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DOMINATION IN CUBES

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ABSTRACT. The graph of the n-dimensional cube is the graph whose vertex set is the set of all n-dimensional Boolean vectors and in which two vertices are adjacent if and only if they differ in exactly one coordinate. In the paper the k-domatic number and the edge-domatic number of these graphs are studied.

The graph Q_n of the n -dimensional cube is the graph whose vertex set is the set of all n-dimensional vectors $(v_1, \ldots, v_n)$, where $v_i = 0$ or $v_i = 2$ for $i = 1, \ldots$ $\ldots, n$, and in which two vertices are adjacent if and only if they differ exactly in one coordinate.

We shall study the edge-domatic number and the k-domatic number of these graphs.

The domatic number of an undirected graph G was introduced by E. Cockayne and S. T. Hedetniemi in [1]. The edge-domatic number and the k-domatic number were introduced by the author of this paper in [2] and [3].

A subset D of the vertex set $V(G)$ of a graph G is called dominating if for each vertex $x \in V(G) - D$ there exists a vertex $y \in D$ adjacent to x. If k is a positive integer and if for each vertex $x \in V(G) - D$ there exists a vertex $y \in D$ whose distance from x in G is at most k, then the set D is called k-dominating. If D is a subset of the edge set $E(G)$ of G and for each edge $e \in E(G) - D$ there exists an edge $f \in D$ having a common end vertex with e, the set D is called a dominating edge set of G.

A partition of $V(G)$, all of whose classes are dominating (or k-dominating) sets in G, is called a domatic (or k-domatic respectively) partition of G. A partition of $E(G)$, all of whose classes are dominating sets in G, is called an edge-domatic partition of G. The maximum number of classes of a domatic (or k-domatic, or edge-domatic) partition of G is called the domatic (or k-domatic, or edge-domatic respectively) number of G. The domatic number of G is denoted by $d(G)$, the k-domatic number by $d_k(G)$, the edge-domatic number by $ed(G)$.

In the following the vector $(v_1, \ldots, v_n)$ will be denoted simply by $v_1 \ldots v_n$. The

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symbol $(v_1 \dots v_n, v'_1 \dots v'_n)$ will denote the edge in Q_n joining the vertices $v_1 \dots v_n, v'_1 \dots v'_n$.

Theorem 1. *Let k, n be integers, let $1 \leqq k \leqq n$. Then*

$$d_k(Q_n) \geqq 2^{k-1} d(Q_{n-k+1}).$$

Proof. Denote $d(Q_{n-k+1}) = p$. Take the cube graph Q_{n-k+1} and choose a domatic partition $\mathscr{D}$ of it with p classes $D_1, \dots, D_p$. Now let M be the set of all ordered k-tuples $(i, h_1, \dots, h_{k-1})$ of integers, where $1 \leqq i \leqq p$ and each h_j is 0 or 1. The cardinality of M is $2^{k-1}p$. Consider the cube graph Q_n. We shall construct a partition $\mathscr{D}^*$ of the vertex set of Q_n whose classes will be D^*_m for all elements $m \in M$. Every vertex $v_1 \dots v_n$ will be in D^*_m such that $m = (v_{n-k+2}, \dots, v_n)$, where i is the number such that $v_1 \dots v_{n-k+1} \in D_i$ in Q_{n-k+1}. We shall prove that $\mathscr{D}^*$ is a k-domatic partition of Q_n. Let $m = (i, h_1, \dots, h_{k-1}) \in M$ and let $\boldsymbol{v} = v_1 \dots v_n$ be a vertex of Q_n. Suppose that $v \notin D_m$. Let $\boldsymbol{w} = v_1 \dots v_{n-k+1} h_1 \dots h_{k-1}$; this is a vertex of Q_n. As the vectors $\boldsymbol{v}$, $\boldsymbol{w}$ differ in at most $k-1$ coordinates, their distance in Q_n is at most $k-1$. In Q_{n-k+1} consider the vertex $\boldsymbol{v}' = v_1 \dots v_{n-k+1}$. This vertex either belongs to D_i or is adjacent to a vertex $\boldsymbol{z}' \in D_i$ in Q_{n-k+1}, because D_i is a dominating set in this graph. In the first case $\boldsymbol{v}$ has the distance at most $k-1$ from a vertex of D_m in Q_n, namely the vertex $v_1 \dots v_{n-k+1} h_1 \dots h_{k-1}$. In the second case let $\boldsymbol{z}$ be the vector obtained from $\boldsymbol{z}'$ by adding $k-1$ coordinates $h_1, \dots, h_{k-1}$ after the coordinates of $\boldsymbol{z}'$. The vertices $\boldsymbol{w}$, $\boldsymbol{z}$ are adjacent in Q_n and therefore the distance between $\boldsymbol{v}$ and $\boldsymbol{z}$ is at most k, while $\boldsymbol{z} \in D_m$. The set D_m is dominating in Q_n. As m was chosen arbitrarily, $\mathscr{D}^*$ is a k-domatic partition of Q_n with $2^{k-1}p = 2^{k-1}d(Q_{n-k+1})$ classes, which implies the assertion. □

It was proved in [4] that if $n = 2^s$, where s is a positive integer, then $d(Q_{n-1}) = d(Q_n) = n$. We have a corollary.

Corollary. *Let s, k be positive integers, let $n = 2^s + k$. Then $d_j(Q_{n-2}) \geqq 2^{s+k-1}$,*

Theorem 2. *Let n be a positive integer divisible by 3. Then*

$$ed(Q_n) \geqq 4n/3.$$

Proof. First consider $n = 3$. There exists an edge-domatic partition of Q_3 consisting of the set {(000, 100), (010, 011), (101, 111)} and the sets obtained from it by the iterations of the peımutation given by $000 \mapsto 100 \mapsto 110 \mapsto 010 \mapsto 000$, $001 \mapsto 101 \mapsto 111 \mapsto 011 \mapsto 001$. (In geometry this permutation is the 90° rotation of the cube around its vertical axis.) This set has $4n/3 = 4$ elements. Now consider the cube graph Q_n, where n is divisible by 3 and $n \geqq 6$. For $i = 1, \dots, n/3$ let F_i be the set of edges which join vertices differing in the $(3i-2)$-th, the $(3i-1)$-th or the $3i$-th coordinate. The sets $F_1, \dots, F_{n/3}$ form a partition of $E(Q_n)$. The spanning subgraph of Q_n having the edge set F_i is a graph having 2^{n-3} connected components which are all isomorphic to Q_3; denote this graph by H_i. The vertex set of each connected component of H_1 consists of vertices for which the coordinates $v_4, \dots, v_n$ are the same. We shall call such a component even (or odd) if among the coordinates $v_4, \dots, v_n$ there is an even (odd, respectively) number of those which are equal to 1. In each even component of H_1 we take the set of edges $(000v_4 \dots v_n, 100v_4 \dots v_n)$, $(010v_4 \dots v_n, 011v_4 \dots v_n)$, $(101v_4 \dots v_n, 111v_4 \dots v_n)$, in each odd component of H_1 we take the set of edges $\{(100v_4 \dots v_n, 110v_4 \dots v_n), (000v_4 \dots v_n, 001v_4 \dots v_n), (011v_4 \dots v_n, 111v_4, \dots v_n)\}$. Let D be the union of all these sets for all connected components of H_1. Consider the set M of vertices of Q_n which are incident with no edge of D. It consists of all vertices $001v_4, \dots v_n$, $110v_4 \dots v_n$, where the number of coordinates equal to 1 among $v_4, \dots, v_n$ is even, and $010v_4 \dots v_n$, $101v_4 \dots v_n$, where this number is odd. It is easy to see that M is an independent set in Q_n. Hence each edge of Q_n is incident with at most one vertex of M and with at least one vertex of $V(Q_n) - M$. This implies that each edge of Q_n either belongs to D, or has a common end vertex with an edge of D and thus D is a dominating set in Q_n. We use the permutation given by $000v_4 \dots v_n \mapsto 100v_4 \dots v_n \mapsto 110v_4 \dots \dots 010v_4 \dots v_n \mapsto 000v_4 \dots v_n$, $001v_4 \dots v_n \mapsto 101v_4 \dots v_n \mapsto 111v_4 \dots v_n \mapsto \mapsto 011v_4 \dots v_n \mapsto 001v_4 \dots v_n$ for any values of $v_1, \dots, v_n$. By this permutation and its iterations from D we obtain four pairwise disjoint dominating edge sets in Q_n (including D itself). Instead of H_1 we may take other H_i and proceed analogously. In this way we obtain an edge-domatic partition of Q_n with $4n/3$ classes, which implies the assertion. □

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