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STATISTICAL MAPS: A CATEGORICAL APPROACH

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Dedicated to Professor Beloslav Riečan on the occasion of his 70th birthday

ABSTRACT. In probability theory, each random variable f can be viewed as channel through which the probability p of the original probability space is transported to the distribution p_f, a probability measure on the real Borel sets. In the realm of fuzzy probability theory, fuzzy probability measures (equivalently states) are transported via statistical maps (equivalently, fuzzy random variables, operational random variables, Markov kernels, observables). We deal with categorical aspects of the transportation of (fuzzy) probability measures on one measurable space into probability measures on another measurable spaces. A key role is played by D-posets (equivalently effect algebras) of fuzzy sets.

1. Introduction

Let $(\Omega, \mathbb{A}, p)$ be a probability space in the classical Kolmogorov sense (i.e. Ω is a set, $\mathbb{A}$ is a σ-field of subsets of Ω, and p is a probability measure on $\mathbb{A}$). A measurable map f of Ω into the real line R, called *random variable*, sends p into a probability measure p_f, called the *distribution* of f, defined on the real Borel sets $\mathbb{B}_R$ via $p_f(B) = p\big(f^{\leftarrow}(B)\big)$, $B \in \mathbb{B}_R$. In fact, f induces a map sending probability measures $P(\mathbb{A})$ on $\mathbb{A}$ into probability measures $P(\mathbb{B}_R)$ on $\mathbb{B}_R$ (each point $\omega \in \Omega$, or $r \in R$ is considered as a degenerated point probability measure). The preimage map $f^{\leftarrow}$, called *observable*, maps $\mathbb{B}_R$ into $\mathbb{A}$ and it is a sequentially continuous Boolean homomorphism. A *statistical map* (also fuzzy random variable or operational r.v.) is a "measurable" map sending probability

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measures $P(\mathbb{A})$ on the measurable space $(\Omega, \mathbb{A})$ into probability measures $P(\mathbb{B})$ on another measurable space $(\Xi, \mathbb{B})$, but it can happen that a point $\omega \in \Omega$ is mapped to a nondegenerated probability measure. Indeed, consider a random walk. Assume that after the first step we can end up in k possible states S_{1j}, $j = 1, \dots, k$, with probabilities p_{1j}, $j = 1, \dots, k$, and assume that in the nth step we can end up in l possible states S_{nj}, $j = 1, \dots, l$, with a given probability for each path. It is natural to distinguish three probability spaces: the input space (S_{1j}, $j = 1, \dots, k$, representing its elementary events), the path space, and the output space (S_{nj}, $j = 1, \dots, l$, representing its elementary events). In general, starting in a given S_{1j}, we can reach more than one final state S_{nj}, hence it is natural to consider a generalized random variable from the probability measures on the input space into the probability measures on the output space sending each S_{1j} (as an elementary event) into a probability measure assigning each subset S of the set $\{S_{nj} : j = 1, \dots, l\}$ of final states the probability that from S_{1j} in the nth step we end up in S. Such models lead to the so-called *fuzzy probability.* The corresponding observable is still *sequentially continuous*, but *sends fuzzy subsets into fuzzy subsets* (the image of a crisp set need not be crisp) and *preserves some operations on fuzzy sets.* The category ID of D-posets of fuzzy sets is suitable for modelling fundamental notions of fuzzy probability theory (cf. [12]). Details about fuzzy probability theory can be found, e.g., in [3], [14], [4], [5], [10], [12], [18], [21]. Note that "a fuzzy random variable" is sometimes used to denote a completely different notion (cf. [20]).

Definition 1.1. *Let $(\Omega, \mathbb{A})$, $(\Xi, \mathbb{B})$ be measurable spaces. Let T be a map of $P(\mathbb{A})$ into $P(\mathbb{B})$ such that, for each $B \in \mathbb{B}$, the assignment $\omega \mapsto \big(T(\delta_\omega)\big)(B)$ yields a measurable map of Ω into $[0,1]$ and*

$$\big(T(m)\big)(B) = \int \big(T(\delta_\omega)\big)(B)\,\mathrm{d}m \tag{BG}$$

for all $m \in P(\mathbb{A})$ and all $B \in \mathbb{B}$. Then T is said to be a ***statistical map*** *(also a fuzzy random variable in the sense of Bugajski and Gudder).*

Observe that if f is a classial measurable map of Ω into Ξ, then the distribution T_f of f (sending a probability p into $p_f = p \circ f^{\leftarrow}$) is a statistical map. Indeed, $\big(T_f(\delta_\omega)\big)(B) = 1$ iff $f(\omega) \in B$ and (BG) means $T_f(m) = m \circ f^{\leftarrow}$, $m \in P(\mathbb{A})$.

Example 1.2. Let $(\Omega, \mathbb{A})$, $(\Xi, \mathbb{B})$ be measurable spaces. For $q \in P(\mathbb{B})$, denote T_q the constant map of $P(\mathbb{A})$ into $P(\mathbb{B})$ sending each $m \in P(\mathbb{A})$ to q. Since for all $\omega \in \Omega$ we have $q(B) = \big(T_q(\delta_\omega)\big)(B)$, condition (BG) yields $q(B) = \big(T_q(m)\big)(B)$ for all $m \in P(\mathbb{A})$. Thus T_q is a statistical map. In a certain sense, T_q generalizes a classical degenerated measurable map. Each T_q, $q \in P(\mathbb{B})$, will be called a *degenerated statistical map.*

In Section 3, under additional asumptions, we shall construct a nondegerated statistical map sending a given $m \in P(\mathbb{A})$ to a given $q \in P(\mathbb{B})$.

Recall (cf. [15], [6]) that a *D-poset* is a quintuple $(E, \leq, \ominus, 0_E, 1_E)$ where E is a set, $\leq$ is a partial order, 0_E is the least element, 1_E is the greatest element, $\ominus$ is partial operation on E such that $a \ominus b$ is defined iff $b \leq a$, and the following axioms are assumed:

(D1) $a \ominus 0_E = a$ for each $a \in E$;

(D2) if $c \leq b \leq a$, then $a \ominus b \leq a \ominus c$ and $(a \ominus c) \ominus (a \ominus b) = b \ominus c$.

If no confusion can arise, then the quintuple $(E, \leq, \ominus, 0_E, 1_E)$ is condensed to E. A map h of a D-poset E into a D-poset F which preserves the D-structure is said to be a *D-homomorphism.*

It is known that D-posets are equivalent to effect algebras introduced in [7]. Interesting results about effect algebras, D-posets, and other quantum structures can be found in [6], [19].

Unless stated otherwise, I will denote the closed unit interval carrying the usual linear order and the usual D-structure: $a \ominus b$ is defined whenever $b \leq a$ and then $a \ominus b = a - b$. Analogously, if X is a set and I^X is the set of all functions on X into I, then we consider I^X as a D-poset in which the partial order and the partial operation $\ominus$ are defined pointwise: $b \leq a$ iff $b(x) \leq a(x)$ for all $x \in X$ and $a \ominus b$ is defined by $(a \ominus b)(x) = a(x) - b(x)$, $x \in X$. A subset $\mathcal{X} \subseteq I^X$ containing the constant functions 0_X, 1_X and closed with respect to the inherited partial operation "$\ominus$" is a typical D-poset we are interested in; we shall call it a *D-poset of fuzzy sets.*

Clearly, if we identify $A \subseteq X$ and the corresponding characteristic function $\chi_A \in I^X$, then each field $\mathbb{A}$ of subsets of X can be considered as a D-poset $\mathbb{A} \subseteq I^X$ of fuzzy sets: $\mathbb{A}$ is partially ordered ($\chi_B \leq \chi_A$ iff $B \subseteq A$) and then $\chi_A \ominus \chi_B$ is defined as $\chi_{A \setminus B}$ provided $B \subseteq A$.

Further, assume that I carries the usual sequential convergence and that I^X and other D-posets of fuzzy sets carry the pointwise sequential convergence. In what follows, we identify I and $I^{\{x\}}$, where $\{x\}$ is a singleton. Let $\mathbb{A}$ be a field of subsets of X considered as a D-poset of fuzzy sets and let p be a probability measure on $\mathbb{A}$. Then p, as a map of $\mathbb{A} \subseteq I^X$ into I, is *sequentially continuous.* It is easy to see that p is a D-homomorphism. On the other hand, for each sequentially continuous D-homomorphism h of $\mathbb{A} \subseteq I^X$ into I there exists a unique probability measure p on $\mathbb{A}$ such that $h = p$. In fact, fields of sets form a distinguished subcategory of the category of D-posets of fuzzy sets.

For more information concerning the σ-additivity and the sequential continuity of measures see [9], [13].

The category ID consists of the reduced D-posets of fuzzy sets carrying the pointwise convergence as objects and the sequentially continuous D-homomorphisms as morphisms. Note that the assumption that all objects of ID are reduced (each two points a, b of the underlying set X are separated by some fuzzy set $u \in \mathcal{X} \subseteq I^X$, i.e. $u(a) \neq u(b)$) plays the same role as the Hausdorff separation axiom T_2: limits are unique and the continuous extensions from dense subobjects are uniquely determined (cf. [17]).

2. Measurable maps and random maps

This section is devoted to classical measurable spaces and measurable maps, resp. classical probability spaces and measure preserving measurable maps (such maps will be called *random maps*). We summarize some basic properties of the coresponding categories and indicate possible generalizations.

By a classial measurable space we understand a pair $(\Omega, \mathbb{A})$, where Ω is a set and $\mathbb{A}$ is a σ-field of so-called measurable subsets of Ω (we could start with a field $\mathbb{A}_0$ of subsets of Ω and then to pass to the generated σ-field $\mathbb{A} = \sigma(\mathbb{A}_0)$; this yields a functor and many results about fields of sets can be translated to the corresponding results about σ-fields). We shall always assume that singletons $\{\omega\}$, $\omega \in \Omega$, are measurable. By a measurable map from a measurable space $(\Omega, \mathbb{A})$ to a measurable space $(\Xi, \mathbb{B})$ we understand a map $f \colon \Omega \to \Xi$ such that for each measurable set B in $\mathbb{B}$ the peimage $f^{\leftarrow}(B) = \big\{\omega \in \Omega : \ f(\omega) \in B\big\}$ is a measurable set in $\mathbb{A}$. Since the characteristic function $\chi_{f^{\leftarrow}(B)}$ is the composition $\chi_B \circ f$ of f and the characteristic function χ_B of B, the measurability of f can be expressed in terms of the composition of f and the characteristic functions of measurable sets (cf. [8]).

The composition of two measurable maps is a measurable map and this leads to the category MS the objects of which are measurable spaces and the morphisms of which are measurable maps. It is known that the category MS has products. Indeed, let $\big\{(\Omega_s, \mathbb{A}_s) : \ s \in S\big\}$ be an indexed family of measurable spaces, then the usual product space $\Big(\prod\limits_{s\in S} \Omega_s, \prod\limits_{s\in S} \mathbb{A}_s\Big)$, together with the indexed family $\{\mathrm{pr}_t : \ t \in S\}$ of projections (pr_t maps $\prod\limits_{s\in S} \Omega_s$ onto Ω_t and sends $\{\omega_s : \ s \in S\}$ to ω_t), is the categorical product in MS. This means that if $(\Omega, \mathbb{A})$ is a measurable space and, for each $s \in S$, f_s is a measurable map of $(\Omega, \mathbb{A})$ into $(\Omega_s, \mathbb{A}_s)$, then there exists a unique measurable map f of $(\Omega, \mathbb{A})$ into

$\Big(\prod\limits_{s\in S}\Omega_s, \prod\limits_{s\in S}\mathbb{A}_s\Big)$ such that $\mathrm{pr}_s \circ f = f_s$ for all $s \in S$. Of course, the projections are measurable and $f(\omega) = \big\{f_s(\omega) : s \in S\big\}$.

The next definition is motivated by [2] (dealing with joint obsevables).

Definition 2.1. *Let $\big\{(\Omega_s, \mathbb{A}_s) : s \in S\big\}$ be an indexed family of measurable spaces, let $(\Omega, \mathbb{A})$ be a measurable space and, for each $s \in S$, let f_s be a measurable map of $(\Omega, \mathbb{A})$ to $(\Omega_s, \mathbb{A}_s)$. Let $(\Xi, \mathbb{B})$ be a measurable space. If for each $s \in S$ there is a measurable map g_s of $(\Xi, \mathbb{B})$ to $(\Omega_s, \mathbb{A}_s)$ and there is a measurable map f of $(\Omega, \mathbb{A})$ to $(\Xi, \mathbb{B})$ such that $g_s \circ f = f_s$, then $(\Xi, \mathbb{B})$ is said to be a* ***joint measurable space with respect to*** *$\{f_s : s \in S\}$; g_s, $s \in S$, are said to be* ***marginal projections*** *and f is said to be a* ***joint measurable map****.*

Observe that $(\Omega, \mathbb{A})$ is a trivial joint measurable space and it is easy to see that joint measurable spaces are plentiful. As the next proposition shows, there is a universal one depending only on $\big\{(\Omega_s, \mathbb{A}_s) : s \in S\big\}$, namely, their pruduct.

Proposition 2.2. *Let $\big\{(\Omega_s, \mathbb{A}_s) : s \in S\big\}$ be an indexed family of measurable spaces, let $(\Omega, \mathbb{A})$ be a measurable space and, for each $s \in S$, let f_s be a measurable map of $(\Omega, \mathbb{A})$ into $(\Omega_s, \mathbb{A}_s)$. Then the product space $\Big(\prod\limits_{s\in S}\Omega_s, \prod\limits_{s\in S}\mathbb{A}_s\Big)$ is a joint measurable space with respect to $\big\{f_s : s \in S\big\}$, the projections $\big\{\mathrm{pr}_t : t \in S\big\}$ are marginal projections, and the joint measurable map is uniquely defined.*

Proof. The assertions follow directly from the properties of a categorical product. □

Let f be a classical measurable map of a measurable space $(\Omega, \mathbb{A})$ into a measurable space $(\Xi, \mathbb{B})$. As already stated in Section 1, the distribution T_f of f is a statistical map. It sends each probability p on $\mathbb{A}$ to the probability $p_f = p \circ f^{\leftarrow}$ on $\mathbb{B}$ and, in particular, it sends each degenerated probability δ_ω, $\omega \in \Omega$ to the degenerated probability $\delta_{f(\omega)}$. Statistical maps sending degenerated (pure) probabilities to degenerated probabilities are called *deterministic* (cf. [2], where $P(\mathbb{A})$ is denoted $M_1^+(\Omega)$ and a statistical map is called an observable). In fact, classical measurable maps are naturally equivalent to deterministic statistical maps (if $\Omega \neq \delta(M_1^+(\Omega))$, then two different classical measurable maps can define the same deterministic statistical map).

In the next section we pass from classical measurable maps to statistical maps. The remaining part of the present section is devoted to measure preserving measurable maps.

Definition 2.3. *Let $(\Omega, \mathbb{A}, p)$ and $(\Xi, \mathbb{B}, q)$ be probability spaces and let f be a measurable map of Ω into Ξ. If $q = p \circ f^{\leftarrow}$, i.e. $q(B) = p\big(f^{\leftarrow}(B)\big)$ for all $B \in \mathbb{B}$, then we say that f* ***preserves measure*** *and f is called a* ***random map*** *of $(\Omega, \mathbb{A}, p)$ to $(\Xi, \mathbb{B}, q)$.*

Denote PS the category of probability spaces and random maps. Clearly, each random variable is a random map.

Example 2.4. Let $\Omega = \{a, b\}$, $\mathbb{A} = 2^{\Omega}$ (as a rule, we identify a subset and its characteristic function; if X is a set, then 2^X denotes the σ-field of all subsets of X), $\Xi = \{a, b\}$, $\mathbb{B} = 2^{\Xi}$, let p be the uniform probability measure on $\mathbb{A}$ (defined by $p\big(\{a\}\big) = p\big(\{b\}\big) = \frac{1}{2}$), and let q be the uniform probability measure on $\mathbb{B}$. We claim that the probability spaces $(\Omega, \mathbb{A}, p)$ and $(\Xi, \mathbb{B}, q)$ do not have a categorical product in the category PS of probability spaces and random maps.

Contrariwise, suppose that $(\Lambda, \mathbb{C}, m)$, together with projections $\mathrm{pr}_{\Omega} \colon \Lambda \to \Omega$ and $\mathrm{pr}_{\Xi} \colon \Lambda \to \Xi$, is their categorical product. Clearly, the usual product $\big(\Omega \times \Xi, 2^{\Omega \times \Xi}, p \times q\big)$ is a probability space and the projection maps $f_{\Omega} \colon \Omega \times \Xi \to \Omega$, sending $(x, y) \in \Omega \times \Xi$ to $x \in \Omega$ and $f_{\Xi} \colon \Omega \times \Xi \to \Xi$, sending $(x, y) \in \Omega \times \Xi$ to $y \in \Xi$, are random maps of $\big(\Omega \times \Xi, 2^{\Omega \times \Xi}, p \times q\big)$ to $(\Omega, \mathbb{A}, p)$ and $(\Xi, \mathbb{B}, q)$, respectively. According to the definition of a categorical product, (cf. [1]) there exists a unique random map f of $\big(\Omega \times \Xi, 2^{\Omega \times \Xi}, p \times q\big)$ to $(\Lambda, \mathbb{C}, m)$ such that $\mathrm{pr}_{\Omega} \circ f = f_{\Omega}$ and $\mathrm{pr}_{\Xi} \circ f = f_{\Xi}$. Denote $A = \mathrm{pr}_{\Omega}^{\leftarrow}\big(\{a\}\big) \in \mathbb{C}$, $B = \mathrm{pr}_{\Omega}^{\leftarrow}\big(\{b\}\big) \in \mathbb{C}$, $C = \mathrm{pr}_{\Xi}^{\leftarrow}\big(\{c\}\big) \in \mathbb{C}$, $D = \mathrm{pr}_{\Xi}^{\leftarrow}\big(\{d\}\big) \in \mathbb{C}$. Then the sets $A \cap C$, $A \cap D$, $B \cap C$, and $B \cap D$ form a measurable partition of Λ (they are mutually disjoint and their union is the set Λ) and $f(a, c) \in A \cap C$, $f(a, d) \in A \cap D$, $f(b, c) \in B \cap C$, $f(b, d) \in B \cap D$. Since the singletons are measurable sets and f is a measurable map, necessarily $m\big(\{f(x, y)\}\big) = (p \times q)\big(\{x, y\}\big) = \frac{1}{4}$ for all $(x, y) \in \Omega \times \Xi$ and $m(A \cap C) = m(A \cap D) = m(B \cap C) = m(B \cap D) = \frac{1}{4}$.

Now, the contradiction follows from the fact that, besides $p \times q$, on $2^{\Omega \times \Xi}$ there exists another measure $r \neq p \times q$ such that $r \circ f_{\Omega}^{\leftarrow} = p$ and $r \circ f_{\Xi}^{\leftarrow} = q$ (the marginal projections of r are p and q, respectively).

Indeed, let r be the probability measure on $2^{\Omega \times \Xi}$ defined by $r\big(\{a, c\}\big) = r\big(\{b, d\}\big) = \frac{3}{8}$ and $r\big(\{b, c\}\big) = r\big(\{a, d\}\big) = \frac{1}{8}$. Clearly, $\big(\Omega \times \Xi, 2^{\Omega \times \Xi}, r\big)$ is a probability space and the projection maps f_{Ω} and f_{Ξ} are random maps of $\big(\Omega \times \Xi, 2^{\Omega \times \Xi}, r\big)$ to $(\Omega, \mathbb{A}, p)$ and $(\Xi, \mathbb{B}, q)$, respectively. Then there is a unique random map g of $\big(\Omega \times \Xi, 2^{\Omega \times \Xi}, r\big)$ to $(\Lambda, \mathbb{C}, m)$ such that $\mathrm{pr}_{\Omega} \circ g = f_{\Omega}$ and $\mathrm{pr}_{\Xi} \circ g = f_{\Xi}$. Again, from $g(a, c) \in A \cap C$, $g(a, d) \in A \cap D$, $g(b, c) \in B \cap C$, and $g(b, d) \in B \cap D$ it follows that $m(A \cap C) = r\big(\{a, c\}\big) = r\big(\{b, d\}\big)\ m(B \cap D) = \frac{3}{8}$ and $m(B \cap C) = r\big(\{b, c\}\big) = r\big(\{a, d\}\big) = m(A \cap D) = \frac{1}{8}$. This is a contradiction.

COROLLARY 2.5. *The category PS of probability spaces and random maps is not productive.*

DEFINITION 2.6. *Let $\{(\Omega_s, \mathbb{A}_s, p_s) : s \in S\}$ be an indexed family of probability spaces, let $(\Omega, \mathbb{A}, p)$ be a probability space and, for each $s \in S$, let f_s be a random map of $(\Omega, \mathbb{A}, p)$ to $(\Omega_s, \mathbb{A}_s, p_s)$. Let $(\Xi, \mathbb{B}, q)$ be a probability space. If for each $s \in S$ there is a random map g_s of $(\Xi, \mathbb{B}, q)$ to $(\Omega_s, \mathbb{A}_s, p_s)$ and there is a random map f of $(\Omega, \mathbb{A}, p)$ to $(\Xi, \mathbb{B}, q)$ such that $g_s \circ f = f_s$, then $(\Xi, \mathbb{B})$ is said to be a* ***joint probability space with respect to*** *$\{f_s : s \in S\}$; f is said to be a* ***joint random map*** *and g_s, $s \in S$, are said to be* ***marginal projections****.*

Observe that $(\Omega, \mathbb{A}, p)$ is a trivial joint probability space. A simple modification of the example above shows that the usual product of probability spaces fails to be a universal joint probability space of the factor probability spaces. The category PS is simply "too big". We shall describe products, hence universal joint probability spaces in a comma category over a fixed "base" probability space $(\Omega_b, \mathbb{A}_b, p_b)$.

Let $(\Omega_b, \mathbb{A}_b, p_b)$ be a probability space. The comma category $PS(p_b)$ of "probability spaces over $(\Omega_b, \mathbb{A}_b, p_b)$" is defined as follows. The objects of $PS(p_b)$ are random maps of the base probability space $(\Omega_b, \mathbb{A}_b, p_b)$: if $(\Omega, \mathbb{A}, p)$ is a probability space and f is a random map of $(\Omega_b, \mathbb{A}_b, p_b)$ to $(\Omega, \mathbb{A}, p)$, then the corresponding object of $PS(p_b)$ is denoted by $\big(f, (\Omega, \mathbb{A}, p)\big)$. Morphisms of $PS(p_b)$ are defined as follows: a morphism of an object $\big(f_1, (\Omega_1, \mathbb{A}_1, p_1)\big)$ to an object $\big(f_2, (\Omega_2, \mathbb{A}_2, p_2)\big)$ is a random map g of $(\Omega_1, \mathbb{A}_1, p_1)$ to $(\Omega_2, \mathbb{A}_2, p_2)$ such that $g \circ f_1 = f_2$.

Let $\{(f_s, (\Omega_s, \mathbb{A}_s, p_s)) : s \in S\}$ be an indexed family of objects of the category $PS(p_b)$. Let $(\Omega, \mathbb{A}) = \Big(\prod_{s\in S} \Omega_s, \prod_{s \in S} \mathbb{A}_s\Big)$, together with the indexed family $\{\mathrm{pr}_t : t \in S\}$ of projections, be the (categorial) product measurable space of the family $\{(\Omega_s, \mathbb{A}_s) : s \in S\}$. Then the unique measurable map f of $(\Omega_b, \mathbb{A}_b)$ to $(\Omega, \mathbb{A})$ such that $\mathrm{pr}_s \circ f = f_s$, for all $s \in S$, is defined by $f(\omega) = \{f_s(\omega) : s \in S\}$. Put $p_f = p_b \circ f^{\leftarrow}$. Since $\mathrm{pr}_s \circ f = f_s$ for all $s \in S$, $(\Omega, \mathbb{A}, p_f)$ is a joint probability space with respect to $\{f_s : s \in S\}$.

PROPOSITION 2.7. *Let $\{(f_s, (\Omega_s, \mathbb{A}_s, p_s)) : s \in S\}$ be an indexed family of objects of the category $PS(p_b)$. Then $\big(f, (\Omega, \mathbb{A}, p_f)\big) = \Big(f, \Big(\prod_{s\in S} \Omega_s, \prod_{s\in S} \mathbb{A}_s, p_f\Big)\Big)$, together with the indexed family $\{\mathrm{pr}_s : s \in S\}$ of projections, is the categorical product (in $PS(p_b)$) of $\{(f_s, (\Omega_s, \mathbb{A}_s, p_s)) : s \in S\}$.*

P r o o f. It follows from the construction of $\Big(\prod\limits_{s\in S}\Omega_s,\prod\limits_{s\in S}\mathbb{A}_s,p_f\Big)$ that $\mathrm{pr}_s\circ f=f_s$ and $p_f\circ\mathrm{pr}_s^{\leftarrow}=p_s$, for all $s\in S$. Thus $\big(f,(\Omega,\mathbb{A},p_f)\big)$ is an object of $PS(p_b)$ and each pr_s, $s\in S$, is a morphism of $PS(p_b)$. Let $\big(g,(\Xi,\mathbb{B},q)\big)$ be an object of $PS(p_b)$ and, for each $s\in S$, let g_s be a morphism of $\big(g,(\Xi,\mathbb{B},q)\big)$ to $\big(f_s(\Omega_s,\mathbb{A}_s,p_s)\big)$, i.e. g_s a random map of $(\Xi,\mathbb{B},q)$ to $(\Omega_s,\mathbb{A}_s,p_s)$ such that $g\circ g_s=f_s$. Clearly, there exists a unique morphism h of $\big(g,(\Xi,\mathbb{B},q)\big)$ to $\big(f,(\Omega,\mathbb{A},p_f)\big)$ such that $\mathrm{pr}_s\circ h=g_s$. Namely, $h(\xi)=\big\{g_s(\xi):\ s\in S\big\}$, $\xi\in\Xi$. This completes the proof. □

COROLLARY 2.8. *Let $\big\{(f_s,(\Omega_s,\mathbb{A}_s,p_s)):\ s\in S\big\}$ be an indexed family of objects of the category $PS(p_b)$ and let $\Big(f,\Big(\prod\limits_{s\in S}\Omega_s,\prod\limits_{s\in S}\mathbb{A}_s,p_f\Big)\Big)$ be their categorical product in $PS(p_b)$. Let $\big(g,(\Xi,\mathbb{B},q)\big)$ be an object of $PS(p_b)$ and, for each $s\in S$, let g_s be a morphism of $\big(g,(\Xi,\mathbb{B},q)\big)$ to $\big(f_s(\Omega_s,\mathbb{A}_s,p_s)\big)$. Then the probability space $\Big(\prod\limits_{s\in S}\Omega_s,\prod\limits_{s\in S}\mathbb{A}_s,p_f\Big)$, together with the marginal projections $\{\mathrm{pr}_s:\ s\in S\}$, is a joint probability space with respect to $\{g_s:\ s\in S\}$ and the joint random map h of $(\Xi,\mathbb{B},q)$ to $\Big(\prod\limits_{s\in S}\Omega_s,\prod\limits_{s\in S}\mathbb{A}_s,p_f\Big)$ is uniquely determined.*

In fact, the construction of the product in $PS(p_b)$ yields the existence of a maximal (remember, not universal) joint probability space. Indeed, it is easy to check that the following holds.

COROLLARY 2.9. *Let $(\Omega,\mathbb{A},p)$ be a probability space. Let $\big\{(\Omega_s,\mathbb{A}_s,p_s):\ s\in S\big\}$ be an indexed family of probability spaces and, for each $s\in S$, let f_s be a random map of $(\Omega,\mathbb{A},p)$ to $(\Omega_s,\mathbb{A}_s,p_s)$. Let $(\Xi,\mathbb{B},q)$ be a joint probability space with respect to $\{f_s:\ s\in S\}$, for each $s\in S$ let g_s be a marginal projection of $(\Xi,\mathbb{B},q)$ to $(\Omega_s,\mathbb{A}_s,p_s)$, and let g be a joint random map of $(\Omega,\mathbb{A},p)$ to $(\Xi,\mathbb{B},q)$. Then $\Big(\prod\limits_{s\in S}\Omega_s,\prod\limits_{s\in S}\mathbb{A}_s,p_f\Big)$, where f is the measurable map of $(\Omega,\mathbb{A})$ to $\Big(\prod\limits_{s\in S}\Omega_s,\prod\limits_{s\in S}\mathbb{A}_s\Big)$ defined by $f(\omega)=\big\{f_s(\omega):\ s\in S\big\}$, $\omega\in\Omega$, is a joint measurable space with respect to $\{f_s:\ s\in S\}$ and there is a unique map h of $(\Xi,\mathbb{B},q)$ to $\Big(\prod\limits_{s\in S}\Omega_s,\prod\limits_{s\in S}\mathbb{A}_s,p_f\Big)$ such that $h\circ g=f$ and $\mathrm{pr}_s\circ h=g_s$ for all $s\in S$.*

Recall that two random variables f, g on $(\Omega,\mathbb{A},p)$ are usually (cf. [16]) said to be equivalent if $p\big(\{\omega\in\Omega:\ f(\omega)\neq g(\omega)\}\big)=0$. Consequently, the distributions p_f of f and p_g of g are the same probabilities on the real Borel sets $\mathbb{B}_R$. This leads to a much coarser equivalence. Namely, in the category PS of probability spaces, each two random maps f, g of $(\Omega, \mathbb{A}, p)$ to $(\Xi, \mathbb{B}, q)$ are equivalent in the sense that $q = p \circ f^{\leftarrow} = p \circ f^{\leftarrow}$. This way we get a quotient category PS^{-}: the objects are probability spaces (the same as the objects of PS); the morphisms are equivalence classes of random maps. Hence in PS^{-} there is at most one morphism of $(\Omega, \mathbb{A}, p)$ to $(\Xi, \mathbb{B}, q)$. More information about quotient categories can be found in [1].

Since statistical maps generalize measurable maps, it might be interesting to generalize the results of this section to statistical maps and, further, to ID-posets.

3. Statistical maps and transportation of probabilities

The theory and applications of statistical maps (see Definition 1.1) is outlined in [4], [5], [14]. An alternative approach to statistical maps is via ID-posets (see [12]). The advantage of this approach is that many technical theorems can be reduced to categorical handling of arrows and diagrams and some generalizations are more natural.

Let $(\Omega, \mathbb{A})$ be a measurable space and let $\mathcal{M}(\mathbb{A})$ be the set of all measurable functions ranging in the closed unit interval $I = [0,1]$. Then $\mathcal{M}(\mathbb{A}) \subseteq I^{\Omega}$ carrying the natural D-poset structure and pointwise sequential convergence (see Introduction) is a distinguished D-poset of fuzzy sets. Identifying each point $\omega \in \Omega$ and the corresponding (degenerated) point probability measure δ_{ω}, for $u \in \mathcal{M}(\mathbb{A})$ define $\mathrm{ev}(u) = u^{*} \in I^{P(\mathbb{A})}$ by $u^{*}(p) = \int u \, \mathrm{d}p$, $p \in P(\mathbb{A})$. Then $\mathcal{M}^{*}(\mathbb{A}) = \{u^{*} : u \in \mathcal{M}(\mathbb{A})\} \subseteq I^{P(\mathbb{A})}$, carrying the inherited difference and convergence structures, becomes an object of ID.

Let $(\Omega, \mathbb{A})$, $(\Xi, \mathbb{B})$ be measurable spaces and let $\mathcal{M}^{*}(\mathbb{A})$, $\mathcal{M}^{*}(\mathbb{B})$ be the corresponding objects of ID. As proved in [12], a map T of $P(\mathbb{A})$ to $P(\mathbb{B})$ is a statistical map iff for each $v^{*} \in \mathcal{M}^{*}(\mathbb{B})$ the composition $v^{*} \circ T$ belongs to $\mathcal{M}^{*}(\mathbb{A})$. Using this fact, it is easy to see that the composition of two statistical maps is a statistical map, too.

Recall that if $\mathcal{X} \subseteq I^{X}$ and $\mathcal{Y} \subseteq I^{Y}$ are ID-posets, then $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ are called ID-maesurable spaces and a map f of X into Y such that $\mathcal{Y} \circ f \subseteq \mathcal{X}$ is said to be measurable.

Definition 3.1. *Let $(\Omega, \mathbb{A})$ be a measurable space. Then $\big(P(\mathbb{A}), \mathcal{M}^{*}(\mathbb{A})\big)$ is said to be an* ***extended measurable space****.*

Denote EMS the category of extended measurable spaces and statistical maps. Statistical maps will also be called extended measurable maps.

Definition 3.2. *Let $(\Omega, \mathbb{A}, p)$ be a probability space. Then $\big(P(\mathbb{A}), \mathcal{M}^*(\mathbb{A}), p\big)$ is said to be an* extended probability space. *Let $(\Xi, \mathbb{B}, q)$ be another probability space and let T be a statistical map of $P(\mathbb{A})$ to $P(\mathbb{B})$ such that $T(p) = q$. Then T is said to be an* ***extended random map*** *of $(\Omega, \mathbb{A}, p)$ to $(\Xi, \mathbb{B}, q)$.*

This section is devoted to the existence of random maps and extended random maps in some simple situations.

Let $(\Omega_1, \mathbb{A}_1, p_1)$ and $(\Omega_2, \mathbb{A}_2, p_2)$ be probability spaces. Answers to the following questions will help to understand the nature of a random map.

Q1. Is there a random map f of $(\Omega_1, \mathbb{A}_1, p_1)$ to $(\Omega_2, \mathbb{A}_2, p_2)$?

Q2. Is there an extended random map of $(\Omega_1, \mathbb{A}_1, p_1)$ to $(\Omega_2, \mathbb{A}_2, p_2)$?

The answer to the second question is YES. Just consider the degenerated statistical map T_{p_2} sending each $m \in P(\mathbb{A})$ to p_2. For discrete probability spaces we shall construct a nondegenerated extended random map.

It is easy to see that (even under additional assuptions) the answer to the first question is NO. Indeed, let $(\Omega, \mathbb{A}, p)$ be a discrete probability space, e.g., let $\Omega = \{\omega_1, \omega_2, \omega_3\}$, let $p\big(\{\omega_i\}\big) > 0$ for all $\omega_i \in \Omega$, let $(\Omega, \mathbb{A}, p) = (\Omega_2, \mathbb{A}_2, p_2)$, let $\Omega_1 = \{a, b\}$, let f be a map of Ω onto Ω_1 defined by $f(\omega_1) = f(\omega_2) = a$, $f(\omega_3) = b$, and let $p_1 = p \circ f^{\leftarrow}$. Then both $(\Omega_1, \mathbb{A}_1, p_1)$ and $(\Omega_2, \mathbb{A}_2, p_2)$ are objects of the comma category $PS(p)$ of probability spaces over $(\Omega, \mathbb{A}, p)$ and there is no random map of $(\Omega_1, \mathbb{A}_1, p_1)$ to $(\Omega_2, \mathbb{A}_2, p_2)$.

Next, let f be a random map of $(\Omega_1, \mathbb{A}_1, p_1)$ to $(\Omega_2, \mathbb{A}_2, p_2)$. What can be said about spaces $(\Omega, \mathbb{A}, p)$ such that f is a morphism of the comma category $PS(p)$ of probability spaces over $(\Omega, \mathbb{A}, p)$?

Definition 3.3. *Let f be a random map of a probability space $(\Omega_1, \mathbb{A}_1, p_1)$ to a probability space $(\Omega_2, \mathbb{A}_2, p_2)$. Let $(\Omega, \mathbb{A}, p)$ be a probability space and, for $i = 1, 2$, let f_i be a random map of $(\Omega, \mathbb{A}, p)$ to $(\Omega_i, \mathbb{A}_i, p_i)$. If $f \circ f_1 = f_2$, then $(\Omega, \mathbb{A}, p)$ together with f_1, f_2 is said to be a* ***base probability space for*** *f and f_1, f_2 are said to be* ***base projections***.

Note that $(\Omega, \mathbb{A}, p)$ is a base probability space for f iff $\big(f_i, (\Omega_i, \mathbb{A}_i, p_i)\big)$, $i = 1, 2$, are objects of the comma category $PS(p)$ and f is a morphism of $PS(p)$.

Next we show that for each random map there is "the best" base probability space. Let f be a random map of a probability space $(\Omega_1, \mathbb{A}_1, p_1)$ to a probability space $(\Omega_2, \mathbb{A}_2, p_2)$. Consider the product set $\Omega_1 \times \Omega_2$ and define $\Omega = \big\{(\xi, \lambda) \in \Omega_1 \times \Omega_2 : \ \lambda = f(\xi)\big\}$, for $A \in \mathbb{A}_1$ put $A_\Omega = \big\{(\xi, f(\xi)) \in \Omega : \ \xi \in A\big\}$, $\mathbb{A} = \big\{A_\Omega : \ A \in \mathbb{A}_1\big\}$, and define a map p on $\mathbb{A}$ by $p(A_\Omega) = p_1(A)$, $A \in \mathbb{A}_1$. Further, define maps f_1 of Ω to Ω_1 and f_2 of Ω to Ω_2 by $f_1\big((\xi, f(\xi))\big) = \xi$ and $f_2\big((\xi, f(\xi))\big) = f(\xi)$. A straightforward proof of the following lemma is omitted.

Lemma 3.4. *$(\Omega, \mathbb{A}, p)$ together with f_1, f_2 is a base probability space for f.*

In what follows, $(\Omega, \mathbb{A}, p)$ will be denoted by $(\Omega_1, \mathbb{A}_1, p_1) \times_f (\Omega_2, \mathbb{A}_2, p_2)$.

Definition 3.5. *Let f be a random map of a probability space $(\Omega_1, \mathbb{A}_1, p_1)$ to a probability space $(\Omega_2, \mathbb{A}_2, p_2)$. Then $(\Omega_1, \mathbb{A}_1, p_1) \times_f (\Omega_2, \mathbb{A}_2, p_2)$ together with f_1, f_2 is said to be the **f-product** of $(\Omega_1, \mathbb{A}_1, p_1)$ and $(\Omega_2, \mathbb{A}_2, p_2)$.*

Theorem 3.6. *Let f be a random map of a probability space $(\Omega_1, \mathbb{A}_1, p_1)$ to a probability space $(\Omega_2, \mathbb{A}_2, p_2)$ and let $(\Omega, \mathbb{A}, p) = (\Omega_1, \mathbb{A}_1, p_1) \times_f (\Omega_2, \mathbb{A}_2, p_2)$ together with f_1, f_2 be their f-product. Let $(\Xi, \mathbb{B}, q)$ together with g_1, g_2 be a base probability space for f. Then there is a unique random map g of $(\Xi, \mathbb{B}, q)$ to $(\Omega_1, \mathbb{A}_1, p_1) \times_f (\Omega_2, \mathbb{A}_2, p_2)$ such that $f_i \circ g = g_i$, $i = 1, 2$.*

P r o o f. Let $\xi \in \Xi$. According to the assumptions we have $f\big(g_1(\xi)\big) = g_2(\xi)$. Define $g(\xi) = \big(g_1(\xi), g_2(\xi)\big) = \big(g_1(\xi), f(g_1(\xi))\big)$. Then $f_i\big(g(\xi)\big) = g_i(\xi)$, $i = 1, 2$. Let $A_\Omega \in \mathbb{A}$. Then $q\big(g^{\leftarrow}(A_\Omega)\big) = q\big(g_1^{\leftarrow}(A)\big) = p_1(A) = p(A_\Omega)$ and hence g is a random map. Clearly, if h is a map of Ξ into Ω such that $f_i \circ h = g_i$, $i = 1, 2$, then $h(\xi) = g(\xi)$. This completes the proof. □

Now, let us turn to extended random maps. Let $(\Omega_b, \mathbb{A}_b, p_b)$ be a probability space. The comma category $EPS(p_b)$ of "extended probability spaces over $(\Omega_b, \mathbb{A}_b, p_b)$" is defined as follows. The objects of $EPS(p_b)$ are exteded random maps of the base probability space $(\Omega_b, \mathbb{A}_b, p_b)$: if $(\Omega, \mathbb{A}, p)$ is a probability space and T is an extended random map of $(\Omega_b, \mathbb{A}_b, p_b)$ to $(\Omega, \mathbb{A}, p)$, then the corresponding object of $EPS(p_b)$ is denoted by $\big(T, (\Omega, \mathbb{A}, p)\big)$. Morphisms of $EPS(p_b)$ are defined as follows: a morphism of an object $\big(T_1, (\Omega_1, \mathbb{A}_1, p_1)\big)$ to an object $\big(T_2, (\Omega_2, \mathbb{A}_2, p_2)\big)$ is an extended random map S of $(\Omega_1, \mathbb{A}_1, p_1)$ to $(\Omega_2, \mathbb{A}_2, p_2)$ such that $S \circ T_1 = T_2$.

Definition 3.7. *Let T be an extended random map of a probability space $(\Omega_1, \mathbb{A}_1, p_1)$ to a probability space $(\Omega_2, \mathbb{A}_2, p_2)$. Let $(\Omega, \mathbb{A}, p)$ be a probability space and, for $i = 1, 2$, let T_i be an extended random map of $(\Omega, \mathbb{A}, p)$ to $(\Omega_i, \mathbb{A}_i, p_i)$. If $T_1 \circ T = T_2$, then $(\Omega, \mathbb{A}, p)$ together with T_1, T_2 is said to be a **base probability space for** T and T_1, T_2 are said to be **base projections**.*

Note that $(\Omega, \mathbb{A}, p)$ is a base probability space for T iff $\big(T_i, (\Omega_i, \mathbb{A}_i, p_i)\big)$, $i = 1, 2$, are objects of the comma category $EPS(p)$ and T is a morphism of $EPS(p)$.

Next we show that for each random map there is "the best" base probability space. Let T be an extended random map of a probability space $(\Omega_1, \mathbb{A}_1, p_1)$ to a probability space $(\Omega_2, \mathbb{A}_2, p_2)$. Consider the product set $P(\mathbb{A}_1) \times P(\mathbb{A}_2)$ and define $\Omega = \big\{(\xi, m) \in \Omega_1 \times P(\mathbb{A}_2) : \ m = T(\xi)\big\}$, for $A \in \mathbb{A}_1$ put $A_\Omega =$

$\{(\xi, T(\xi)) \in \Omega : \xi \in \Omega_1\}$, $\mathbb{A} = \{A_\Omega : A \in \mathbb{A}_1\}$. The one-to-one map of Ω to Ω_1 sending $(\xi, T(\xi))$ to ξ, hence sending each $A_\Omega \in \mathbb{A}$ to $A \in \mathbb{A}_1$, extends to a one-to-one map T_1, sending $(q, T(q))$ to q, of $\{(q, T(q)) \in P(\mathbb{A}_1) \times P(\mathbb{A}_2) : q \in P(\mathbb{A}_1)\}$ onto $P(\mathbb{A}_1)$. Since $\mathbb{A}$ an $\mathbb{A}_1$ are isomorphic, $P(\mathbb{A})$ and $\{(q, T(q)) \in P(\mathbb{A}_1) \times P(\mathbb{A}_2) : q \in P(\mathbb{A}_1)\}$ can be identified. Then T_1 becomes a map of $P(\mathbb{A})$ onto $P(\mathbb{A}_1)$. Denote p the unique probability measure on $\mathbb{A}$ which corresponds to p_1 and define $T_2 = T \circ T_1$. A straightforward proof of the next lemma is omitted.

Lemma 3.8. *$(\Omega, \mathbb{A}, p)$ together with T_1, T_2 is a base probability space for T.*

In what follows, $(\Omega, \mathbb{A}, p)$ will be denoted by $(\Omega_1, \mathbb{A}_1, p_1) \otimes_T (\Omega_2, \mathbb{A}_2, p_2)$.

Definition 3.9. *Let T be an extended random map of a probability space $(\Omega_1, \mathbb{A}_1, p_1)$ to a probability space $(\Omega_2, \mathbb{A}_2, p_2)$. Then $(\Omega_1, \mathbb{A}_1, p_1) \otimes_T (\Omega_2, \mathbb{A}_2, p_2)$ together with T_1, T_2 is said to be the **T-product** of $(\Omega_1, \mathbb{A}_1, p_1)$ and $(\Omega_2, \mathbb{A}_2, p_2)$.*

Theorem 3.10. *Let T be an extended random map of a probability space $(\Omega_1, \mathbb{A}_1, p_1)$ to a probability space $(\Omega_2, \mathbb{A}_2, p_2)$ and let $(\Omega, \mathbb{A}, p) = (\Omega_1, \mathbb{A}_1, p_1) \otimes_T (\Omega_2, \mathbb{A}_2, p_2)$ together with T_1, T_2 be their T-product. Let $(\Xi, \mathbb{B}, r)$ together with S_1, S_2 be a base probability space for T. Then there is a unique random map S of $(\Xi, \mathbb{B}, r)$ to $(\Omega_1, \mathbb{A}_1, p_1) \otimes_T (\Omega_2, \mathbb{A}_2, p_2)$ such that $T_i \circ S = S_i$, $i = 1, 2$.*

Proof. It follows directly from the construction of the T-product $(\Omega, \mathbb{A}, p)$ that $S(m) = (S_1(m), S_2(m)) = (S_1(m), T(S_1(m)))$, $m \in P(\mathbb{B})$, defines a unique map S of $P(\mathbb{B})$ to $P(\mathbb{A})$ such that $T_i \circ S = S_i$, $i = 1, 2$. Further, since $\mathbb{A}$ and $\mathbb{A}_1$ are isomorphic and, for each $q \in P(\mathbb{A}_1)$, T_1 sends $(q, T(q))$ to q, S is an extended random map. This completes the proof. □

Finally, let us reconsider questions Q1 and Q2. Let $(\Omega_1, \mathbb{A}_1, p_1)$ and $(\Omega_2, \mathbb{A}_2, p_2)$ be probability spaces. As already shown, even under the additional assumption that there is a probability space $(\Omega, \mathbb{A}, p)$ and there are random maps f_i of $(\Omega, \mathbb{A}, p)$ to $(\Omega_i, \mathbb{A}_i, p_i)$, $i = 1, 2$, the answer to Q1 is NO and, trivially, the answer to Q2 in YES. It is natural to ask

- Q3. Let T be an extended random map of $(\Omega_1, \mathbb{A}_1, p_1)$ to $(\Omega_2, \mathbb{A}_2, p_2)$. Is there a a probability space $(\Omega, \mathbb{A}, p)$ and are there random maps f_i of $(\Omega, \mathbb{A}, p)$ to $(\Omega_i, \mathbb{A}_i, p_i)$, $i = 1, 2$?
- Q4. Let $(\Omega, \mathbb{A}, p)$, $(\Omega_i, \mathbb{A}_i, p_i)$, $i = 1, 2$, be probability spaces and let f_i be a random map of $(\Omega, \mathbb{A}, p)$ to $(\Omega_i, \mathbb{A}_i, p_i)$, $i = 1, 2$. Is there a nondegenerated extended random map T of $(\Omega_1, \mathbb{A}_1, p_1)$ to $(\Omega_2, \mathbb{A}_2, p_2)$?

Again, the answer to Q3 is YES. Indeed, it suffices to put $\Omega = \Omega_1 \times \Omega_2$, $\mathbb{A} = \mathbb{A}_1 \times \mathbb{A}_2$, $p = p_1 \times p_2$ and $f_i = \mathrm{pr}_i$, $i = 1, 2$.

We have a positive answer to Q4 only under an additional assumption. The nondegenerated extended random map T in question is constructed via conditional probabilities. In terminology and notation related to conditional probability we generally follow [16]. For the reader's convenience we recall some basic notions needed in the sequel.

Let Λ be a set and let $\mathcal{F} = \{f_s : s \in S\}$ be a family of real-valued functions on Λ. Let $\mathbb{C}$ be the minimal σ-field of subsets of Λ containing all preimages $f_s^{\leftarrow}(B)$, $s \in S$, of Borel subsets B. Then we say that $\mathbb{C}$ *is induced by* $\mathcal{F}$.

Let $(\Omega, \mathbb{A}, p)$ be a probability space, let $\mathbb{B}$ be a σ-field contained in $\mathbb{A}$, let $p_{\mathbb{B}}$ be the restriction of p to $\mathbb{B}$, and let $\mathcal{E}$ be the family of all $\mathbb{A}$-measurable functions whose integral (hence indefinite integral) exists. Then the *conditional expectation* $E^{\mathbb{B}}(f)$ of $f \in \mathcal{E}$ given $\mathbb{B}$ is a $\mathbb{B}$-measurable function, defined up to a $p_{\mathbb{B}}$-equivalence by

$$\int_B E^{\mathbb{B}}(f)\,\mathrm{d}p_{\mathbb{B}} = \int_B f\,\mathrm{d}p = \int \chi_B f\,\mathrm{d}p\,, \qquad B \in \mathbb{B}\,.$$

The restriction of the conditional expectation $E^{\mathbb{B}}$ to the family of indicators of events (i.e. characteristic functions of sets in $\mathbb{A}$) is called *conditional probability given* $\mathbb{B}$.

Let $\mathcal{F} = \{f_s : s \in S\}$ be a family of random variables on $(\Omega, \mathbb{A}, p)$, let $\mathbb{A}_{\mathcal{F}} \subseteq \mathbb{A}$ be the σ-field of subsets of Ω induced by $\mathcal{F}$, and let $\mathbb{B}$ be a σ-field contained in $\mathbb{A}$. Let $p^{\mathbb{B}}$ be a mapping of $\Omega \times \mathbb{A}_{\mathcal{F}}$ into $[0,1] = I$ such that

(1) For each $A \in \mathbb{A}_{\mathcal{F}}$, $p^{\mathbb{B}}(\omega, A)$ is $\mathbb{B}$-measurable;

(2) for each $\omega \in \Omega$, $p^{\mathbb{B}}(\omega, A)$ is a probability measure on $\mathbb{A}_{\mathcal{F}}$;

(3) for each $A \in \mathbb{A}_{\mathcal{F}}$ and each $B \in \mathbb{B}$, $\int_B p^{\mathbb{B}}(\omega, A)\,\mathrm{d}p = p(A \cap B)$.

Then $p^{\mathbb{B}}$ is said to be a *conditional distribution of* $\mathcal{F}$ given $\mathbb{B}$. The existence theorem (cf. [16, p. 361, Theorem A]) states that if $\mathcal{F}$ is countable, then the conditional distribution $p^{\mathbb{B}}$ of $\mathcal{F}$ given $\mathbb{B}$ exists.

Theorem 3.11. *Let $(\Omega_i, \mathbb{A}_i, p_i)$, $i = 1, 2$, be probability spaces and assume that $\mathbb{A}_2$ is induced by a countable family $\mathcal{G} = \{g_k : k \in \mathbf{N}\}$ of random variables. Then there is a nondegenerated extended random map T of $(\Omega_1, \mathbb{A}_1, p_1)$ to $(\Omega_2, \mathbb{A}_2, p_2)$.*

Proof. Consider the product $(\Omega, \mathbb{A}, p) = (\Omega_1 \times \Omega_2, \mathbb{A}_1 \times \mathbb{A}_2, p_1 \times p_2)$. Let $\mathbb{B}$ be the σ-field of subsets of $\Omega_1 \times \Omega_2$ generated by cylinders $B \times \Omega_2$, $B \in \mathbb{A}_1$. For $k \in \mathbf{N}$, define $f_k \colon \Omega \to R$ by $f_k(\xi, \lambda) = g_k(\lambda)$ and put $\mathcal{F} = \{f_k : k \in \mathbf{N}\}$. Then $\mathbb{A}_{\mathcal{F}} = \{\Omega_1 \times C : C \in \mathbb{A}_2\} \subseteq \mathbb{A}$ is a σ-field induced by the countable family $\mathcal{F}$ and, according to the existence theorem, there exists a conditional distribution $p^{\mathbb{B}}$ of $\mathcal{F}$ given $\mathbb{B}$. For each $A \in \mathbb{A}_{\mathcal{F}}$, $p^{\mathbb{B}}(\cdot, A)$ is $\mathbb{B}$-measurable and hence $p^{\mathbb{B}}\big((\xi,\lambda_1),A\big) = p^{\mathbb{B}}\big((\xi,\lambda_2),A\big)$ whenever $\lambda_1,\lambda_2 \in \Omega_2$. For $\xi \in \Omega_1$ and $C \in \mathbb{A}_2$ define $T_0(\xi,C) = p^{\mathbb{B}}\big((\xi,\cdot),\Omega_1\times C\big)$. This yields a map T_0 on $\Omega_1\times\mathbb{A}_2$ into $[0,1] = I$ such that $T_0(\cdot,C)$ is $\mathbb{A}_1$-measurable, $T_0(\xi,\cdot)$ is a probability measure on $\mathbb{A}_2$, i.e. T_0 is a Markov kernel. It is known (cf. [4], [5]) that T_0 determines a unique statistical map T of $P(\mathbb{A}_1)$ into $P(\mathbb{A}_2)$. Further, for each $C \in \mathbb{A}_2$ we have

$$\int\limits_{\Omega_1\times\Omega_2} p^{\mathbb{B}}\big((\xi,\lambda),\Omega_1\times C\big)\,\mathrm{d}p = p\big((\Omega_1\times\Omega_2)\cap(\Omega_1\times C)\big) = p_2(C)$$

and hence

$$\big(T(p_1)\big)(C) = \int T_0(\xi,C)\,\mathrm{d}p_1 = \int\limits_{\Omega_1\times\Omega_2} p^{\mathbb{B}}\big((\xi,\lambda),\Omega_1\times C\big)\,\mathrm{d}p = p_2(C)\,.$$

Consequently, $T(p_1) = p_2$ and T is a nondegenerated extended random map of $(\Omega_1,\mathbb{A}_1,p_1)$ to $(\Omega_2,\mathbb{A}_2,p_2)$. This completes the proof. □

Let T be a statistical map of $(\Omega_1,\mathbb{A}_1)$ to $(\Omega_2,\mathbb{A}_2)$. If both $(\Omega_i,\mathbb{A}_i)$, $i = 1,2$, are discrete (finite or infinite), then there is a natural way how to represent T via conditional probabilities (see [14], [18]). In fact, consider the product $(\Omega,\mathbb{A}) = (\Omega_1,\mathbb{A}_1)\times(\Omega_2,\mathbb{A}_2)$, together with the projections pr_1, pr_2. Note that $\mathbb{A}$ is the σ-field of all subsets of Ω. Let p be a probability measure on $\mathbb{A}$ defined by $p\big(\{(\xi,\lambda)\}\big) = \big(T(\delta_\xi)\big)\big(\{\lambda\}\big)p_1\big(\{\xi\}\big)$. Then $\big(T(\delta_\xi)\big)\big(\{\lambda\}\big)$ means the "conditional probability" of $\mathrm{pr}_2^{\leftarrow}\big(\{\lambda\}\big)$ given $\mathrm{pr}_1^{\leftarrow}\big(\{\xi\}\big)$. Indeed, $p\big(\{(\xi,\lambda)\}\big) = p\big(\big(\mathrm{pr}_1^{\leftarrow}(\{\xi\})\big)\cap \big(\mathrm{pr}_2^{\leftarrow}(\{\lambda\})\big)\big)$ and $p_1\big(\{\xi\}\big) = p\big(\big(\mathrm{pr}_1^{\leftarrow}(\{\xi\})\big)\big)$. We do not know whether anything similar holds in the general case (cf. Problem 1).

Definition 3.12. *Let $(\Omega,\mathbb{A},p)$ be a probability space. Let f be a measurable map of $(\Omega,\mathbb{A})$ to a measurable space $(\Omega_1,\mathbb{A}_1)$ and let g be a measurable map of $(\Omega,\mathbb{A})$ to a measurable space $(\Omega_2,\mathbb{A}_2)$; denote $\mathbb{B}_f = \big\{f^{\leftarrow}(A) : A \in \mathbb{A}_1\big\}$, $\mathbb{A}_g = \big\{g^{\leftarrow}(A) : A \in \mathbb{A}_2\big\}$. Let p^f be a function on $\Omega\times\mathbb{A}_g$ into $[0,1] = I$ such that*

(1) *For each $A \in \mathbb{A}_g$, $p^f(\cdot,A)$ is $\mathbb{B}_f$-measurable;*
(2) *for each $\omega \in \Omega$, $p^f(\omega,\cdot)$ is a probability measure;*
(3) *for each $B \in \mathbb{B}_f$ and each $A \in \mathbb{A}_g$, $\int\limits_B p^f(\omega,A)\,\mathrm{d}p = p(A\cap B)$.*

Then p^f is said to be a ***conditional distribution of*** *g* ***given*** *f.*

Observe the following simple fact. For each $A \in \mathbb{A}_g$, if $\xi \in \Omega_1$ and $\omega,\omega' \in f^{\leftarrow}\big(\{\xi\}\big)$, then $p^f(\omega,A) = p^f(\omega',A)$. Indeed, p^f is $\mathbb{B}_f$-measurable, each $\{\xi\}$, $\xi \in \Omega_1$, is measurable, and $\mathbb{B}_f$ is induced by f.

Definition 3.13. *Let T be an extended random map of $(\Omega_1, \mathbb{A}_1, p_1)$ to $(\Omega_2, \mathbb{A}_2, p_2)$. Let $(\Omega, \mathbb{A}, p)$ be a probability space, let f be a random map of $(\Omega, \mathbb{A}, p)$ to a probability space $(\Omega_1, \mathbb{A}_1, p_1)$, let g be a random map of $(\Omega, \mathbb{A}, p)$ to a probability space $(\Omega_2, \mathbb{A}_2, p_2)$, and let p^f be a conditional distribution of g given f. If $T(\xi) = p^f(\omega, \cdot)$ for each $\xi \in \Omega_1$ and each $\omega \in f^{\leftarrow}(\{\xi\})$, then $(\Omega, \mathbb{A}, p)$ together with f, g, p^f is said to be a* ***conditional base probability space for*** T *and f, g are said to be* ***conditional base projections***.

Problem 1. *Let T be an extended random map of a probability space $(\Omega_1, \mathbb{A}_1, p_1)$ to a probability space $(\Omega_2, \mathbb{A}_2, p_2)$. Does there exist conditional base probability space for T?*

Definition 3.14. *Let T be an extended random map of a probability space $(\Omega_1, \mathbb{A}_1, p_1)$ to a probability space $(\Omega_2, \mathbb{A}_2, p_2)$ and let $(\Omega, \mathbb{A}, p)$ together with f, g, p^f be a conditional base probability space for T such that if $(\Gamma, \mathbb{C}, q)$ together with u, v, p^u is another base probability space for T, then*

1. *There is a unique random map h of $(\Gamma, \mathbb{C}, q)$ to $(\Omega, \mathbb{A}, p)$ such that $f \circ h = u$, $g \circ h = v$;*
2. *the conditional distribution p^u is uniquely determined by p^f and h: $p^u(\gamma, v^{\leftarrow}(C)) = p^f(h(\gamma), g^{\leftarrow}(C))$ for all $\gamma \in \Gamma$ and $C \in \mathbb{A}_2$.*

Then $(\Omega, \mathbb{A}, p)$ together with f, g, p^f is said to be the ***conditional T-product*** *of $(\Omega_1, \mathbb{A}_1, p_1)$ and $(\Omega_2, \mathbb{A}_2, p_2)$.*

Problem 2. *Let T be an extended random map of a probability space $(\Omega_1, \mathbb{A}_1, p_1)$ to a probability space $(\Omega_2, \mathbb{A}_2, p_2)$ and let $(\Gamma, \mathbb{C}, q)$ together with u, v, p^u be a conditional base probability space for T. Does there exist the conditional T-product of $(\Omega_1, \mathbb{A}_1, p_1)$ and $(\Omega_2, \mathbb{A}_2, p_2)$?*

Remark 3.15. The results of this section can be described as "transportation of probability measures along arrows in simple diagrams within the categories of measurable spaces and probability spaces". It might be useful to develop a reasonable categorical theory (notions, theorems, counterexamples, ...) about diagrams in which arrows are statistical maps and extended random maps in suitable subcategories of EMS and EPS. In fact, f-products and T-products are limits of one-arrow diagrams. Further, observe that to a measurable map f of a measurable space $(\Omega, \mathbb{A})$ to a measurable space $(\Xi, \mathbb{B})$ there corresponds a sequentially continuous D-homomorphism $f^{\leftarrow}$ of the D-poset of fuzzy sets $\mathbb{B}$ to the D-poset of fuzzy sets $\mathbb{A}$, each probability measure p on $\mathbb{A}$ is a sequentially continuous D-homomorphism of $\mathbb{A}$ to the trivial D-poset of fuzzy sets I, and each probability measure q on $\mathbb{B}$ is a sequentially continuous D-homomorphism of $\mathbb{B}$ to the trivial D-poset of fuzzy sets I. Hence "the transportation of p to q

along f" can be viewed as a commutative diagram $p \circ f^{\leftarrow} = q$ in the category ID of D-posets of fuzzy sets. Similarly, to each statistical map T of $P(\mathbb{A})$ to $P(\mathbb{B})$ there corresponds a sequentially continuous D-homomorphism $T^{\triangleleft}$ of the D-poset of fuzzy sets $\mathcal{M}^*(\mathbb{B})$ to the D-poset of fuzzy sets $\mathcal{M}^*(\mathbb{A})$ sending $v^* \in \mathcal{M}^*(\mathbb{B})$ to $v^* \circ T \in \mathcal{M}^*(\mathbb{A})$. Again, each probability measure p on $\mathbb{A}$ is transported along T via $p \circ T^{\triangleleft}$ to a probability measure on $\mathbb{B}$ and the transportation can be viewed as a commutative diagram. Of course, this means that the category ID provides a domain in which "transportation of probabilities" can be developed in a natural way.

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