Tomasz Natkaniec

On the maximum and the minimum of quasi-continuous functions

Mathematica Slovaca, Vol. 42 (1992), No. 1, 103--110

Persistent URL: http://dml.cz/dmlcz/130075

Mathematica Slovaca

Math. Slovaca, 42 (1992), No. 1, 103–110

ON THE MAXIMUM AND THE MINIMUM OF QUASI–CONTINUOUS FUNCTIONS

TOMASZ NATKANIEC

ABSTRACT. Some results concerning the maximum and the minimum of quasi-continuous functions are presented.

I. Let us establish some of the terminology to be used. $\mathbb{R}$ denotes the *real line*. Let (X, τ) be a *topological space*. A real function f defined on X is said to be *quasi-continuous at a point* $x_0 \in X$ iff for every $\varepsilon > 0$ and for any neighbourhood $U \in \tau$ of the point x_0 there exists an open set V such that $\emptyset \neq V \subset U$ and $|f(x) - f(x_0)| < \varepsilon$ for each $x \in V$ [1]. By $C(f)$ and $Q(f)$ we will denote the *set of all continuity points of a function* f and the *set of all quasi-continuity points of* f, respectively. Furthermore, let $A(f) = X \setminus Q(f)$. A real function $f\colon X \to \mathbb{R}$ is *quasi-continuous* on X iff f is quasi-continuous at every point of X. The symbols $\mathcal{C}$, $\mathcal{Q}$ stand for the families of all *continuous* and *quasi-continuous functions*, respectively.

A family $\mathcal{A}$ of real functions $f\colon X \to \mathbb{R}$ is a lattice iff $\min(f,g) \in \mathcal{A}$ and $\max(f,g) \in \mathcal{A}$ for $f, g \in \mathcal{A}$. If $\mathcal{B}$ is a family of real functions, then the symbol $\mathcal{L}(\mathcal{B})$ stands for the *lattice generated by* $\mathcal{B}$, i.e. the smallest lattice of functions containing $\mathcal{B}$. Evidently, we have $\mathcal{L}(\mathcal{A}) \subset \mathcal{L}(\mathcal{B})$ if $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{L}(\mathcal{L}(\mathcal{A})) = \mathcal{L}(\mathcal{A})$.

The presented paper contains three theorems. In the first we generalize some result of Z. Grande and L. Sołtysik from [6]. They proved that if a function $f\colon X \to \mathbb{R}$ is not upper (lower) semi-continuous, then there exists a quasi-continuous function $g\colon X \to \mathbb{R}$ such that $\max(f,g)$ ($\min(f,g)$) is not quasi-continuous. Now we shall prove that for every non-continuous function $f\colon X \to \mathbb{R}$ there exist two quasi-continuous functions $g, h\colon X \to \mathbb{R}$ for which $\max(f,g)$ and $\min(f,h)$ are not quasi-continuous. In the second part we describe the lattice generated by the family of all quasi-continuous functions $f\colon X \to \mathbb{R}$ (for some class of topological spaces X). This generalizes one result from [4] (for $X = \mathbb{R}$). Notice that in the proof in [4] the completeness of $\mathbb{R}$ plays the key

AMS Subject Classification (1991): Primary 54C30. Secondary 26A15.
Key words: Quasi-continuity, Lattice, Continuity, Darboux property.

role. Now this assumption is not necessary. The last proposition characterizes $\max(g_0, g_1)$ for quasi-continuous functions $g_0, g_1 \colon \mathbb{R} \to \mathbb{R}$. This is a complement of results from [3] and [7], where functions which can be expressed as a sum and as a product of a finite number of quasi-continuous functions were characterized.

II. Let

$$\mathcal{M}_{\min} = \{f \colon X \to \mathbb{R} \colon \ \forall g \in \mathcal{Q} \quad \min(f,g) \in \mathcal{Q}\}$$
and
$$\mathcal{M}_{\max} = \{f \colon X \to \mathbb{R} \colon \ \forall g \in \mathcal{Q} \quad \max(f,g) \in \mathcal{Q}\}.$$

The equality $\mathcal{M}_{\max} \cap \mathcal{M}_{\min} = \mathcal{C}$ is shown in [6]. Now we shall prove that $\mathcal{M}_{\max} = \mathcal{M}_{\min} = \mathcal{C}$.

PROPOSITION 1. *We have the equalities* $\mathcal{M}_{\max} = \mathcal{C} = \mathcal{M}_{\min}$.

Proof. $\mathcal{M}_{\max} \subset \mathcal{C}$. Notice that $\mathcal{M}_{\max} \subset \mathcal{Q}$. Indeed, if $f \notin \mathcal{Q}$, then there exists a point $x_0 \in A(f)$. Let us put $g \colon X \to \mathbb{R}$, $g(x) = f(x_0) - 1$. Then $g \in \mathcal{Q}$ and the function $\max(f,g)$ is not quasi-continuous at x_0.

Let $f \in \mathcal{M}_{\max}$. We shall prove that $f(x_0) = \overline{\lim}_{x \to x_0} f(x) = \underline{\lim}_{x \to x_0} f(x)$ for any $x_0 \in X'$ (where X' denotes the *set of all accumulation points of* X). We shall consider two cases.

(a) Suppose that $f(x_0) < c < \overline{\lim}_{x \to x_0} f(x)$. Then for every neighbourhood U of x_0 there exists an open set V_U such that $V_U \subset U$ and $f(x) \geq c$ for $x \in V_U$. Let $\mathcal{B}_0$ be a *basis of* (X, τ) *at the point* x_0. We define the function $g \colon X \to \mathbb{R}$ as follows:

$$g(x) = \begin{cases} f(x_0) & \text{for } x \in \overline{\bigcup_{U \in \mathcal{B}_0} V_U}, \\ c & \text{otherwise.} \end{cases}$$

Observe that g is a quasi-continuous function and the set $\{x \in X \colon \max(f,g)(x) < c\}$ is nowhere-dense. Thus the function $\max(f,g)$ is not quasi-continuous at the point x_0.

(b) Now we suppose that $\underline{\lim}_{x \to x_0} f(x) < c < f(x_0)$. Then for every neighbourhood $U \in \mathcal{B}_0$ there exists an open set V_U such that $V_U \subset U$ and $f(x) \leq c$ for $x \in V_U$. We put

$$h(x) = \begin{cases} c & \text{for } x \in \overline{\bigcup_{U \in \mathcal{B}_0} V_U}, \\ f(x_0) + 1 & \text{otherwise.} \end{cases}$$

Notice that the function h is quasi-continuous, $\max(f,h)(x_0) = f(x_0)$ and $\max(f,h)(x) \in \{c\} \cup \langle f(x_0) + 1, \infty)$ for $x \notin \operatorname{Fr}\Big(\bigcup_{U \in \mathcal{B}_0} V_U\Big)$. Hence $\max(f,h)$ is

not quasi-continuous at the point x_0.
The cases (a) and (b) imply the continuity of the function f.

The inclusion $\mathcal{C} \subset \mathcal{M}_{\max}$ follows easily from the results of [6] but for the sake of completeness we give here the following proof.

Assume that $f \in \mathcal{C}$, $g \in \mathcal{Q}$ and $x_0 \in X$. Fix $\varepsilon > 0$ and an open set U with $x_0 \in U$. We shall consider two cases.

(a) $f(x_0) \neq g(x_0)$. Let $d = |f(x_0) - g(x_0)|$. We can choose a neighbourhood V of x_0 such that $V \subset U$ and $|f(x) - f(x_0)| < \min(d/2, \varepsilon)$ and an open set $W \subset V$ such that $|g(x) - g(x_0)| < \min(d/2, \varepsilon)$. Then $|\max(f,g)(x) - \max(f,g)(x_0)| < \varepsilon$ for $x \in W$.

(b) $f(x_0) = g(x_0)$. Let V be a neighbourhood of x_0 such that $V \subset U$ and $|f(x) - f(x_0)| < \varepsilon/2$ for $x \in V$, and let $W \subset V$ be an open set such that $|g(x) - g(x_0)| < \varepsilon/2$ for $x \in W$. Then $|\max(f,g)(x) - \max(f,g)(x_0)| < \varepsilon$ for $x \in W$.
Thus $\mathcal{M}_{\max} = \mathcal{C}$.

Since $-f \in \mathcal{M}_{\max}$ iff $f \in \mathcal{M}_{\min}$, we obtain the equality $\mathcal{M}_{\min} = \mathcal{C}$.

III. The lattices generated by quasi-continuous functions defined on $\mathbb{R}$ with the Euclidean topology and the density topology are studied in [4], [5]. Now we improve these results.

LEMMA 1. *Let (X, τ) be a regular, dense-in-itself space with a countable basis (notice that such spaces must be metrizable). If A is a nowhere dense subset of X, then there exists a sequence $(K_{n,m})_{n\in\mathbb{N},\ m\leq n}$ of open sets such that:*

(1) *if $\overline{K}_{n,m} \cap \overline{K}_{i,j} \neq \emptyset$, then $n = i$ and $m = j$,*
(2) $\forall x \in \overline{A}\ \ \forall U \in \tau\ \ (x \in U \implies \forall m\ \exists n \geq m\ \ \overline{K}_{n,m} \subset U)$,
(3) *if $x \notin \overline{A}$, then there exists $U \in \tau$ such that $x \in U$ and the set $\{(n,m),\ U \cap \overline{K}_{n,m} \neq \emptyset\}$ has at most one element.*

Remarks. The condition (2) implies that $\overline{A} \subset \overline{\bigcup_{n\geq m} K_{n,m}}$ for each $m \in \mathbb{N}$. From (3) it follows that the set $\overline{A} \cup \bigcup_{n,m} \overline{K}_{n,m}$ is closed.

Proof. Let $\mathcal{B} = (B_n)_n$ be a countable basis of (X, τ) and let $(W_n)_n$ be a sequence of open sets such that $\overline{A} = \bigcap_{n\in\mathbb{N}} W_n$ and $W_1 \supset W_2 \supset \ldots$. Such sequence exists because every closed set in a regular space with a countable basis is a G_δ set. Let $(G_n)_n$ be a sequence of all sets from $\mathcal{B}$ such that $G_n \cap \overline{A} \neq \emptyset$ for each $n \in \mathbb{N}$. For every number $n \in \mathbb{N}$ we chose (inductively) a non-empty, open set K_n such that $\overline{K}_n \subset G_n \cap W_n \setminus (\overline{A} \cup \bigcup_{i<n} \overline{K}_i)$. It is possible because

the set $G_n \cap W_n \setminus (\overline{A} \cup \bigcup_{i<n} \overline{K}_i)$ is non-empty, open and X i regular. Chosen in this way the sets K_n have the following properties:

(i) $\overline{K}_n \cap \overline{A} = \emptyset$ for $n \in \mathbb{N}$ and $\overline{K}_n \cap \overline{K}_m = \emptyset$ for $n \neq m$,

(ii) for every $x \in \overline{A}$ and for every neighbourhood U of x the set $\{n : \overline{K}_n \subset U\}$ is infinite,

(iii) if $x \notin \overline{A}$, then there exists a neighbourhood U of x for which the set $\{n : \overline{K}_n \cap U \neq \emptyset\}$ has at most one element.

Evidently, (i) and (ii) hold. We shall verify (iii). Let $x \notin \overline{A}$. There exists $n_0 \in \mathbb{N}$ and a neighbourhood V of x such that $V \cap W_{n_0} = \emptyset$. Thus if $V \cap \overline{K}_n \neq \emptyset$, then $n < n_0$. If $x \in \overline{K}_m$ for some $m < n_0$, then $U = V \setminus \bigcup_{\substack{n<n_0 \\ n \neq m}} \overline{K}_n$. If $x \notin \overline{K}_n$ for every $n < n_0$, then $U = V \setminus \bigcup_{n<n_0} \overline{K}_n$. Fix $n \in \mathbb{N}$.

We choose (inductively) a sequence $(K_{n,m})_{m \leq n}$ of nonempty, open subsets of X such that

(iv) $\overline{K}_{n,m} \subset K_n$ for $m \in \mathbb{N}$ and $\overline{K}_{n,m} \cap \overline{K}_{n,t} = \emptyset$ for $m \neq t$.

The construction of $(K_{n,m})_{m \leq n}$ is the following. Fix a point $x_0 \in K_n$. Let $(D_n)_n$ be a basis of (X, τ) at x_0. We choose a sequence $(x_m, U_m, K_{n,m}) \in K_n \times \tau \times \tau$ $(m \leq n)$ such that

(v) $x_1 \in K_n \setminus \{x_0\}$, $x_0 \in U_1 \subset \overline{U}_1 \subset K_n \cap D_1 \setminus \{x_1\}$, $x_1 \in K_{n,1} \subset \overline{K}_{n,1} \subset K_n \setminus \overline{U}_1$,

(vi) $x_{m+1} \in U_m \setminus \{x_0\}$, $x_0 \in U_{m+1} \subset \overline{U}_{m+1} \subset U_m \cap D_{m+1} \setminus \{x_{m+1}\}$, $x_{m+1} \in K_{n,m+1} \subset \overline{K}_{n,m+1} \subset U_m \setminus \overline{U}_{m+1}$.

Chosen in this way the sequence $(K_{n,m})_{m \leq n, n \in \mathbb{N}}$ has the desired properties.

LEMMA 2. *The family $\mathcal{N}$ of all functions $f\colon X \to \mathbb{R}$ for which the set $A(f) = X \setminus Q(f)$ is nowhere dense forms a lattice.*

Proof. (For $X = \mathbb{R}$ see [4].) Let $f, g \in \mathcal{N}$ and $h = \max(f, g)$. It is enough to prove that the set $C = A(h) \setminus \overline{A(f) \cup A(g)}$ is nowhere dense. Let U be an open set such that $U \cap \overline{A(f) \cup A(g)} = \emptyset$ and $x_0 \in U \cap C$. Then there exists $\varepsilon > 0$ and a neighbourhood U_1 of x_0 such that for every open set $\emptyset \neq V \subset U_1$ there exists a point $x \in V$ such that $|h(x) - h(x_0)| \geq \varepsilon$. Assume that $h(x_0) - f(x_0) \geq g(x_0)$. Let $\emptyset \neq V \subset U_1$ be an open set such that $|f(x) - f(x_0)| < \frac{\varepsilon}{2}$ for each $x \in V$ and let $x_1 \in V$ be a p int for which $|h(x_1) - f(x_0)| \geq \varepsilon$. Then $h(x_1) - g(x_1) > f(x_1)$ and there exi t an op n set

$\emptyset \neq W \subset V$ such that $|g(x) - g(x_1)| < \frac{\varepsilon}{2}$ for $x \in W$. Thus $g(x) > f(x)$ for $x \in W$ and $h|_W = g|_W$ is quasi-continuous at each point $x \in W$.

PROPOSITION 2. *Let X_0 be a set of all isolated points of X. If the subspace $X' = X \setminus X_0$ satisfies the assumptions of Lemma 1, then $\mathcal{L}(\mathcal{Q}) = \mathcal{N}$.*

Proof. Of course, $\mathcal{Q} \subset \mathcal{N}$ and by Lemma 2 we have $\mathcal{L}(\mathcal{Q}) \subset \mathcal{N}$. We shall prove that $\mathcal{N} \subset \mathcal{L}(\mathcal{Q})$. Let $f \in \mathcal{N}$, $A = A(f)$ and let $(K_{n,m})_{m \leq n}$ be a sequence of open sets which satisfies the conditions (1)–(4) from Lemma 1. Let $(w_n)_n$ be a sequence of all rationals. We define functions g_i $(i = 0,1,2,3)$ as follows:

$$g_i(x) = \begin{cases} f(x) & \text{for } x \in \overline{A} \cup X_0, \\ w_m & \text{for } x \in \bigcup_{n \in \mathbb{N}} \overline{K}_{n,4m+i},\ m \in \mathbb{N}, \\ f(x) & \text{otherwise.} \end{cases}$$

Then g_i, $i = 1,2,3,0$ are quasi-continuous. It is enough to verify that g_i is quasi-continuous at every point $x_0 \in \overline{A}$. Fix $i = 0$, $x_0 \in \overline{A}$, a neighbourhood U of x_0 and $\varepsilon > 0$. There exists $m \in \mathbb{N}$ such that $w_m \in (f(x_0)-\varepsilon, f(x_0)+\varepsilon)$ and there exists $n \in \mathbb{N}$ such that $4m \leq n$ and $\overline{K}_{n,4m} \subset U$. Then $|g_0(x)-g_0(x_0)| < \varepsilon$ for $x \in K_{n,4m}$. Thus g_0 is quasi-continuous. Similarly we verify quasi-continuity of g_i for $i = 1,2,3$. Since $f = \min(\max(g_0,g_1), \max(g_2,g_3))$, $f \in \mathcal{L}(\mathcal{Q})$.

Remark 1. Observe that we have also $f = \max(h_1,h_2)$, where $h_1 = \min(\max(g_0,g_1), g_2)$ and $h_2 = \min(\max(g_0,g_2), g_1)$ but there exists a function $f \in \mathcal{N}$ such that $f \neq \max(g,h)$ and $f \neq \min(g,h)$ for each $g,h \in \mathcal{Q}$ (e.g. $f(x) = x$ for $x \in \{-1,1\}$ and $f(x) = 0$ otherwise).

IV.

LEMMA 3. *Let (X,τ) satisfy all assumptions of Proposition 2. If $g_0, g_1 \colon X \to \mathbb{R}$ are quasi-continuous and $f = \max(g_0,g_1)$, then the set $A(f)$ of all points at which f is not quasi-continuous is nowhere dense and $\overline{\lim}_{\substack{x \to x_0 \\ x \in Q(f)}} f(x) \geq f(x_0)$ for each $x_0 \in A(f)$.*

Remark. Notice that if (X,τ) is the real line with the Euclidean topology, then $\overline{\lim}_{\substack{x \to x_0 \\ x \in Q(f)}} f(x) = \overline{\lim}_{\substack{x \to x_0 \\ x \in C(f)}} f(x)$ [4].

Proof. By Proposition 2 the set $A(f)$ is nowhere dense. Let us suppose that $x_0 \in A(f)$, $f(x_0) = g_0(x_0)$ and $\overline{\lim}_{\substack{x \to x_0 \\ x \in Q(f)}} f(x) < f(x_0)$. Then there exist a neighbourhood U of x_0 and a constant $c \in \mathbb{R}$ such that $f(x_0) > f(x_0) - c > \overline{\lim\limits_{\substack{x \to x_0 \\ x \in Q(f)}}} f(x)$ and therefore $f(x_0) - f(x) > c$ for each $x \in U \cap Q(f)$. Since $f(x) \geq g_0(x)$, we obtain $g_0(x_0) - g_0(x) > c$ for each $x \in U \cap Q(f)$. Since the set $Q(f)$ is dense in U, g_0 is not quasi-continuous.

PROPOSITION 3. *For every function $f \colon \mathbb{R} \to \mathbb{R}$ the following conditions are equivalent:*

(1) *$f = \max(g_0, g_1)$ for some functions $g_0, g_1 \in \mathcal{Q}$,*
(2) *the set $A(f)$ is nowhere dense and $\overline{\lim\limits_{\substack{x \to x_0 \\ x \in C(f)}}} f(x) \geq f(x_0)$ for each $x_0 \in A(f)$.*

P r o o f. The implication (1) $\Longrightarrow$ (2) follows from Lemma 3.

(2) $\Longrightarrow$ (1): Let $(I_n)_n$ be the sequence of all components of the set $\mathbb{R} \setminus \overline{A}(f)$, $I_n = (a_n, b_n)$ and $m_n = \min\left(\sup\limits_{I_n} f - \frac{1}{n}, n\right)$ for each $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ we choose 3 sequences of pairwise disjoint, closed subintervals $J_{n,i,m} \subset I_n$ ($i = 0, 1, 2$) with the following properties:

(i) the end-points of $J_{n,i,m}$ are continuity points of f,
(ii) $\overline{\bigcup\limits_{i,k} J_{n,i,k}} \setminus \bigcup\limits_{i,k} J_{n,i,k} = \{a_n, b_n\}$,
(iii) if $i = 0$ and $k = 1, 2, \ldots, n$, then $f(x) \geq m_n$ for $x \in \bigcup\limits_k J_{n\,0,k}$,
(iv) $J_{n,1,k} \underset{k \to \infty}{\searrow} a_n$ and $J_{n,2,k} \underset{k \to \infty}{\nearrow} b_n$ (The symbol $J_{n,1,k} \underset{k \to \infty}{\searrow} a_n$ means $x < y$ if $x \in J_{n,1,k}$, $y \in J_{n,1,t}$ and $k > t$, and $\{a_n\} = \overline{\bigcup\limits_k J_{n,1,k}} \setminus \bigcup\limits_k J_{n,1,k}$),
(v) if $i \in \{1, 2\}$ and $k = 1, 2, \ldots,$ then $\operatorname*{osc}\limits_{J_{n,i\,k}} f < \frac{1}{k}$,
(iv) if $\overline{\lim\limits_{\substack{x \to a_n^+ \\ x \in C(f)}}} f(x) \geq f(a_n)$, then $\inf\limits_{J_{n,1,k}} f \geq f(a_n) - \frac{1}{k}$ and if $\overline{\lim\limits_{x \to b_n^-}} f(x) \geq f(b_n)$ then $\inf\limits_{J_{n,2\,k}} f \geq f(b_n) - \frac{1}{k}$.

Lèt $(w_n)_n$ be a sequence of all rationals. We define functions $g_i \colon \mathbb{R} \to \mathbb{R}$ as follows.

$$g_i(x) = \begin{cases} f(x) & \text{for } x \in \overline{A(f)}, \\ w_k & \text{if } x \in J_{n,0,2k-i} \text{ and } w_k \le m_n,\ k = 1,2,\dots,E\left(\frac{n}{2}\right), \\ f(a_n) - \frac{1}{k} & \text{if } x \in J_{n,1,2k-i} \text{ and } \inf\limits_{J_{n,1,2k-i}} f + \frac{1}{k} \ge f(a_n), \\ f(b_n) - \frac{1}{k} & \text{if } x \in J_{n,2,2k-i} \text{ and } \inf\limits_{J_{n,2,2k-i}} f + \frac{1}{k} \ge f(b_n), \\ f(x) & \text{otherwise.} \end{cases}$$

It is easy to see that $f = \max(g_0, g_1)$. We shall verify that $g_0 \in \mathcal{Q}$. It is enough to prove that g_0 is quasi-continuous at every point $x_0 \in \overline{A(f)}$. Then $\varlimsup\limits_{\substack{x \to x_0 \\ x \in C(f)}} f(x) \ge f(x_0)$. Fix $\varepsilon > 0$ and a neighbourhood U of x_0. We shall consider two cases.

(a) x_0 is an end-point of some interval I_n (e.g. $x_0 = a_n$) and $\varlimsup\limits_{\substack{x \to a_n \\ x \in C(f)}} f(x) \ge f(x_0)$. Then there exists $k \in \mathbb{N}$ such that $\frac{1}{k} < \varepsilon$, $J_{n,1,2k} \subset U$ and $\inf\limits_{J_{n,1,2k}} f + \frac{1}{k} \ge f(x_0)$. We have $g_0(x) = f(a_n) - \frac{1}{k}$ for $x \in J_{n,1,2k}$ and hence $|g_0(x) - g_0(x_0)| < \varepsilon$ for $x \in J_{n,1,2k}$.

(b) There exists a subsequence $(I_{t_n})_n$ of the sequence $(I_n)_n$ such that $I_{t_n} \xrightarrow[n]{} x_0$ and $\lim\limits_{n \to \infty} m_{t_n} = \varlimsup\limits_{\substack{x \to x_0 \\ x \in C(f)}} f(x) \ge f(x_0)$. Let w_k be a rational number such that $\varepsilon > f(x_0) - w_k > \frac{1}{2}\varepsilon$. There exists n_0 such that $t_{n_0} \ge k$, $I_{t_{n_0}} \subset U$ and $f(x_0) - m_{t_{n_0}} < \frac{\varepsilon}{2}$. Then $J_{t_{n_0},0,2k} \subset U$ and $w_k \le m_{t_{n_0}}$ and consequently, $g_0(x) = w_k$ for $x \in J_{t_{n_0},0,2k}$. Thus $|g_0(x) - g_0(x_0)| < \varepsilon$ for $x \in J_{t_{n_0},0,2k}$. This finishes the proof of quasi-continuity of g_0 at the point x_0. Similarly we can verify that g_1 is quasi-continuous.

Remark 2. Obviously, a function being the maximum of quasi-continuous functions must be pointwise discontinuous.

If f is a function of the Baire class α $(\alpha \ge 1)$ or Lebesque measurable, then the functions g_0 and g_1 defined in the proof of Proposition 3 belong to the adequate class.

We shall verify this fact in the case when f is a Baire 1 function. Let $G \subset \mathbb{R}$ be an open set. Then $g_0^{-1}(G) = f^{-1}(G) \cap \Big(\overline{A(f)} \cup \Big(\mathbb{R} \setminus \Big(\overline{A(f)} \cup \bigcup\limits_{n,i,k} J_{n,i,k}\Big)\Big)\Big) \cup B$, where B is a sum of countably many closed intervals of a type $J_{n,i,k}$ (thus B is a F_σ set). Since the set $\overline{A(f)} \cup \bigcup_{n,i,k} J_{n,i,k}$ is closed, the set $\overline{A(f)} \cup \Big(\mathbb{R} \setminus \Big(\overline{A(f)} \cup \bigcup_{n,i,k} J_{n,i,k}\Big)\Big)$ is F_σ and consequently, $g_0^{-1}(G)$ is a F_σ set.

Remark 3. Similarly as in Lemma 3 we can prove the following implication. If $f = \max(g,h)$ for some quasi-continuous functions with the Darboux property $g,h\colon \mathbb{R} \to \mathbb{R}$, then

($*$) the set $A(f)$ is nowhere dense and $\min\big(\overline{\lim}_{\substack{x \to x_0^- \\ x \in C(f)}} f(x), \overline{\lim}_{\substack{x \to x_0^+ \\ x \in C(f)}} f(x)\big) \geq f(x_0)$

for each $x_0 \in \mathbb{R}$ (See [2], Theorem 3).
We are not able to prove that the condition ($*$) implies that there exist quasi-continuous functions with the Darboux property $g,h\colon \mathbb{R} \to \mathbb{R}$ such that $f = \max(g,h)$.

Acknowledgment. This paper was partially written when the author was visiting the Departamento de Matematicas Fundamentales de UNED in Madrid. The author would like to express his gratitude to the Department for the hospitality.

REFERENCES

[1] BLENDSOE, W. W.: *Neighbourly functions*, Proc. Am. Math. Soc. **3** (1952), 114–115.

[2] BRUCKNER, A. M.—CEDER, J. G.—PEARSON, T. L.: *On Darboux functions*, Rev. Roum. Math. Pures Appl. **19** (1974), 977–988.

[3] GRANDE, Z.: *Sur les fonctions cliquish*, Časopis Pěst. Mat. **110** (1985), 225–236.

[4] GRANDE, Z.—NATKANIEC, T.: *Lattices generated by τ-quasi-continuous functions*, Bull. Polish Acad. Sci. Math. **34** (1986), 525–530.

[5] GRANDE, Z.—NATKANIEC, T.—STROŃSKA, E.: *Algebraic structures generated by d-quasi-continuous functions*, Bull. Polish Acad. Sci. Math. **35** (1987), 717–723.

[6] GRANDE, Z.—SOLTYSIK, L.: *Some remarks on quasi-continuous real functions*, Problemy Mat. WSP Bydgoszcz **z.10** (1988), 79–87.

[7] NATKANIEC, T.: *Products of quasi-continuous functions*, Math. Slovaca **40** (1990), 401–405.

[8] NEUBRUNN, T.: *Quasi-continuity*, Real Anal. Exch. **14** (1988-89), 259–306.

Received September 22, 1989

Instytut Matematyki WSP
ul. Chodkiewicza 30
85-064 Bydgoszcz
Poland