n}}{\mathrm{e}^{\gamma+1}} y_{n-1}^{\gamma} = 0, \qquad n \ge 1, \tag{E$_1$}$$

where $0 \le p < 1$, and $\gamma \ge 1$ is the quotient of odd positive integers, satisfies the hypotheses of part (i) of Theorem 3 with $\delta = -1$ and part (ii) with $\delta = +1$. Here, $\{y_n\} = \{\mathrm{e}^{-n}\}$ is a nonoscillatory solution which converges to 0 as $n \to \infty$. Equation (E_1) also satisfies the hypotheses of part (i) of Theorem 2 provided $p > -1/\mathrm{e}$ and m is odd, or $p < -1/\mathrm{e}$ and m is even. The equation

$$\Delta^m[y_{n-m+1} + p y_{n-m}] - \Big(\frac{p}{\mathrm{e}} + 1\Big)(\mathrm{e}-1)^m\,\mathrm{e}^{(1-\gamma)n}\,\mathrm{e}^{\gamma+1-m}\,y_{n-1}^{\gamma} = 0, \qquad n \ge 1, \tag{E$_2$}$$

with m odd, $0 \le p < 1$, and $\gamma \le 1$ the ratio of odd positive integers, satisfies the hypotheses of Theorem 3(ii) for $\delta = -1$ and has the nonoscillatory solution $\{y_n\} = \{\mathrm{e}^n\}$ satisfying $\mathrm{e}^n \to \infty$ as $n \to \infty$. If $p < -\mathrm{e}$, then Theorem 2(i) holds, and if $-\mathrm{e} < p \le 0$, then Theorem 2(ii) holds. In each case, $\{y_n\} = \{\mathrm{e}^n\}$ is an unbounded nonoscillatory solution. As an example of an equation satisfying the hypotheses of Theorem 3 and having an oscillatory solution, consider

$$\Delta^m[y_{n-m+1} + p y_{n-m-1}] + \delta(1+p)2^m y_{n-\alpha} = 0, \qquad n \ge 1, \tag{E$_3$}$$

where $0 \le p < 1$. If $\delta = +1$ and α is even, or $\delta = -1$ and α is odd, then $\{y_n\} = \{(-1)^n\}$ is an oscillatory solution of (E_3). If $p \le 0$, then equation (E_3)

can also be used to construct examples of equations satisfying Theorem 2 and having oscillatory solutions.

Remark. Theorem 3(i) generalizes Theorem 5 in [24], and Theorem 8 in [27], and Theorem 3(ii) generalizes part of Corollary 1(b) in [10].

For our next result, we will need a stronger version of condition (3), namely, that there exists a constant $P_6 < 0$ such that

$$-1 < P_6 \leq p_n \leq 0. \tag{8}$$

THEOREM 4. *Suppose that conditions* (1) *and* (8) *hold, and m is either even or odd. If $\delta = +1$, then any solution $\{y_n\}$ of* (E) *is either oscillatory or satisfies $y_n \to 0$ as $n \to \infty$, while if $\delta = -1$, then either $\{y_n\}$ is oscillatory, $y_n \to 0$, or $|y_n| \to \infty$ as $n \to \infty$.*

Proof. Suppose that $\{y_n\}$ is a nonoscillatory solution of (E) such that $y_{n-m+1-k} > 0$ and $y_{n-l} > 0$ for $n \geq N_1 \geq N_0 \geq 0$. Lemma 1(g) implies that (5) holds if $\delta = +1$ and either (5) holds or $|y_n| \to \infty$ as $n \to \infty$ if $\delta = -1$. Suppose (5) holds. If either m is even and $\delta = +1$ or m is odd and $\delta = -1$, (8) and Lemma 1(d) imply that $z_n < 0$ eventually. It then follows that $y_n \leq -P_6 y_{n-k}$ for $n \geq N_2$ for some $N_2 \geq N_1$. Hence, $y_{n+k} \leq (-P_6)^2 y_{n-k}$, and by induction, we have that $y_{n+jk} \leq (-P_6)^{j+1} y_{n-k}$ for every positive integer j. Since $0 < -P_6 < 1$, this implies that $y_n \to 0$ as $n \to \infty$.

If m is even and $\delta = -1$ or m is odd and $\delta = +1$, then (8) and Lemma 1(d) imply $0 < z_n < A_1$ for some constant $A_1 > 0$ and sufficiently large n, and so $0 < y_n < -P_6 y_{n-k} + A_1$. If $\{y_n\}$ is unbounded, then there exists an increasing sequence $\{\alpha_i\}$ such that $y_{\alpha_i} \to \infty$ as $i \to \infty$, and $y_{\alpha_i} = \max\{y_n : N_1 \leq n \leq \alpha_i\}$. For each i, $y_{\alpha_i} < -P_6 y_{\alpha_i - k} + A_1 \leq -P_6 y_{\alpha_i} + A_1$, or $(P_6 + 1) y_{\alpha_i} \leq A_1$. In view of (8), this is impossible. Therefore, $\{y_n\}$ is bounded, and there exists a constant $A_2 > 0$ such that $\limsup_{n\to\infty} y_n = A_2$. Thus, there is an increasing sequence $\{\beta_j\}$ such that $y_{\beta_j} \to A_2$ as $j \to \infty$. From (8), we have

$$-P_6 y_{\beta_j - k} \geq y_{\beta_j} - z_{\beta_j}.$$

Since $A_2 > 0$, there exists $\varepsilon > 0$ such that $(1 - P_6)\varepsilon < (1 + P_6) A_2$, and so $0 < -P_6(A_2 + \varepsilon) < A_2 - \varepsilon$. But for all sufficiently large j, $y_{\beta_j - k} < A_2 + \varepsilon$, so we have

$$A_2 - \varepsilon > -P_6 y_{\beta_j - k} \geq y_{\beta_j} - z_{\beta_j}$$

for all such j. As $j \to \infty$, this contradicts $y_{\beta_j} \to A_2$ as $j \to \infty$ since $z_{n_j} \to 0$ as $j \to \infty$. □

Remark. Notice that if $\delta = +1$, Theorem 4 implies that unbounded solutions must be oscillatory. Theorem 4 generalizes Corollary 2.1(v) in [18], Theorem 3.4 in [20], Theorem 4 in [24], Theorems 2 and 4 in [27], and a part of Corollary 1(b) in [10].

EXAMPLE. Equation (E_1) provides examples of all the different cases in Theorem 4 depending on whether $-1/\mathrm{e} < p \le 0$ or $-1 < p < -1/\mathrm{e}$. For $-1 < p \le 0$, equation (E_2) satisfies the hypotheses of Theorem 4 with $\delta = -1$ and has an unbounded nonoscillatory solution. Similarly, for $-1 < p \le 0$, if $\delta = +1$ and α is even, or $\delta = -1$ and α is odd, (E_3) yields equations satisfying Theorem 4 and having oscillatory solutions.

THEOREM 5. *Suppose that* (1) *and* (4) *hold. If*

(i) *m is even and $\delta = -1$,*

or

(ii) *m is odd and $\delta = +1$,*

then any solution $\{y_n\}$ of (E) *is either oscillatory or $|y_n| \to \infty$ as $n \to \infty$.*

Proof. Let $\{y_n\}$ be a nonoscillatory solution of (E) such that $y_{n-m+1-k} > 0$ and $y_{n-l} > 0$ for $n \ge N_1 \ge N_0$. Part (h) of Lemma 1 implies (6) holds, so $\delta z_n \to -\infty$ as $n \to \infty$. Now for sufficiently large n, (4) implies that $P_2 y_{n-k} \le z_n \le y_n$, and hence $y_n \to \infty$ as $n \to \infty$. □

EXAMPLE. If m is even and $-\mathrm{e} \le p \le -1$ or m is odd and $p \le -\mathrm{e}$, then equation (E_2) satisfies the hypotheses of Theorem 5 and has the unbounded nonoscillatory solution $\{y_n\} = \{\mathrm{e}^n\}$ with $\mathrm{e}^n \to \infty$ as $n \to \infty$.

Remark. Theorem 5 generalizes Corollary 1(a) in [10], Theorem 4.3 in [20], and Theorems 2 and 7 in [24].

Next, we obtain a result on the behavior of the bounded solutions of (E) for the case when p_n is bounded above away from -1. Assume that there exists a constant P_7 such that

$$P_2 \le p_n \le P_7 < -1 . \tag{9}$$

THEOREM 6. *Suppose conditions* (1) *and* (9) *hold. If m is even and $\delta = +1$, or if m is odd and $\delta = -1$, then any bounded solution $\{y_n\}$ of* (E) *is either oscillatory or satisfies $y_n \to 0$ as $n \to \infty$.*

Proof. Assume that $\{y_n\}$ is a bounded nonoscillatory solution of (E) with $y_{n-m+1-k} > 0$ and $y_{n-l} > 0$ for $n \ge N_1 \ge N_0$. Lemma 1(e) implies that either (5) or (6) holds. If (6) holds, then the argument used in the proof of Theorem 4 shows that $y_n \to \infty$ as $n \to \infty$ contradicting $\{y_n\}$ being bounded. Therefore (5) holds. Now Lemma 1(c) implies that if m even and $\delta = +1$ or m is odd and $\delta = -1$, then $\delta z_n < 0$ and $\{\delta z_n\}$ is increasing to zero as $n \to \infty$. Since $\{y_n\}$ is bounded, $\limsup_{n\to\infty} y_n = l$ is nonnegative and finite. If $l > 0$, then there exists an increasing sequence $\{n_j\}$ such that $n_1 > N_1$, and $n_j \to \infty$ and $y_{n_j-k} \to l$ as $j \to \infty$. Let $c = P_7 + 1 < 0$, $\varepsilon = -cl/8 > 0$, $d = cl/8P_7 > 0$

and $\lambda = -3cl/4 > 0$. Then, there exists $N_2 \geq N_1$ such that $\delta z_{n_j} > -\varepsilon$ and $y_{n_j-k} > l - d > 0$ for $j \geq N_2$. Hence, for $j \geq N_2$ we have

$$-\varepsilon < \delta z_{n_j} < y_{n_j} + P_7(l-d).$$

It follows that

$$-y_{n_j} < P_7 l - P_7 d + \varepsilon = (c-1)l - cl/4 = -\lambda - l,$$

so $l + \lambda < y_{n_j}$ for $j \geq N_2$. This contradicts $\limsup_{n\to\infty} y_n = l > 0$. Thus, $\limsup_{n\to\infty} y_n = 0$, and so $y_n \to 0$ as $n \to \infty$. □

Remark. Theorem 6 generalizes Theorem 2.3 in [18] and Theorem 9 in [27].

EXAMPLE. If $p < -1$ and m is either even or odd, then equation (E_1) satisfies the hypotheses of Theorem 6 and has the solution $\{y_n\} = \{\mathrm{e}^{-n}\}$. Also, if $-\mathrm{e} < p < -1$ and m is odd, then equation (E_2) shows that under the hypotheses of Theorem 6, it is possible for equation (E) to have unbounded solutions.

Remark. Equation (E_3) provides examples of equations satisfying the hypotheses of Theorems 5 and 6 and having oscillatory solutions. That is, under the conditions given here, it is not possible to obtain results on the limiting behavior of all solutions of (E).

Our next two results require a stronger condition on the function F, namely, that there exists a constant $B > 0$ such that

$$|F(n,u)| \geq B|u| \qquad \text{for all } n \geq N_0 \text{ and all } u. \tag{10}$$

In addition, we ask that there exists $P_8 > 0$ such that

$$0 \leq p_n \leq P_8. \tag{11}$$

THEOREM 7. *Let conditions* (10) *and* (11) *hold, m be even, and $\{y_n\}$ be a solution of* (E).

(i) *If $\delta = +1$, then $\{y_n\}$ is oscillatory,*

while

(ii) *if $\delta = -1$ and $\{y_n\}$ is bounded, then either $\{y_n\}$ is oscillatory or $y_n \to 0$ as $n \to \infty$.*

Proof. Suppose $\{y_n\}$ is a solution of (E) such that $y_{n-m+1-k} > 0$ and $y_{n-l} > 0$ for $n \geq N_1 \geq N_0$. By part (a) of Lemma 1, $\{\delta\Delta^{m-1}z_n\}$ is decreasing and satisfies $\Delta^{m-1}z_n \to \delta L \geq -\infty$ as $n \to \infty$. If $\delta L < 0$, then $\{z_n\}$ is eventually negative, which contradicts (11). Hence, $\delta L \geq 0$, and by (10), we have

$$|\Delta^{m-1} z_{N_1-m+1}| \geq \delta \Delta^{m-1} z_{N_1-m+1} = \delta L + \sum_{i=N_1}^{\infty} F(i, y_{i-l})$$

$$\geq \delta L + B \sum_{i=N_1}^{\infty} y_{i-l} .$$

If $\delta = +1$, $\Delta^{m-1} z_n$ is bounded above; if $\delta = -1$, the boundedness assumption on y_n implies that $|\Delta^{m-1} z_n|$ is bounded. In either case, the series on the right hand side of the above inequality converges, and so $y_n \to 0$ as $n \to \infty$. This in turn implies $z_n \to 0$ as $n \to \infty$. By Lemma 1(d), $z_n < 0$ if m is even and $\delta = +1$, so we get a contradiction in this case. □

EXAMPLE. If m is even and $p \geq 0$, then α in equation (E_3) can be chosen so that the hypotheses of Theorem 7 are satisfied and (E_3) has the oscillatory solution $\{y_n\} = \{(-1)^n\}$. In addition, for m even, $\gamma = 1$, and $p \geq 0$, equation (E_1) satisfies Theorem 7(ii) and has the bounded nonoscillatory solution $\{y_n\} = \{e^{-n}\}$ which converges to zero. Equation (E_2) with m even, $\gamma = 1$, and $p \geq 0$ satisfies part (ii) of Theorem 7 and has an unbounded nonoscillatory solution.

THEOREM 8. *Let conditions* (10) *and* (11) *hold, m be odd, and $\{y_n\}$ be a solution of* (E). *If $\delta = +1$, then either $\{y_n\}$ is oscillatory or $y_n \to 0$ as $n \to \infty$, while if $\delta = -1$ and $\{y_n\}$ is bounded, then $\{y_n\}$ is oscillatory.*

Proof. As in the proof of Theorem 7, for any nonoscillatory solution $\{y_n\}$ we have $y_n \to 0$ and $z_n \to 0$ as $n \to \infty$. But since m is odd, if $\delta = -1$, Lemma 1(d) contradicts $z_n > 0$. □

EXAMPLE. If m is odd, $\gamma = 1$, and $p \geq 0$, equation (E_1) satisfies the hypotheses of Theorem 8. Here, $\{y_n\} = \{e^{-n}\}$ is a solution. With m odd, $\delta = -1$, α odd, and $p \geq 0$, equation (E_3) satisfies Theorem 8 and has the bounded oscillatory solution $\{y_n\} = \{(-1)^n\}$. This also shows that the hypotheses of Theorem 8 are not sufficient to ensure that oscillatory solutions of (E) tends to zero as $n \to \infty$.

Remark. Theorem 8 generalizes Theorem 2 in [26] and part of Corollary 1(b) in [10].

EXAMPLE. As a final example, consider the equation

$$\Delta^m[y_{n-m+1} + p y_{n-m}] + (-1)^\beta (e+1)^m (1-p/e)\, e^{\beta+1-m} y_{n-\beta} = 0, \qquad n \geq 1+\beta. \tag{E_4}$$

For any value of the nonnegative integer β, equation (E_4) has the unbounded oscillatory solution $\{y_n\} = \{(-1)^n e^n\}$. Hence, by appropriately choosing the parity of β, it is possible to obtain examples of equation (E) which have unbounded oscillatory solutions for any values of m, δ, and p.

3. Concluding remarks

We conclude this paper with a few suggestions for further research. First, by examining Theorems 4–6, we see that $p_n \equiv -1$ behaves as a bifurcation point for the behavior of nonoscillatory solutions of (E). Moreover, if $p_n \equiv -1$ and either

(a) $\delta = +1$ and m is even,

or

(b) $\delta = -1$ and m is odd,

the behavior of nonoscillatory solutions, if any, is not fully understood. If (a) holds, then Theorem 2(i) tells us that $\liminf\limits_{n\to\infty} |y_n| = 0$, and if (b) holds, Theorem 2(ii) says that either $|y_n| \to \infty$ as $n \to \infty$ or $\liminf\limits_{n\to\infty} |y_n| = 0$. In fact, when (9) ($P_2 \le p_n \le P_7 < -1$) and either (a) or (b) holds, we are unable to rule out the possibility of equation (E) having a solution $\{y_n\}$ with $\limsup\limits_{n\to\infty} |y_n| = \infty$ and $\liminf\limits_{n\to\infty} |y_n| = 0$ (see Theorem 6). Further study of this situation is needed.

Secondly, when $p_n \ge 1$, the results here require the additional hypothesis (10). Without this added condition some, albeit minimal, information about the behavior of solutions is obtainable from Theorem 2. It would be interesting to see the conclusions of Theorems 7–8 reached without this added assumption.

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