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ON CONNECTIVITY POINTS

ĽUBOMÍR SNOHA

Recently, Rosen [5] has proved that if f is a bounded real-valued function with the domain an open interval, then the set of points at which f is connected and the set of points at which f is Darboux are G_δ sets.

This theorem remains true for the set of points at which f belongs to class $\mathcal{U}$ and for the set of points at which f belongs to class $\mathcal{U}_0$ (definitions see below). The proofs of these results are similar to the proof of the theorem of Rosen and are therefore omitted.

In [4], Lipiński has shown that the following conjecture of Ceder is right: If C and D are G_δ sets with $C \subset D$, then there exists a function f in the second Baire class such that the set of points at which f is continuous and the set of points at which f is Darboux are the sets C and D, respectively.

In the present paper it is shown that if $A \subset R$ is a G_δ set, then there exists a bounded real-valued function f such that f is discontinuous at every point and the set of points at which f is connected is the set A. Similarly for the classes $\mathcal{C}ted$ and $\mathcal{D}$, $\mathcal{D}$ and $\mathcal{U}$, $\mathcal{U}$ and $\mathcal{U}_0$ (definitions see below).

We shall use the following notations and definitions. Let Q be the set of all rational numbers. The cardinal number of the set M is denoted by $\overline{\overline{M}}$, c is the cardinality of the continuum. Let Ω be the first ordinal number of the cardinality of the continuum. The set M is said to be c-dense in itself, provided $\overline{\overline{I \cap M}} = c$ for every open interval I which meets the set M. All functions in this paper are of the type $R \to R$. No distinction is made between a function and its graph. The symbol $f|M$ denotes the restriction of the function f to the set M. For any subset M of the plane $R \times R$, $(M)_X$ denotes the X-projection of M. For any point z of the X axis, M_z denotes the set of points of M which have the X coordinate equal to z. We denote closed and open intervals with end points a and b by $\langle a, b \rangle$ and (a, b), respectively. However, (a, b) may also be a point of the plane. For any function f and any $x \in R$ we write

$$R(f, x) = \bigcap_{n=1}^{\infty} f\left(\left(x - \frac{1}{n}, x + \frac{1}{n}\right)\right), \quad R^+(f, x) = \bigcap_{n=1}^{\infty} f\left(\left\langle x, x + \frac{1}{n}\right)\right)$$

and similarly $R^-(f, x)$. We denote the closure of the set M by $\overline{M}$. Further, $\operatorname{Fr}(M) = \overline{M} \cap \overline{(R \setminus M)}$. A continuum is a compact connected set.

Let $\mathscr{C}$ ($\mathscr{C}ted$, $\mathscr{D}$) denote the class of continuous (connected, Darboux) functions. We denote the uniform closure of the class $\mathscr{D}$ by $\mathscr{U}$. Let $\mathscr{U}_0$ be the class of functions which are Darboux in the sense of Radakovič. The definitions and the properties of the classes $\mathscr{C}ted$, $\mathscr{D}$, $\mathscr{U}$ and $\mathscr{U}_0$ see in [1, 2, 3].

In [1], Bruckner and Ceder describe what it means for a real function to be Darboux at a point, and later in [3], Garrett, Nelms, and Kellum introduce the idea of a function connected at a point.

Definition 1. *A function f is said to be connected from the left (right) at a point z (we write $f \in Cted^-(z)$ ($f \in \mathscr{C}ted^+(z)$)) if and only if*

(i) *$(z, f(z))$ is a limit point of f from the left (right)*

(ii) *whenever (z, a) and (z, b) are two limit points of f from the left (right), then the continuum M contains a point of f whenever $(M)_X$ is a non-degenerate set with right (left) end point z and M_z is a subset of the vertical open interval with end points (z, a) and (z, b).*

The function f is connected at a point z (we write $f \in \mathscr{C}ted(z)$) if f is connected from both the left and the right at z.

If each such M is a horizontal interval instead, then one obtains the definitions of Darboux from the left (right) at a point and Darboux at a point. We write $f \in \mathscr{D}^-(z)$ ($f \in \mathscr{D}^+(z)$) and $f \in \mathscr{D}(z)$.

Definition 2. *We define the class $\mathscr{U}^+(z)$ just as the class $\mathscr{D}^+(z)$, but we write "contains a point of the closure of $f|\langle z, \infty)\backslash A$ whenever the set $A \subset R$ has the cardinality less than c" instead of "contains a point of f". Similarly we define $\mathscr{U}^-(z)$ and $\mathscr{U}(z)$.*

Definition 3. *If we replace "contains a point of f" in the definition of the class $\mathscr{D}^+(z)$ by "contains a point of the closure of $f|\langle z, \infty)$", we obtain the definition of the class $\mathscr{U}_0^+(z)$. The classes $\mathscr{U}_0^-(z)$ and $\mathscr{U}_0(z)$ are similarly defined.*

It can be proved that $f \in \mathscr{C}ted$ if and only if $f \in \mathscr{C}ted(x)$ for every $x \in R$ (see [3]), and similarly for $\mathscr{D}$ (see [1]), $\mathscr{U}$ and $\mathscr{U}_0$.

Further, $\mathscr{C}(x) \subsetneqq \mathscr{C}ted(x) \subsetneqq \mathscr{D}(x) \subsetneqq \mathscr{U}(x) \subsetneqq \mathscr{U}_0(x)$ for every $x \in R$. ($\mathscr{C}(x)$ denotes the class of functions which are continuous at x.)

Lemma 1. *Let I be an interval and let X be a first category set. Then the set $I\backslash X$ has the cardinality of continuum.*

Proof. Let $X = \bigcup_{n=1}^{\infty} X_n$, where X_n are nowhere dense sets. If $A = I\backslash \bigcup_{n-1}^{\infty} \overline{X}_n$ were countable, then $\bigcup_{n=1}^{\infty} \overline{X}_n$ would be a second category set. Hence A is uncountable. On the other hand A is a Borel set, hence A has cardinality c.

Lemma 2. *Let $X \subset R$ be a first category set. Then there exists a transfinite* *sequence $\{A_\xi\}_{\xi<\Omega}$ of the subsets of the set $R\setminus X$ such that the sets A_ξ are countable, dense in R and mutually disjoint.*

Proof. Let

$$I_0, I_1, \ldots, I_n, \ldots \quad (I_i \neq I_j \text{ for } i \neq j)$$

be a sequence of all non-degenerate closed intervals with rational end points. Since the set X contains no interval, all the sets

$$I_0\setminus X, \quad I_1\setminus X, \ldots, I_n\setminus X, \ldots \tag{1}$$

are non-empty. Thus there exists a set

$$A_0 = \{a_{00}, a_{01}, \ldots, a_{0i}, \ldots : a_{0i} \in I_i\setminus X \text{ for every } i\}.$$

Clearly, A_0 is countable and dense in R. Let us suppose $\{A_\alpha\}_{\alpha<\varphi}$, where $\varphi < \Omega$ is defined. We are going to define the set A_φ. Since the ordinal number φ has cardinality less than c and all the sets A_α, $\alpha<\varphi$ are countable, the set $\bigcup_{\alpha<\varphi} A_\alpha$ has cardinality less than c. On the other hand, according to lemma 1, all the sets (1) have cardinality c. Thus all the sets

$$I_0\setminus\Big(X\cup\bigcup_{\alpha<\varphi} A_\alpha\Big), \quad I_1\setminus\Big(X\cup\bigcup_{\alpha<\varphi} A_\alpha\Big), \ldots, I_n\setminus\Big(X\cup\bigcup_{\alpha<\varphi} A_\alpha\Big), \ldots$$

are non-empty. Then there exists a set

$$A_\varphi = \Big\{a_{\varphi 0}, a_{\varphi 1}, \ldots, a_{\varphi i}, \ldots : a_{\varphi i} \in I_i\setminus\Big(X\cup\bigcup_{\alpha<\varphi} A_\alpha\Big) \text{ for every } i\Big\}.$$

Clearly, the set A_φ is countable, dense in R and disjoint with every set A_α, $\alpha<\varphi$, which finishes the proof.

Lemma 3. *Let $B \subset R$ be an uncountable Borel set. Then there exist non-empty, bounded, perfect, nowhere dense and mutually disjoint subsets $P_k (k=1, 2, \ldots)$ and $Q_k (k=1, 2, \ldots)$ of the set B such that for every open interval I for which $\overline{\overline{B\cap I}} = c$ there exists such n that $P_n \subset I\cap B$ and $Q_n \subset I\cap B$ and the sets P_n and Q_n are nowhere dense in $B\cap I$.*

Proof. Since the continuum-hypothesis holds for Borel sets, we can write $B = B_1\cup B_2$, where the set B_1 is countable and the set B_2 is a Borel set c-dense in itself. Now it suffices to use the proof of lemma 4.1 in [2].

Theorem 1. *Let $A \subset R$ be a G_δ set. Then there exists a bounded function f: $R\to R$ such that f is discontinuous at every point and $f \in \mathscr{C}ted(x)$ if and only if $x \in A$.*

Proof. The complement of the set A is an F_σ set. Thus there exist closed sets F_k $(k=1, 2, \ldots)$ such that

$$R \setminus A = F_1 \cup F_2 \cup \ldots \quad \text{where} \quad F_1 \subset F_2 \subset \ldots$$

Let us put $F_0 = \emptyset$ and $B_k = F_k \setminus F_{k-1}$ $(k = 1, 2, \ldots)$. Since the boundary of a closed set is nowhere dense, the boundary of every B_k is nowhere dense. Hence

$$B = \bigcup_{k=1}^{\infty} ((B_k \cap \operatorname{Fr}(B_k)) \cup ((\operatorname{int} B_k) \cap Q))$$

is a first category set. According to lemma 2 there exists a transfinite sequence $\{A_\xi\}_{\xi<\Omega}$ of the subsets of the set $R \setminus B$ such that the sets A_ξ are countable, dense in R and mutually disjoint.

Further, let $\{M_\xi\}_{\xi<\Omega}$ be a transfinite sequence of all continua which are subsets of the set $R \times \langle 0, 1 \rangle$ and have non degenerate X projection.

Let us define a function f: $R \to R$ as follows:

$$f(x) = \begin{cases} \min \{y \in R : (x, y) \in M_\xi\} & \text{if } x \in A_\xi,\ \xi < \Omega \text{ and the set } \{y \in R: (x, y) \in M_\xi\} \text{ is non empty} \\ 1 + \dfrac{1}{k} & \text{if there exists such natural } k \text{ that } x \in (B_k \cap \operatorname{Fr}(B_k)) \cup ((\operatorname{int} B_k) \cap Q) \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, f is a bounded function. We shall prove the other properties.

The function f is discontinuous at every point, because it meets every continuum from the sequence $\{M_\xi\}_{\xi<\Omega}$.

Let us take $z \in A$. We are going to show that $f \in \mathscr{C}ted(z)$. First of all $f(z) \in \langle 0, 1 \rangle$. It follows from the definition of the function f that $(z, f(z))$ is a limit point of f from both the right and the left. Further, the function f attains values greater than or equal to $1 + \frac{1}{k}$ on the set F_k only. Moreover F_k is closed and disjoint with A, hence for every natural k there exists a neighbourhood of the point z such that all the values of the function f are less than $1 + \frac{1}{k}$ in this neighbourhood. It follows from this that $a, b \in \langle 0, 1 \rangle$ whenever (z, a) and (z, b) are limit points of f. Let M be a continuum such that $(M)_X$ is a non-degenerate set with left end-point z and M_z is a subset of the vertical open interval with end-points (z, a) and (z, b). If the continuum M is a subset of the set $R \times \langle 0, 1 \rangle$, then $M = M_\xi$ for some ξ, which implies that f meets M. If M is not a subset of the set $R \times \langle 0, 1 \rangle$, then there exists a continuum $M^* \subset M$ such that M^* is a subset of the set $R \times \langle 0, 1 \rangle$ and $(M^*)_X$ is a non degenerate set. Then f meets M^* and hence f meets M. We have proved $f \in \mathscr{C}ted^+(z)$. Similarly $f \in \mathscr{C}ted^-(z)$.

Let us take $z \notin A$. We are going to show $f \notin \mathscr{C}ted(z)$. There exists exactly one natural k such that $z \in B_k$. Clearly, $z \in (B_k \cap \operatorname{Fr}(B_k)) \cup ((\operatorname{int} B_k) \cap Q)$ or

$z \in (\operatorname{int} B_k) \cap (R \setminus Q)$. In the first case we have $f(z) = 1 + \frac{1}{k}$. In the second case every neighbourhood of z contains some points of the set $(\operatorname{int} B_k) \cap Q$. Thus we have $1 + \frac{1}{k} \in R(f, z)$ in both cases. Further, the function f meets every continuum from the sequence $\{M_\xi\}_{\xi < \Omega}$, whence $\langle 0, 1 \rangle \subset R(f, z)$. Hence $R(f, z)$ is not an interval (see definition of f), which implies $f \notin \mathscr{C}ted(z)$. This completes the proof of theorem 1.

Theorem 2. *Let $A \subset R$ be a G_δ set. Then there exist bounded functions u, v, w in the second Baire class such that*
(i) *$u \notin \mathscr{C}ted(x)$ for every $x \in R$ and $u \in \mathscr{D}(y)$ if and only if $y \in A$*
(ii) *$v \notin \mathscr{D}(x)$ for every $x \in R$ and $v \in \mathscr{U}(y)$ if and only if $y \in A$*
(iii) *$w \notin \mathscr{U}(x)$ for every $x \in R$ and $w \in \mathscr{U}_0(y)$ if and only if $y \in A$.*

Proof. Let the sets B_k $(k = 1, 2, \ldots)$ have the same meaning as they have in the proof of theorem 1.

According to lemma 3 there exists a sequence of bounded, perfect and nowhere dense (and non-empty in the case of uncountability of A) sets $\{P_n\}_{n=1}^{\infty}$, $P_n \subset A$ $(n = 1, 2, \ldots)$ such that every interval intersecting the set A in the set of cardinality of continuum contains at least one of the sets P_n. Similarly for every natural k there exist sequences of bounded perfect and nowhere dense (and non-empty in the case of uncountability of B_k) sets $\{P_{kn}\}_{n=1}^{\infty}$, $P_{kn} \subset B_k$ $(n = 1, 2, \ldots)$ and $\{Q_{kn}\}_{n=1}^{\infty}$, $Q_{kn} \subset B_k$ $(n = 1, 2, \ldots)$ such that for every interval I intersecting the set B_k in the set of cardinality of continuum there exists a natural n with the property: $P_{kn} \subset I \cap B_k$ and $Q_{kn} \subset I \cap B_k$. Moreover, each two of the sets P_n, P_{kn}, Q_{kn} $(n, k = 1, 2, \ldots)$ are disjoint.

Let us define the sets J and J_k $(k = 1, 2, \ldots)$ as follows. Let $J = \emptyset$ or let J be the set of all natural numbers if the set A is countable or uncountable, respectively. Similarly for J_k and B_k.

Let h_n $(n \in J)$ and h_{kn} $(k = 1, 2, \ldots$ and $n \in J_k)$ be the Cantor step functions defined on P_n and P_{kn} such that $h_n(P_n) = h_{kn}(P_{kn}) = \langle 0, 1 \rangle$. (Such functions can be easily defined because each of the sets P_n, P_{kn} is homeomorphic to the Cantor set.)

Let us define the function $g\colon R \to R$ as follows:

$$g(x) = \begin{cases} h_n(x) & \text{if there exists such } n \text{ that } x \in P_n \\ h_{kn}(x) & \text{if there exist such } k, n \text{ that } x \in P_{kn} \\ 1 + \dfrac{1}{k} & \text{if there exists such } k \text{ that } x \in B_k \setminus \bigcup\limits_{n=1}^{\infty} P_{kn} \\ 0 & \text{otherwise.} \end{cases}$$

Let us notice some properties of the function g. First of all, every interval intersects at least one of the sets A and B_k $(k = 1, 2, \ldots)$ in the set of cardinality of continuum. Consequently, every interval contains at least one non-empty set out of P_n $(n=1, 2, \ldots)$ or P_{kn} $(k, n=1, 2, \ldots)$. Hence for every $z \in R$ we have

$$\langle 0, 1\rangle \subset (R^+(g, z) \cap R^-(g, z)). \tag{2}$$

Further we are going to prove that $g \in \mathscr{D}(z)$ if and only if $z \in A$. Let $z \in A$. Clearly $g(z) \in \langle 0, 1\rangle$ and $(z, g(z))$ is a limit point of g from both the right and the left. In a similar way as in the case of the function f (see proof of theorem 1) it can be proved that $a, b \in \langle 0, 1\rangle$ whenever (z, a) and (z, b) are limit points of g. Now it follows from (2) that $g \in \mathscr{D}(z)$.

Let $z \notin A$, i.e. $z \in B_k$ for exactly one natural k. There are two possibilities. If $z \in B_k \setminus \bigcup_{n=1}^{\infty} P_{kn}$, then $g(z) = 1 + \frac{1}{k}$. If $z \in P_{kn}$ for some natural n, then each neighbourhood of z contains at least one set out of Q_{kn} $(n=1, 2, \ldots)$. It is a consequence of the fact that all the sets P_{kn} are taken out of a subset of the set B_k c-dense in itself (see proof of the lemma 3). Hence $1 + \frac{1}{k} \in R(g, z)$ in both above mentioned cases. Thus $R(g, z)$ is not an interval and consequently $g \notin \mathscr{D}(z)$.

The function g belongs to the Baire class 2, because the sets $g^{-1}((a, \infty))$ and $g^{-1}((-\infty, a))$ are of the type $G_{\delta\sigma}$ for every $a \in R$. In fact, it is easy to see that $g^{-1}\left(1 + \frac{1}{k}\right)$ is a $G_{\delta\sigma}$ set for every natural k, $g^{-1}(1)$ is countable and $g^{-1}(0)$ is also a $G_{\delta\sigma}$ set. Further, for every $a, b \in (0, 1)$, $a < b$ we have

$$g^{-1}((a, b)) = \bigcup_{n \in J} h_n^{-1}((a, b)) \cup \bigcup_{k=1}^{\infty} \bigcup_{n \in J_k} h_{kn}^{-1}((a, b)).$$

The functions h_n $(n \in J)$ and h_{kn} $(k=1, 2, \ldots$ and $n \in J_k)$ defined on the closed sets P_n and P_{kn} are continuous. In view of it $g^{-1}((a, b))$ is a $G_{\delta\sigma}$ set, which completes the proof that g is Baire 2.

(i) Let us define the function u: $R \to R$ as follows:

$$u(x) = \begin{cases} g(x) & \text{if } g(x) \neq \frac{1}{2} + \frac{1}{4} \sin x \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

That the function u is connected at no point follows from (2), (3) and from the fact that the graph of the restriction of the function $s(x) = \frac{1}{2} + \frac{1}{4} \sin x$ to a closed interval is a continuum.

We have shown that $g \in \mathscr{D}(z)$ if and only if $z \in A$. It is easy to see that this property remains true in the case of the function u.

The function g is in the second Baire class. Further, $g \cap s$ is a planar F_σ set. The sets $(g \cap s) \cap ((-\infty, \infty) \times (a, \infty))$ and $(g \cap s) \cap ((-\infty, \infty) \times (-\infty, a))$ also are of the type F_σ for every $a \in R$. Therefore $(g \cap s)^{-1}((a, \infty))$ and $(g \cap s)^{-1}((-\infty, a))$ are F_σ sets for every $a \in R$. Now it is not difficult to show that u is a Baire 2 function.

(ii) Let us define the function v: $R \to R$ as follows:

$$v(x) = \begin{cases} g(x) & \text{if } g(x) \neq \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

That the function v is Darboux at no point follows from (2) and (4).

Let $z \in A$. We are going to prove that $v \in \mathscr{U}^+(z)$. Clearly $v(z) \in \left\langle 0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right\rangle$ and $(z, v(z))$ is a limit point of v from both the right and the left. Similarly as in the case of the function f (see proof of theorem 1) we can prove that $a, b \in \langle 0, 1\rangle$ whenever (z, a) and (z, b) are limit points of v. Let $y \in (0, 1)$, let $\delta > 0$ and let $M \subset R$ be a set of cardinality less than c. It suffices to prove that there exists a point x_0 such that $(x_0, y) \in \overline{v|(\langle z, z+\delta\rangle \setminus M)}$. The interval $\langle z, z+\delta\rangle$ contains at least one non-empty set from among the sets P_n $(n = 1, 2, \ldots)$ or P_{kn} $(k, n = 1, 2, \ldots)$. Let, e.g., there exists such an n that $\emptyset \neq P_n \subset \langle z, z+\delta\rangle$ (in the remaining case the proof is analogous). Denote $P_n^* = P_n \setminus (h_n^{-1}(\frac{1}{2}) \cup M)$. The function h_n is defined on the set P_n, the planar set h_n is closed and c-dense in itself and $\overline{\overline{h_n^{-1}(\frac{1}{2}) \cup M}} < c$. These facts imply $\overline{h_n|P_n^*} = h_n$. However, $h_n|P_n^* = v|P_n^*$. Thus it suffices to choose $x_0 \in h_n^{-1}(y)$. This ends the proof that $v \in \mathscr{U}^+(z)$. In the same way we can prove that $v \in \mathscr{U}^-(z)$.

Now let $z \notin A$, i.e. $z \in B_k$ for exactly one natural k. Similarly as in the case of the function g it can be proved that $1 + \frac{1}{k} \in R(v, z)$. Further $1 \in R(v, z)$ but no number from the interval $\left(1 + \frac{1}{k+2}, 1 + \frac{1}{k+1}\right)$ is a value of v. Therefore $v \notin \mathscr{U}(z)$.

The function v is Baire 2 because it differs from the Baire 2 function g on a countable set only.

(iii) Let us define the function w: $R \to R$ as follows:

$$w(x) = \begin{cases} g(x) & \text{if } g(x) \in Q \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Denote the set $w^{-1}\left(\left(\frac{1}{3}, \frac{2}{3}\right)\right)$ by the letter S. Clearly $\overline{\overline{S}} < c$, because S is countable. In view of (2) and (5) the numbers 0 and 1 are contained in $R(w, z)$ for every $z \in R$. But no number from the interval $\left(\frac{1}{3}, \frac{2}{3}\right)$ is a value of the function $w|R \setminus S$. Therefore $w \notin \mathscr{U}(z)$ for every $z \in R$.

Let $z \in A$. We are going to prove that $w \in \mathscr{U}_0^+(z)$. Clearly $w(z) \in \langle 0, 1 \rangle \cap Q$ and $(z, w(z))$ is a limit point of w from both the right and the left. Further, $a, b \in \langle 0, 1 \rangle$ whenever (z, a) and (z, b) are limit points of w. Let $y \in (0, 1)$ and let $\delta > 0$. It suffices to prove that there exists a point x_0 such that $(x_0, y) \in \overline{w|\langle z, z+\delta \rangle}$. The interval $\langle z, z+\delta \rangle$ contains at least one non-empty set from among the sets P_n $(n = 1, 2, \ldots)$ or P_{kn} $(k, n = 1, 2, \ldots)$, which the function w maps onto the set $\langle 0, 1 \rangle \cap$Q. From this and from the fact that $\langle z, z+\delta \rangle \times \{y\}$ is a planar compact set, it follows that there exists such an x_0. This ends the proof that $w \in \mathscr{U}_0^+(z)$. In the same way we can prove that $w \in \mathscr{U}_0^-(z)$.

Now let $z \notin A$. That $w \notin \mathscr{U}_0(z)$ can be proved similarly as $v \notin \mathscr{U}(z)$.

We are going to prove that w is a Baire 2 function. The set of all values of w is countable. Hence it suffices to prove that $w^{-1}(a)$ is a $G_{\delta\sigma}$ set for every value a of w. It is not difficult. In fact, $w^{-1}(a)$ is countable for every $a \in (0, 1 \rangle \cap Q$. Further, $w^{-1}\left(1 + \frac{1}{k}\right)$ is a G_δ set for every natural k. Finally we have

$$w^{-1}(0) = \left(A \setminus \bigcup_{n \in J} h_n^{-1}(Q \setminus \{0\})\right) \cup \bigcup_{k=1}^{\infty} \bigcup_{n \in J_k} h_{kn}^{-1}((R \setminus Q) \cup \{0\}).$$

The set A is of the type G_δ and $Q \setminus \{0\}$ is countable. Further, $(R \setminus Q) \cup \{0\}$ is a G_δ set and the functions h_{kn} ($k = 1, 2, \ldots$ and $n \in J_k$) defined on the closed sets P_{kn} are continuous. From these facts it follows that $w^{-1}(0)$ is a $G_{\delta\sigma}$ set, which finishes the proof that w is Baire 2.

Theorem 2 is proved.

We finish our paper by the following problem, which does not seem to be easily solvable:

Problem. Characterize the sets $A \subset R$ for which there exists a function φ: $R \to R$ such that the following conditions are fulfilled simultaneously:

(i) $\varphi \in \mathscr{C}(x)$ at no point $x \in R$

(ii) $\varphi \in \mathscr{C}ted(y)$ if and only if $y \in A$

(iii) $\varphi \in \mathscr{D}(z)$ at every point $z \in R$.

(Such a function φ does not exist if $A = R \setminus \{x_0\}$, where $x_0 \in R$. Hence, the family of all such sets A is not identical with the family of all G_δ sets.)

This problem can be generalized to the one characterizing the sets $A_1 \subset A_2 \subset A_3 \subset A_4 \subset A_5 \subset R$, for which there exists a function ψ: $R \to R$ such that the sets A_1, A_2, A_3, A_4, A_5 are the sets of points at which ψ belongs to $\mathscr{C}$, $\mathscr{C}ted$, $\mathscr{D}$, $\mathscr{U}$, $\mathscr{U}_0$, respectively.

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О ТОЧКАХ СВЯЗНОСТИ

Ľubomír Snoha

Резюме

В статье доказывается следующая теорема: Если $A \subset R$ — множество типа G_δ, то существует ограниченная функция f: $R \to R$, разрывная в каждой точке и связная во всех точках множества A и только в этих точках.