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A MATRIX WITH AN APPLICATION TO THE MOTION OF AN ABSORBING MARKOV CHAIN I

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ABSTRACT. The determinant $\Delta_m(u, z|_j) = |\mathbf{I}-\mathbf{SQ}'|$, where $\mathbf{Q}'$ is the transpose of a tridiagonal matrix $\mathbf{Q}$ of order $m\times m$ with r on the main diagonal and p and q on the first upper and lower diagonals respectively, $p+r+q=1$, $\mathbf{I}$ is an $m\times m$ identity matrix, and $\mathbf{S}$ stands for the $m\times m$ diagonal matrix whose jth diagonal element is z and whose other diagonal elements are all equal to u, is evaluated. The result is applied to an absorbing Markov chain to find the P.G.F. of $\nu^k(j \mid i)$, the total number of visits to state j, starting at i, before k is reached. Explicit expressions for the P.D., the mean, and the variance of $\nu^k(j \mid i)$ are derived. The limiting forms of these results are also given.

1. Introduction

Consider a stochastic process which makes transitions from one to another of a finite number of available states $\{0, 1, \dots, N\}$ in accordance with an absorbing Markov chain, whose transition probability matrix is given by $\mathbf{P} = \{p(i,j)\}_{i,j=0}^{N}$. Whenever the chain enters the state i, the next state j to which it will move is selected with probability $p(i,j)$ such that

$$\left.\begin{aligned} p(i,i+1) &= p \\ p(i,i) &= r \\ p(i,i-1) &= q \end{aligned}\right\} \quad 0 < i < N, \quad p+r+q = 1 .$$

We assume that the states 0 and N are both absorbing, while each of the states in $T_{N-1} = \{1, 2, \dots, N-1\}$ is transient. We further assume that $\boldsymbol{p}^{(0)} = \left(p_1^{(0)}, p_2^{(0)}, \dots, p_{N-1}^{(0)}\right)$ be the vector of initial state occupation probabilities. Various properties of the motion of an absorbing Markov chain have been considered

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in numerous textbooks, among them we mention Parzen (1962) [14], Cox and Miller (1965) [3], Feller (1967) [7], Kemeny and Snell (1976) [11], Srinivasan and Mehata (1976) [15], and Iosifescu (1980) [9], and references cited there. Theoretical formula for the universal probability generating function of the frequency count of a Markov chain has been derived by Good (1961) [8] (see also Bhat (1961) [2] and Neuts (1964) [13]). In this paper, we evaluate the determinant $\Delta_{N-1}(u, z|_j) = |\mathbf{I} - \mathbf{SQ}'|$, where $\mathbf{Q}'$ is the transpose of a tridiagonal matrix $\mathbf{Q}$ obtained by omitting the first and last row and column of $\mathbf{P}$, and $\mathbf{S}$ stands for the $(N-1)\times(N-1)$ diagonal matrix $\operatorname{diag}(u, \dots, u, z, u, \dots, u)$ with z being the jth component. It is not readily available in the literature on either matrix theory or probability theory. Using the result, explicit expression for the probability generating function (P.G.F.) of $\nu^k(j \mid i)$, the total number of visits to a state j, starting at i, before k, $k \in T^*_{N-1} = \{0, N\}$ is reached, is obtained. The probability distribution (P.D.), the mean, and the variance of $\nu^k(j \mid i)$ and the limiting forms of the results are also given. By an alternative method similar to the extrapolation method of Kemperman (1961) [12], Barnett (1964) [1] has derived similar formulae for simple random walk in the special case $r = 0$.

2. Derivation of an explicit expression for $\Delta_m(u, z|_j)$

Let us denote by $\Delta_m(u, z|_j)$ the determinant $|\mathbf{I} - \mathbf{SQ}'|$, where $\mathbf{Q}'$ is the transpose of a tridiagonal matrix $\mathbf{Q}$ of order $m \times m$ with r on the main diagonal and p, q on the first upper and lower diagonals respectively, $p + r + q = 1$, and $\mathbf{S}$ be the $m \times m$ diagonal matrix $\operatorname{diag}(u, \dots, u, z, u, \dots, u)$ whose jth diagonal element is z and all other diagonal elements are equal to u. Then, $\Delta_m(u, z|_j)$ must satisfy the difference equation:

$$\Delta_m(u, z|_j) = (1-ur)\Delta_{m-1}(u, z|_j) - pqu^2\Delta_{m-2}(u, z|_j) \qquad \text{for } j \in T_m,\ m > 2, \tag{1}$$

and for $m = 1$ and $m = 2$ we have

$$\Delta_1(u, z|_j) = \begin{cases} 1 - zr & \text{if } j = 1, \\ 1 - ur & \text{if } j \neq 1 \end{cases} \tag{1a}$$

and

$$\Delta_2(u, z|_j) = \begin{cases} (1 - ur)(1 - zr) - pquz & \text{if } j = 1, 2, \\ (1 - ur)^2 - pqu^2 & \text{if } j \neq 1, 2. \end{cases} \tag{1b}$$

The difference equation (1) can be rewritten in the equivalent form

$$\begin{pmatrix} \Delta_m(u, z|_j) \\ \Delta_{m-1}(u, z|_j) \end{pmatrix} = \mathbb{E} \begin{pmatrix} \Delta_{m-1}(u, z|_j) \\ \Delta_{m-2}(u, z|_j) \end{pmatrix} \qquad \text{for } j \in T_m,\ m > 2,$$

where

$$\mathbb{E} = \begin{pmatrix} 1-ur & -pqu^2 \\ 1 & 0 \end{pmatrix}.$$

It follows after $j-2$ iterations that

$$\begin{pmatrix} \Delta_m(u,z|_j) \\ \Delta_{m-1}(u,z|_j) \end{pmatrix} = \mathbb{E}^{j-2} \begin{pmatrix} \Delta_{m-j+2}(u,z|_j) \\ \Delta_{m-j+1}(u,z|_j) \end{pmatrix} \qquad \text{for} \quad j \in T_m, \quad m > 2. \qquad (2)$$

Hence,

$$\begin{pmatrix} \Delta_m(u,z|_j) \\ \Delta_{m-1}(u,z|_j) \end{pmatrix} = \mathbb{E}^{j-2} \begin{pmatrix} 1-ur & -pquz \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Delta_{m-j+1}(u,z|_j) \\ \Delta_{m-j}(u,z|_j) \end{pmatrix}$$
$$\text{for} \quad j \in T_m - \{1,m\}, \quad m > 2, \qquad (3)$$

where $\Delta_{m-j}(u,z|_j)$ is $(m-j)\times(m-j)$ tridiagonal determinant with $1-ur$ on the main diagonal and $-qu$ and $-pu$ on the first upper and lower diagonals respectively. The determinant $\Delta_{m-j+1}(u,z|_j)$ is the same as $\Delta_{m-j}(u,z|_j)$ except that the first row is replaced by $1\times(m-j+1)$ row vector $(1-zr, -qz, 0, \dots, 0)$. It follows from (3) immediately that

$$\begin{pmatrix} \Delta_m(u,z|_j) \\ \Delta_{m-1}(u,z|_j) \end{pmatrix} =$$
$$= \mathbb{E}^{j-2} \begin{pmatrix} 1-ur & -pquz \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1-zr & -pquz \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Delta_{m-j}(u,z|_j) \\ \Delta_{m-j-1}(u,z|_j) \end{pmatrix} \qquad (4)$$
$$= \mathbb{E}^{j-2} \begin{pmatrix} 1-ur & -pquz \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1-zr & -pquz \\ 1 & 0 \end{pmatrix} \mathbb{E}^{m-j-2} \begin{pmatrix} \Delta_2(u,z|_j) \\ \Delta_1(u,z|_j) \end{pmatrix}$$
$$\text{for} \quad j \in T_m - \{1,m\}, \quad m > 2.$$

Let λ_i be the ith eigenvalue of $\mathbb{E}$, and $\boldsymbol{v}_i$ the corresponding post-eigenvector. Then

$$\mathbb{E}\boldsymbol{v}_i = \boldsymbol{v}_i \lambda_i, \qquad i \in T_2,$$

that is $\mathbb{E}\mathbf{V} = \mathbf{V}\boldsymbol{\Lambda}$, where $\mathbf{V} = (\boldsymbol{v}_1, \boldsymbol{v}_2)$, $\boldsymbol{\Lambda} = (\delta_{i,j}\lambda_i)_{i,j\in T_2}$. While the columns of $\mathbf{V}$ are the post-eigenvectors, the rows of $\mathbf{V}^{-1}$ are the pre-eigenvectors, and we have $\mathbb{E} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1}$ or, more generally, $\mathbb{E}^k = \mathbf{V}\boldsymbol{\Lambda}^k\mathbf{V}^{-1}$, $k = 0,1,2,\dots$, where $\boldsymbol{\Lambda}^k = (\delta_{i,j}\lambda_i^k)_{i,j\in T_2}$, and

$$\mathbf{V} = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}, \qquad \mathbf{V}^{-1} = (\lambda_1 - \lambda_2)^{-1} \begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix}$$

with

$$\lambda_1 = \frac{1}{2}\left[(1-ur) + \sqrt{(1-ur)^2 - 4pqu^2}\right],$$
$$\lambda_2 = \frac{1}{2}\left[(1-ur) - \sqrt{(1-ur)^2 - 4pqu^2}\right].$$

The eigenvalues λ_1 and λ_2 are distinct, except for $u = 1 - \left(\sqrt{p} \pm \sqrt{q}\right)^2$, but $p > 0$, $q > 0$ implies that neither of these points belongs to the interval $0 < u < 1$. Then we see that

$$\mathbb{E}^k = (\lambda_1 - \lambda_2)^{-1} \begin{pmatrix} \lambda_1^{k+1} - \lambda_2^{k+1} & -\lambda_2\lambda_1^{k+1} + \lambda_1\lambda_2^{k+1} \\ \lambda_1^k - \lambda_2^k & -\lambda_2\lambda_1^k + \lambda_1\lambda_2^k \end{pmatrix}. \tag{5}$$

Substituting in formula (4) with appropriate forms from (5), by some simple calculations, we deduce that

$$\Delta_m(u, z|_j) = (\lambda_1 - \lambda_2)^{-1}\{(1-zr)A_{j,m-j+1} - pquz[A_{j,m-j} + A_{j-1,m-j+1}]\}, \tag{6}$$

where

$$A_{x,y} = (\lambda_1^x - \lambda_2^x)(\lambda_1^y - \lambda_2^y).$$

The case $j = m$ follows immediately from formula (2). If we replace j by $m-j+1$ in the result, we obtain the case $j = 1$. Hence, formula (6) is working for any j, $j \in T_m$. We see that, with the appropriate change of notation, expression (6) agrees with Kac (1945) [10] and Weesakul (1961) [16] in the case of r zero and $\mathbf{S}$ is the $m \times m$ diagonal matrix $\operatorname{diag}(u, \cdot, u, \cdot, u)$.

3. The probability distribution of $\nu^k(j \mid i)$

We introduce the following counting process on the state space $\mathbb{N}^{N-1} \times T_{N-1}$, where $\mathbb{N}^{N-1}$ is the set of all $(N-1)$-tuples of non-negative integers, associated with the absorbing Markov chain. We say that the associated process is in state $[(n), j]$, where (n) is the vector $(n_1, n_2, \ldots, n_{N-1}) \in \mathbb{N}^{N-1}$ and $j \in T_{N-1}$ if and only if, after $n_1 + n_2 + \cdots + n_{N-1}$ transitions, the process is in state j (before k, $k \in T^*_{N-1}$ is reached) and has made $n_1, n_2, \ldots, n_{N-1}$ visits to the states $1, 2, \ldots, N-1$ respectively. We let $P[(n), j]$ denote the probability that the state $[(n), j]$ is reached. We introduce the joint probability generating function (joint P.G.F.)

$$\mathbb{H}(z) = \mathbb{H}(z_1, z_2, \ldots, z_{N-1}) \sum_{(n)'\boldsymbol{e} \geq 0} P[(n), j](z)^{(n)}, \tag{7}$$

where z_i, $i \in T_{N-1}$, are $N-1$ dummy complex variables chosen so as to make the $N-1$ series convergent, and $\boldsymbol{e}$ is the column vector of 1's of order $N-1$.

Following Good (1961) [8] (see also Neuts (1964) [13] and Iosifescu (1980) [9]), we deduce that

$$\mathbb{H}(z) = \left(p_1^{(0)} z_1, \ldots, p_{N-1}^{(0)} z_{N-1}\right)(\mathbf{I} - \mathbf{QS})^{-1}(f), \tag{8}$$

where $\mathbf{Q}$ is the truncated form of the transition matrix $\mathbf{P}$ obtained by omitting the first and last row and column, $\mathbf{S} = (\delta_{i,j} z_i)_{i,j \in T_{N-1}}$ and $(f) = (\mathbf{I} - \mathbf{Q})\boldsymbol{e}$.

If the chain starts in any given state i, then $\mathbf{P}^{(0)}$ has zero components in all, but unit probability mass in ith position. We see that formula (8) becomes

$$\mathbb{H}_i(z) = (0,\dots,0,z_i,0,\dots,0)(\mathbf{I}-\mathbf{QS})^{-1}(f)\,. \tag{9}$$

For varying the starting point i, we obtain a system of $N-1$ equations

$$\boldsymbol{G}(z) = \mathbf{S}(\mathbf{I}-\mathbf{QS})^{-1}(f)\,, \tag{10}$$

where $\boldsymbol{G}(z)$ is the transpose of the $1\times N-1$ row vector

$$\big(\mathbb{H}_1(z),\mathbb{H}_2(z),\dots,\mathbb{H}_{N-1}(z)\big)\,.$$

Many interesting generating functions can be derived from formula (10) through an appropriate choice of the matrix $\mathbf{S}$; the matrix $(\mathbf{I}-\mathbf{QS})$ is non-singular (see Neuts (1964) [13]).

Explicit expression for the P.G.F. of $\nu^k(j\mid i)$, the total number of visits to state j, starting at i, before k, $k\in T^*_{N-1}$ is reached, may be obtained from (10) by setting $\mathbf{S}$ equal to $\operatorname{diag}(1,\dots,1,z,1,\dots,1)$ with z being the jth component and using formula (6) with $m=N-1$ and $u=1$. It is easy verified that

$$H_i(z) = \begin{cases} z\big[qC_{j,1}+pC_{j,N-1}\big] & \text{if } i=j\,,\ j\in T_{N-1}\,,\\ qC_{i,1}+pC_{i,N-1} & \text{if } i\neq j\,,\ i,j\in T_{N-1}\,,\end{cases} \tag{11}$$

where $C_{x,y}$ denotes the (x,y)th element of the inverse matrix $(\mathbf{I}-\mathbf{QS})^{-1}$, and

$$C_{i,1} = D\begin{cases} q^{j-1}d_{1,N-j} & \text{if } i=j\,,\\ q^{i-1}\big[d_{j-i,N-j}-z(d_{1,N-i}-d_{j-i,N-i})\big] & \text{if } i<j\,,\\ zq^{i-1}d_{1,N-i} & \text{if } i>j\,,\end{cases}$$

$$C_{i,N-1} = D\begin{cases} p^{N-j-1}d_{1,j} & \text{if } i=j\,,\\ p^{N-i-1}zd_{1,i} & \text{if } i<j\,,\\ p^{N-i-1}\big[d_{i-j,j}-z(d_{1,i}-d_{i-j,j})\big] & \text{if } i>j\,,\end{cases}$$

$$D = \big[d_{j,N-j}-z(d_{j,N-j}-d_{1,N})\big]^{-1} \qquad \text{and}$$
$$d_{x,y} = A_{x,y}\big|_{u=1} = (p^x-q^x)(p^y-q^y)\,.$$

Thus, from (11), we have

$$H_i(z) = D\begin{cases} q^i d_{j-i,N-j} - z[q^i d_{j-i,N-j}-d_{1,N}] & \text{if } i\le j\,,\\ p^{N-i}d_{i-j,j} - z[p^{N-i}d_{i-j,j}-d_{1,N}] & \text{if } i\ge j\end{cases} \tag{12}$$

for $p\neq q$, $p+q+r=1$, and

$$H_i(z) = B\begin{cases} (j-i)(N-j) - z\big[(j-i)(N-j) - pN\big] & \text{if } i \le j\,, \\ j(i-j) - z\big[j(i-j) - pN\big] & \text{if } i \ge j \end{cases} \tag{13}$$

for $p = q$, $2p + r = 1$, where $B = j(N-j) - z\big[j(N-j) - pN\big]$.

We see that with the appropriate change of notation the expressions (12) and (13) agree with that of Barnett (1964) [1] in the case r zero.

The probability distribution (P.D.) of $\nu^k(j \mid i)$, $k \in T^*_{N-1}$, is nothing, but the coefficient of z^{n_j} in $H_i(z)$, where j occurs exactly n_j times. Hence, we obtain from formulae (12) and (13) that

$$\operatorname{pr}\big(\nu^k(j \mid i) = n_j\big) = \\ = d^{-1}_{j,N-j}\begin{cases} q^i d_{j-i,N-j} & \text{if } n_j = 0\,,\ j \ge i\,, \\ p^{N-i} d_{i-j,j} & \text{if } n_j = 0\,,\ j \le i\,, \\ C[1 - q^i d_{j-i,N-j} d^{-1}_{j,N-j}] & \text{if } n_j = 1,2,\dots\,,\ j \ge i\,, \\ C[1 - p^{N-i-1} d_{i-j,j} d^{-1}_{j,N-j}] & \text{if } n_j = 1,2,\dots\,,\ j \le i \end{cases} \tag{14}$$

for $p \ne q$, $p + q + r = 1$, where

$$C = d_{1,N}(1 - d^{-1}_{j,N-j} d_{1,N}) \quad \text{and} \quad d_{x,y} = (p^x - q^x)(p^y - q^y)\,;$$

and

$$\operatorname{pr}\big(\nu^k(j \mid i) = n_j\big) = \begin{cases} 1 - \dfrac{i}{j} & \text{if } n_j = 0\,,\ j \ge i\,, \\ \dfrac{i-j}{N-j} & \text{if } n_j = 0\,,\ j \le i\,, \\ \dfrac{i\omega}{j} & \text{if } n_j = 1,2,\dots\,,\ j \ge i\,, \\ \dfrac{(N-i)\omega}{N-j} & \text{if } n_j = 1,2,\dots\,,\ j \le i \end{cases} \tag{15}$$

for $p = q$, $2p + r = 1$, where $\omega = \dfrac{pN}{j(N-j)}\left(1 - \dfrac{pN}{j(N-j)}\right)^{n_j}$.

The same value, (15), can also be obtained by using L'Hospital's rule with limit as $p \to q$ in (14).

It may be observed from formulae (14) and (15) that the probability distribution $\operatorname{pr}\big(\nu^k(j \mid i) = n_j\big)$ is geometric with modified first term, it will be geometric at the starting point i, $i \in T_{N-1}$, since the first term vanishes in this case.

4. The expected value and the variance of $\nu^k(j \mid i)$

The expected values of $\nu^k(j \mid i)$, $k \in T^*_{N-1}$, may be obtained by differentiating formulae (12) and (13) with respect to z, and evaluating the result at $z = 1$, and are found to be

$$E\big[\nu^k(j \mid i)\big] = \frac{1}{(p-q)(1-a^N)} \begin{cases} a^{(i-j)}\big(1-a^j\big)\big(1-a^{N-i}\big) & \text{if } j \le i\,, \\ \big(1-a^i\big)\big(1-a^{N-j}\big) & \text{if } j \ge i \end{cases} \tag{16}$$

for $p \neq q$, $a = q/p$ and $p + q + r = 1$; and

$$E\big[\nu^k(j \mid i)\big] = \frac{1}{p^N} \begin{cases} j(N-i) & \text{if } j \le i\,, \\ i(N-j) & \text{if } j \ge i \end{cases} \tag{17}$$

when $p = q$, $2p + r = 1$.

Formulae (16) and (17) in the case $r = 0$ agree with those given, for example, by Parzen (1962) [14], Barnett (1964) [1], and Iosifescu (1980) [9].

The second moment may be obtained from

$$E\big[\nu^k(j \mid i)^2\big] = \frac{\mathrm{d}^2}{\mathrm{d}z^2} H_i(z)\big|_{z=1} + E\big[\nu^k(j \mid i)\big]$$

and is found to be

$$E\big[\nu^k(j \mid i)^2\big] = \frac{\big[2(1-a^j)(1-a^{N-j}) - p(1-a^N)(1-a)\big]}{(p-q)^2(1-a^N)^2} \cdot \begin{cases} (1-a^i)(1-a^{N-j}) & \text{if } j \ge i\,, \\ a^{i-j}(1-a^j)(1-a^{N-i}) & \text{if } j \le i \end{cases} \tag{18}$$

for $p \neq q$, $a = q/p$ and $p + q + r = 1$; and

$$E\big[\nu^k(j \mid i)^2\big] = \frac{\big[2j(N-j) - pN\big]}{(pN)^2} \begin{cases} j(N-i) & \text{if } j \le i\,, \\ i(N-j) & \text{if } j \ge i \end{cases} \tag{19}$$

for $p = q$, $2p + r = 1$.

The value of $\mathrm{Var}\big[\nu^k(j \mid i)\big]$ follows immediately from formulae (16)–(19), yielding

$$\mathrm{Var}\big[\nu^k(j \mid i)\big] = \\ = \frac{1}{(p-q)^2(1-a^N)^2} \begin{cases} (1-a^i)(1-a^{N-j}) \\ \cdot\big[(1-a^i)(1-a^{N-j}) - (1-a^N)(p-q)\big] & \text{if } j \ge i\,, \\ a^{i-j}(1-a^j)(1-a^{N-i}) \\ \cdot\big[(1-a^j)(1-a^{N-i})a^{i-j} - (p-q)(1-a^N)\big) & \text{if } j \le i \end{cases} \tag{20}$$

for $p \neq q$, $a = q/p$ and $p+q+r=1$; and

$$\operatorname{Var}\left[\nu^k(j \mid i)\right] = \frac{1}{(pN)^2}\begin{cases} i(N-j)\left[i(N-j)-pN\right] & \text{if } j \geq i, \\ j(N-i)\left[j(N-i)-pN\right] & \text{if } j \leq i \end{cases} \tag{21}$$

for $p = q$, $2p + r = 1$.

5. The probability distribution of $\nu^0(j \mid i)$ and its first two moments

The analogous results for $\nu^0(j \mid i)$, the total number of visits to state j, starting at i, before zero is reached (0 is the single absorbing barrier), may be immediately obtained as the limiting form of those given in the previous sections. In particular, letting $N \to \infty$ in (12)–(21), we get the P.G.F. of $\nu^0(j \mid i)$ is

$$H_i(z) = \frac{1}{(1-a^j) - z\left[(1-a^j) - p + q\right]} \cdot \begin{cases} a^i(1-a^{j-i}) - z\left[a^i(1-a^{j-i}) - p + q\right] & \text{if } j \geq i, \\ (1-a^j)(1-a^{i-j}) - z\left[(1-a^j)(1-a^{i-j}) - p + q\right] & \text{if } j \leq i \end{cases} \tag{22}$$

for $p > q$, $p+q+r=1$, $a = q/p$;

$$H_i(z) = \frac{1}{(1-b^j) - z\left[(1-b^j) - p + q\right]} \cdot \begin{cases} (1-b^{j-i}) - z\left[(1-b^{j-i} - p + q\right] & \text{if } j \geq i, \\ z(q-p) & \text{if } j \leq i \end{cases} \tag{23}$$

for $p < q$, $p+q+r=1$, $b = p/q$; and

$$H_i(z) = \frac{1}{j - z(j-p)}\begin{cases} (j-i) - z\left[(j-i) - p\right] & \text{if } j \geq i, \\ pz & \text{if } j \leq i \end{cases} \tag{24}$$

for $p = q$, $2p + r = 1$.

The probability distribution of $\nu^0(j \mid i)$ is

$$\operatorname{pr}\left(\nu^0(j \mid i) = n_j\right) = \begin{cases} a^i(1-a^{j-i})(1-a^j)^{-1} & \text{if } n_j = 0,\ j \geq i, \\ (1-a^{j-i}) & \text{if } n_j = 0,\ j \leq i, \\ \varrho(1-a^i)(1-a^j)^{-1} & \text{if } n_j = 1,2,\dots,\ j \geq i, \\ \varrho a^{i-j} & \text{if } n_j = 1,2,\dots,\ j \leq i \end{cases} \tag{25}$$

for $p > q$, $p + q + r = 1$, $a = q/p$ and $\varrho = \frac{p-q}{1-a^j}\left(1 - \frac{p-q}{1-a^j}\right)^{n_j - 1}$;

$$\mathrm{pr}\big(\nu^0(j \mid i) = n_j\big) = \begin{cases} (1-b^{j-i})(1-b^j)^{-1} & \text{if } n_j = 0,\ j \geq i, \\ 0 & \text{if } n_j = 0,\ j \leq i, \\ \varrho_1 b^{j-i}(1-b^i)(1-b^j)^{-1} & \text{if } n_j = 1, 2, \dots,\ j \geq i, \\ \varrho_1 & \text{if } n_j = 1, 2, \dots,\ j \leq i \end{cases} \tag{26}$$

for $p > q$, $p + q + r = 1$, $b = q/p$ and $\varrho_1 = \frac{q-p}{1-b^j}\left(1 - \frac{q-p}{1-b^j}\right)^{n_j - 1}$;
and, when $p = q$, $2p + r = 1$, we have

$$\mathrm{pr}\big(\nu^0(j \mid i) = n_j\big) = \begin{cases} 1 - i/j & \text{if } n_j = 0,\ j \geq i, \\ 0 & \text{if } n_j = 0,\ j \leq i, \\ \varrho_2(i/j) & \text{if } n_j = 1, 2, \dots,\ j \geq i, \\ \varrho_2 & \text{if } n_j = 1, 2, \dots,\ j \leq i, \end{cases} \tag{27}$$

where $\varrho_2 = \frac{p}{j}\left(1 - \frac{p}{j}\right)^{n_j - 1}$. Thus, the distribution of $\nu^0(j \mid i)$ is geometric for any $j \in T_N - \{0\}$, $j = i$, and will be modified geometrically for any $j \in T_N - \{0\}$, $j \neq i$.

The mean and the variance of $\nu^0(j \mid i)$ are

$$E\big[\nu^0(j \mid i)\big] = \frac{1}{p-q}\begin{cases} (1-a^i) & \text{if } j \geq i,\ p > q, \\ a^{i-j}(1-a^j) & \text{if } j \leq i,\ p > q, \\ b^{j-i}(b^i - 1) & \text{if } j \geq i,\ p < q, \\ (b^i - 1) & \text{if } j \leq i,\ p < q, \end{cases} \tag{28}$$

and

$$E\big[\nu^0(j \mid i)\big] = \frac{1}{p}\begin{cases} j & \text{if } j \leq i,\ p = q, \\ i & \text{if } j \geq i,\ p = q, \end{cases} \tag{29}$$

where $a = b^{-1} = q/p$, $p + q + r = 1$; and

$$\mathrm{Var}\big(\nu^0(j \mid i)\big) = $$

$$= \frac{1}{(p-q)^2}\begin{cases} (1-a^i)(r + 2q - 2a^j + a^i) & \text{if } j \geq i,\ p > q, \\ (1-a^j)\big[(2 - a^{(i-j)})(1-a^j) - p + q\big] & \text{if } j \leq i,\ p > q, \\ (1-b^i)(1 + 2p + r - b^j - b^{j-i})b^{j-i} & \text{if } j \geq i,\ p < q, \\ (1-b^j)(1 - b^j - q + p) & \text{if } j \leq i,\ p < q, \end{cases} \tag{30}$$

and

$$\mathrm{Var}\big(\nu^0(j \mid i)\big) = \frac{1}{p^2}\begin{cases} i(2j - i - p) & \text{if } j \geq i,\ p = q, \\ j(j - p) & \text{if } j \leq i,\ p = q, \end{cases} \tag{31}$$

where $a = b^{-1} = q/p$, $p + q + r = 1$.

It is interesting to note that formulae (22)–(24), (28) and (29), in the case $r = 0$, are of the same forms as that obtained in Barnett (1964) [1] (see also Iosifescu (1980) [9]).

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