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THE RELATION BETWEEN A FLOW AND ITS DISCRETIZATION

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ABSTRACT. It is proved that the h-time map of a hyperbolic flow and its h-discretization are uniformly topologically conjugate for each small positive h.

Introduction. Let $\Phi(t,x)$ be the flow generated by the equation

$$x' = Ax + g(x), \tag{1}$$

where $A \in \mathcal{L}(\mathbb{R}^m)$, A has no eigenvalues on the imaginary axis, $g \in C^1(\mathbb{R}^m, \mathbb{R}^m)$, $g(0) = 0$, $\sup|g| < \infty$, $|\mathrm{D}g(x)| \le b$ for each $x \in \mathbb{R}^m$ and b sufficiently small. The equation (1) has the discretization

$$x_{n+1} = x_n + h \cdot Ax_n + h \cdot g(x_n), \quad h \neq 0$$

which gives us the mapping

$$G(h,x) = x + h \cdot Ax + h \cdot g(x). \tag{2}$$

It is not difficult to see that $I + h \cdot A$ has no eigenvalues on the unit circle for each small $h \neq 0$. Hence, if moreover $g(x) = o(|x|)$ as $x \to 0$, then the mapping (2) has local stable and unstable manifolds W_h^s, W_h^u for the fixed point 0, respectively. Recently the author of this paper [1] has shown that the manifolds W_h^s, W_h^u tend to W^s, W^u as $h \to 0$, $h > 0$, where W^s, W^u are local stable, unstable manifolds of (1) for the fixed point 0, respectively.

The purpose of this paper is to show that the mapping $\Phi(h,\cdot)$ and $G(h,\cdot)$ are uniformly topologically conjugate for each small positive h, i.e. the following theorem holds:

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THEOREM 1. *For sufficiently small* b *and a compact set* $K \subset \mathbb{R}^m$ *there is a number* $\delta > 0$ *and a* C^0*-mapping*

$$H\colon (0,\delta) \to C^0(\mathbb{R}^m,\mathbb{R}^m) = \{f\colon \mathbb{R}^m \to \mathbb{R}^m,\ f \text{ is continuous}\}$$

such that

$$\Phi(h,\cdot)\cdot H(h,\cdot) = H(h,\cdot)\cdot G(h,\cdot) \qquad \text{on} \quad K$$

and

i) $H(h,\cdot)$ *is a homeomorphism,*
ii) $\sup\limits_{(0,\delta)\times K} |H(\cdot,\cdot)| < \infty, \quad \sup\limits_{(0,\delta)\times K} |H^{-1}(\cdot,\cdot)| < \infty.$

If $K = B_q = \{x,\ |x| \le q\}$ *for* q *large, then* $B_{q/2} \subset \bigcap\limits_{(0,\delta)} H(\cdot,K)$.

Proof. We divide the proof into several steps.

Step 1.

By the *Hartman-Grobman theorem* [2, p. 115] there is an $H_1 \in C^0_{\mathrm{B}}(\mathbb{R}^m,\mathbb{R}^m) = \{f \in C^0(\mathbb{R}^m,\mathbb{R}^m),\ f \text{ is bounded, i.e. } \sup|f| < \infty\}$ such that

$$\Phi(h,\cdot)\cdot(I+H_1) = (I+H_1)\cdot \mathrm{e}^{h\cdot A}$$

and $(I+H_1)^{-1} = I + H_1$ for some $\bar H_1 \in C^0_{\mathrm{B}}(\mathbb{R}^m,\mathbb{R}^m)$.
Let E^s, E^u be stable and unstable subspaces of A, respectively.

Step 2.

LEMMA 2. *There is a* $\delta_1 > 0$ *and a* C^0*-mapping*

$$H_3\colon (0,\delta_1) \to C^0_{\mathrm{B}}(\mathbb{R}^m,\mathbb{R}^m)$$

such that

$$\big(I + H_3(h,\cdot)\big)\cdot(I + h\cdot A) = G(h,\cdot)\cdot\big(I + H_3(h,\cdot)\big), \tag{3}$$

where $I + H_3(h,\cdot)$ *is a homeomorphism for each* $h \in (0,\delta_1)$ *and* $\big(I + H_3(h,\cdot)\big)^{-1} = I + \bar H_3(h,\cdot)$, $\bar H_3(h,\cdot) \in C^0_{\mathrm{B}}(\mathbb{R}^m,\mathbb{R}^m)$. *Moreover*

$$\sup_{(0,\delta_1)\times\mathbb{R}^m} |H_3(\cdot,\cdot)| < \infty, \qquad \sup_{(0,\delta_1)\times\mathbb{R}^m} |\bar H_3(\cdot,\cdot)| < \infty.$$

Proof of Lemma 2. We shall follow [2, Theorem 5.15.]. We can rewrite the equation (3) in the form

$$\begin{aligned} H_3^s &= (I+hA)^s \cdot H_3^s \cdot (I+hA)^{-1} + h\cdot g^s \cdot (I+H_3)\cdot(I+hA)^{-1} \\ H_3^u &= \big((I+hA)^u\big)^{-1} \cdot H_3^u \cdot (I+hA) - h\cdot\big((I+hA)^u\big)^{-1}\cdot g^u\cdot(I+H_3), \end{aligned} \tag{4}$$

where for any mapping $S\colon \mathbb{R}^m \to \mathbb{R}^m$ we write $S^s = P_s S$, $S^u = P_u S$ and P_u, P_s are projections to E^u, E^s, respectively. We solve (4) in the space $C^0_{\mathrm{B}}(\mathbb{R}^m, \mathbb{R}^m)$. It is clear that the mapping

$$T_h\colon C^0_{\mathrm{B}}(\mathbb{R}^m, \mathbb{R}^m) \to C^0_{\mathrm{B}}(\mathbb{R}^m, \mathbb{R}^m)$$
$$T_h(H) = \Big((I + hA)^s \cdot H^s \cdot (I + hA)^{-1}, \big((I + hA)^u\big)^{-1} \cdot H^u \cdot (I + hA)\Big)$$

has the property

$$|T_h(H) - T_h(F)| \le (1 - c \cdot h) \cdot |H - F| \tag{5}$$

for some constant $c > 0$, small positive h and each $H, F \in C^0_{\mathrm{B}}(\mathbb{R}^m, \mathbb{R}^m)$. Indeed, we can choose norms $\|\cdot\|_1$, $\|\cdot\|_2$ on the space E^s, E^u respectively [3, p. 145] such that

$$\|(I + h \cdot A)^s\|_1 \le (1 - h \cdot c)$$
$$\left\|\big((I + h \cdot A)^u\big)^{-1}\right\|_2 \le (1 - h \cdot c)$$

for each small positive h and we put

$$|f| = \sup_{\mathbb{R}^m}\big(\|f^s(\cdot)\|_1 + \|f^u(\cdot)\|_2\big)$$

for each $f \in C^0_{\mathrm{B}}(\mathbb{R}^m, \mathbb{R}^m)$.

Hence (4) has the form

$$H = T_h(H) + h \cdot F_h(H),$$

where $F_h(H) = \Big(g^s \cdot (I + H) \cdot (I + hA)^{-1}, -\big((I + hA)^u\big)^{-1} \cdot g^u \cdot (I + H)\Big)$. Thus

$$H = h \cdot (I - T_h)^{-1} \cdot F_h(H). \tag{6}$$

Since by (5) and the Banach fixed point theorem

$$|(I - T_h)^{-1}| \le \frac{c}{h}$$

we can apply uniformly the implicit function theorem to (6) for each small positive h. (Note that b is sufficiently small). Hence (3) has a unique solution. On the other hand, let us consider the equation

$$(I + H) \cdot (I + h \cdot A + h \cdot g) = (I + h \cdot A) \cdot (I + H)$$

which is equivalent to

$$H^s - (I + hA)^s \cdot H^s \cdot (I + hA + hg)^{-1} = -hg^s \cdot (I + hA + hg)^{-1}$$
$$H^u - \left((I + hA)^u\right)^{-1} \cdot H^u \cdot (I + hA + hg) = h \cdot \left((I + hA)^u\right)^{-1} \cdot g^u.$$

Since this equation is similar to (4) we obtain by the above results that this equation has a unique solution $H(h, \cdot) \in C^0_{\mathrm{B}}(\mathbb{R}^m, \mathbb{R}^m)$ for each small positive h. Using a standard procedure [2, Theorem 5.15] we have $I + H = (I + H)^{-1}$, where H is a solution of (4). This gives us the proof of Lemma 2.

Step 3.

In the last step we try to find a homeomorphism $I + H_4(h, \cdot)\colon \mathbb{R}^m \to \mathbb{R}^m$ such that

$$\mathrm{e}^{h \cdot A}\big(I + H_4(h, \cdot)\big) = \big(I + H_4(h, \cdot)\big) \cdot (I + h \cdot A) \qquad \text{on} \quad K \subset \mathbb{R}^m \tag{7}$$

for $h > 0$ small and $H_4(h, \cdot) \in C^0_{\mathrm{B}}(\mathbb{R}^m, \mathbb{R}^m)$, K is a compact set. Since

$$\mathrm{e}^{h \cdot A} = I + h \cdot A + f(h \cdot A),$$

where $f(x) = \mathrm{e}^x - 1 - x$, we have $f(h \cdot A) = O(h^2)$ as $h \to 0$. Without loss of generality we can suppose $K = B_q$ for q large. Since $f(h \cdot A) \notin C^0_{\mathrm{B}}(\mathbb{R}^m, \mathbb{R}^m)$, we modify $f(h \cdot A)$ in the following way

$$\tilde{f}(h, x) = s(x) \cdot f(h \cdot A)x,$$

where s is a function having the property

i) $s \in C^\infty$
ii) $s = 1$ on B_L
iii) $s = 0$ on B_{2L}

for $L \gg q$ sufficiently large. Thus a modified equation of (7) has the form

$$\big(I + h \cdot A + \tilde{f}(h, \cdot)\big) \cdot \big(I + H_4(h, \cdot)\big) = \big(I + H_4(h, \cdot)\big) \cdot (I + h \cdot A). \tag{8}$$

To solve (8) we follow the above step 2. Hence (8) has the form

$$H_4 = T_h(H_4) + O(h^2)$$

and

$$H_4 = (I - T_h)^{-1} \cdot O(h^2).$$

We see that H_4 exists for each small positive h and $H_4(h,\cdot) \to 0$ as $h \to 0$. It follows also by the step 2 that

$$\big(I + H_4(h,\cdot)\big)^{-1} = I + \bar{H}_4(h,\cdot), \qquad \bar{H}_4(h,\cdot) \in C^0_{\mathrm{B}}(\mathbb{R}^m, \mathbb{R}^m)$$

and $\bar{H}_4(h,\cdot) \to 0$ as $h \to 0$.

Summing up we see that

$$\big(I + H_1(\cdot)\big) \cdot \big(I + H_4(h,\cdot)\big) \cdot \big(I + \bar{H}_3(h,\cdot)\big)$$

is the desired mapping $H(h,\cdot)$ satisfying

$$\Phi(h,\cdot) \cdot H(h,\cdot) = H(h,\cdot) \cdot G(h,\cdot) \qquad \text{on} \quad K. \tag{9}$$

Indeed, since $L \gg q$ is large, $H_4(h,\cdot) = O(h)$, $\bar{H}_3(h,\cdot)$ is bounded and h is small we have

$$\big(I + H_4(h,\cdot)\big) \cdot \big(I + \bar{H}_3(h,\cdot)\big)K \subset B_L,$$

and thus $\tilde{f}(h,\cdot) = f(h \cdot A)$ on $\big(I + H_4(h,\cdot)\big) \cdot \big(I + \bar{H}_3(h,\cdot)\big)K$. Moreover, $H_1(\cdot)$ is also bounded on $\mathbb{R}^m$. These facts imply both

$$B_{q/2} \subset \bigcap_{(0,\delta)} H(\cdot, K)$$

for δ small and (9).

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