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Dedicated to Professor B. Riečan on the occasion of his 70th birthday

RIEMANN AVERAGE TRUTH-VALUE OF ŁUKASIEWICZ FORMULAS

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ABSTRACT. We give a purely algebraic necessary and sufficient condition for a finitely additive measure on a finitely generated free MV-algebra to coincide with the Riemann integral.

1. Preliminaries: states, spectra, bases and statement of main results

Intuitively, a finitely additive measure in Łukasiewicz infinite-valued propositional logic is a method to measure the average truth-value $\bar{\varphi}$ of any formula φ. Since φ must only depend on the meaning of φ, any such averaging map $^-$ is defined on Lindenbaum algebras of formulas, i.e., on MV-algebras. In [6] finitely additive measures on MV-algebras were investigated using the following terminology:

A *state* of an MV-algebra A is a function $\sigma\colon A \to [0,1]$ such that

(i) $\sigma(0) = 0$,
(ii) $\sigma(1) = 1$,
(iii) for all $a, b \in A$ if $a \odot b = 0$, then $\sigma(a) + \sigma(b) = \sigma(a \oplus b)$ (Additivity).

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For applications to MV-algebraic probability theory see the handbook chapter [11] and references therein. As a notable example of a state, the integral can be naturally defined on every free MV-algebra, once the latter is concretely represented as an algebra of McNaughton functions (see [6]).

Our purpose in this paper is to give a purely algebraic characterization of the integral, among all possible states of the free n-generated MV-algebra Free_n. We refer to [1] for background on MV-algebras.

Notation. For any MV-algebra A we shall denote by $\mathcal{M}(A)$ its maximal ideal space. $\mathcal{M}(A)$ comes equipped with the *spectral* topology: a basis of closed sets for $\mathcal{M}(A)$ is given by the *zero-sets* Z_a of all elements $a \in A$, i.e., by the sets $Z_a = \{J \in \mathcal{M}(A) : a \in J\}$ for arbitrary $a \in A$. As is well known, $\mathcal{M}(A)$ is a nonempty compact Hausdorff space.

For any compact Hausdorff space X we denote by $\mathcal{C}(X)$ the MV-algebra of all continuous $[0,1]$-valued functions on X with pointwise operations.

As usual, Γ denotes the categorical equivalence between MV-algebras and abelian lattice-ordered groups with (strong) order-unit ([4], [1]).

The d-*disk* $\mathcal{D}^d$ is defined by $\mathcal{D}^d = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : \sum x_i^2 \leq 1\}$, equipped with the natural topology of $\mathbb{R}^d$, $d = 1, 2, 3, \ldots$. By $\mathcal{D}^0$ we understand the discrete topological space with one element.

Our algebraic characterization of the Riemann integral will be given in terms of the following definition:

Definition 1.1. A *basis* in an MV-algebra A is a set $B = \{b_1, \ldots, b_u\}$ of nonzero elements of A, together with integers $0 < m_1, \ldots, m_u$ such that

(i) B generates A,

(ii) in *the* abelian lattice-ordered group G with order unit 1 such that $A = \Gamma(G, 1)$ we have $\sum_{i=1}^{u} m_i b_i = 1$,

(iii) for every $k = 1, 2, \ldots, u$, the one-set of each k-cluster of B is homeomorphic to the disk $\mathcal{D}^{k-1}$.

Here, by a k-*cluster* of B we understand a k-element subset $C \subseteq B$ such that $\bigwedge_{b \in C} b \neq 0$; the *one-set* of C is the subspace of $\mathcal{M}(A)$ consisting of maximal ideals J of A such that, in the quotient MV-algebra A/J, $\bigoplus_{b \in C} m_b b/J = 1$.

The integers m_i are called the *multiplicities* of B.

Lemma 1.2.

(i) ([1; 1.2.10, 3.5.1, 7.2.6]) *For any MV-algebra A and ideal $I \in \mathcal{M}(A)$, there is an isomorphism $I^{\natural}$ of the quotient A/I onto a uniquely determined subalgebra of $[0,1]$. The isomorphism is unique.*

(ii) ([1; 3.6]) *If in addition, A is semisimple, the map $A \ni a \mapsto a^* \in [0,1]^{\mathcal{M}(A)}$ defined by $a^*(I) = I^{\natural}(a/I) \in [0,1]$ is an isomorphism of A onto a* separating *subalgebra A^* of $\mathcal{C}(\mathcal{M}(A))$; in other words, whenever I, J are distinct maximal ideals of A, there is a function $a^* \in A^*$ such that $a^*(I) \neq a^*(J)$.*

NOTATION. When dealing with a semisimple MV-algebra we shall tacitly identify any element $a \in A$ with its corresponding function $a^* \colon \mathcal{M}(A) \to [0,1]$ by writing $a(J)$ rather than $a^*(J)$ or a/J.

PROPOSITION 1.3. *Let $B = \{b_1, \dots, b_u\}$ be a basis in a semisimple MV-algebra A with multiplicities $m_1, \dots, m_u$. Then $m_i = 1/\max b_i$ for each $i = 1, \dots, u$. The maximum value $\max b_i$ is attained by b_i at precisely one point in $\mathcal{M}(A)$, namely the only element I_i in the one-set of the 1-cluster $\{b_i\}$.*

Proof. Let $(G,1)$ correspond to A via Γ. Direct inspection using Lemma 1.2 shows that G is the lattice-ordered group of real-valued functions over $\mathcal{M}(A)$ generated by A. Each $b_i \in B$ belongs to G. The one-set of the 1-cluster $\{b_i\}$ is the singleton $\{I_i\} \subseteq \mathcal{M}(A)$, with $m_i b_i(I_i) = 1$. Since we also have $\sum_j m_j b_j(I) = 1$, then all b_j's with $j \neq i$ must vanish at I_i. Thus $m_i = 1/b_i(I_i)$. One now easily checks that $b_i(I_i)$ is the maximum value of b_i, and that this value is attained only at I_i (for otherwise, B would not separate points of $\mathcal{M}(A)$; since B generates A, the latter, too, would not separate points, against Lemma 1.2). □

The proof of the following proposition shall be given in the next section; as usual, for any elements $a, b \in A$, $a \ominus b$ stands for $a \odot \neg b$:

PROPOSITION 1.4. *Let $B = \{b_1, \dots, b_u\}$ be a basis in* Free_n *with multiplicities $m_1, \dots, m_u$. Let $\{b_i, b_j\}$ be a 2-cluster. Let D be obtained from B by removing b_i and b_j and adding the three elements $b_i^{\downarrow} = b_i \ominus (b_i \wedge b_j)$, $b_j^{\downarrow} = b_j \ominus (b_i \wedge b_j)$ and $b^{\wedge} = b_i \wedge b_j$. Then D is a basis in* Free_n. *The multiplicities of D are as given by Proposition* 1.3

NOTATION AND TERMINOLOGY. The above transformation $B \mapsto D$ is known as the *binary starring of B at $\{b_i, b_j\}$*. We write $E \preceq^* D$ to mean that E is obtained from the basis D via a finite sequence of binary starring operations. The above Proposition 1.4 ensures that E is a basis.

Our main result is as follows:

THEOREM 1.5. *For each $n = 1, 2, 3, \dots$, precisely one state ς of* Free_n *satisfies the condition*

$$(\forall B)(\exists D \preceq^* B)(\forall E \preceq^* D)(\forall h \in E)\Big(\varsigma(h) = \tfrac{\max h}{(n+1)!} \sum_{C \in E(h)} \prod_{k \in C} \max k\Big), \quad (1)$$

where $E(h)$ is the set of maximal clusters C of E such that h C.

To evaluate ς, arbitrarily choose a free generating set S of Free *and a one-one map β of S onto the set $\{\xi_1,\ldots,\xi_n\}$ of projection functions $\xi_i\colon [0,1]^n \to [0,1]$. Let $\tilde{}$: $\mathrm{Free}_n \to \mathcal{C}([0,1]^n)$ be the canonical homomorphism extending β. Then for each $f \in \mathrm{Free}_n$*

$$\varsigma(f) = \int_{[0,1]^n} \tilde{f}, \qquad 2)$$

independently of the choice of S and β. Thus in particular, ς is invariant inde all automorphisms of Free_n, *and is* faithful, *i.e.,*

$$(\forall f \in \mathrm{Free}_n)\big(f > 0 \implies \varsigma(f) > 0\big);$$

further, $\varsigma(f)$ is a rational number, and so is $\lim_{n\to\infty} \varsigma(\underbrace{f \oplus \cdots \quad f}_{n \text{ times}})$.

Remark. In the light of (2), the state ς is called the *Riemann integral* o Free_n.

2. The proofs

Background on Schauder bases and unimodular triangulations. ([1; 9.1 2]) Let $\mathcal{S}$ be a rational simplicial complex over some closed subspac W of $[0,1]^n$. In other words, W is the point-set union of the simplexes in $\mathcal{S}$. We also say that W is the *support* of $\mathcal{S}$. The rationality of $\mathcal{S}$ means that the coordinates of every simplex in $\mathcal{S}$ are rationals. Let v be a vertex of $\mathcal{S}$. Then $v = (r_1/s_1,\ldots,r_n/s_n)$ for uniquely determined integers $r_i, s_i > 0$ such that $s_i \neq 0$ and r_i and s_i are relatively prime. The least common multiple of the set $\{s_i\}$ is called the *denominator* of v, written $\mathrm{den}(v)$. Passing to *homogeneous coordinates*, we obtain the integer vector

$$\mathbf{v} = \left(\tfrac{\mathrm{den}\,(v)}{s_1} r_1,\ldots,\tfrac{\mathrm{den}\,(v)}{s_n} r_n,\, \mathrm{den}\,(v)\right) \in \mathbb{Z}^{n+1}. \qquad 3$$

Let S be an m-dimensional simplex in $\mathcal{S}$ with vertices $v_0,\ldots,v_m$, $0 \leq m \leq n$. For each $j = 0,\ldots,m$ let us again write $v_j = (r_1^j/s_1^j,\ldots,r_n^j/s^j)$, with r_i^j and s_i^j relatively prime integers ≥ 0, and $s_i^j \neq 0$. By definition, writing in homogeneous coordinates, we obtain the vector $\mathbf{v}_j = (w_1^j, \ldots, w_n^j, \mathrm{den}(v_j))$ $\mathbb{Z}^{n+1}$, where the w_i^j are suitable integers ≥ 0 as in (3) above. We say that S is *unimodular* iff the set of integer vectors $\{\mathbf{v}_0,\ldots,\mathbf{v}_m\}$ is extendible to a basis of the free abelian group $\mathbb{Z}^{n+1}$. A rational simplicial complex $\mathcal{S}$ over W is said to be *unimodular* iff all its simplexes are unimodular. In this case we also say that $\mathcal{S}$ is a *unimodular triangulation* of W. Unimodular triangulations are the affine counterparts of nonsingular fans in toric algebraic geometry ([2], [7]).

The *Schauder hat* at vertex v in a unimodular triangulation $\mathcal{S}$ over $W \subseteq [0,1]^n$ is the uniquely determined continuous piecewise-linear function $h_v\colon W \to [0,1]$ that takes the value $1/\operatorname{den}(v)$ at v, vanishes at all other vertices of $\mathcal{S}$, and is linear (in the affine sense) on each simplex of $\mathcal{S}$. We denote by $B_{\mathcal{S}}$ the set of Schauder hats of $\mathcal{S}$.

LEMMA 2.1. *Let $\mathcal{S}$ be a unimodular triangulation with support $W \subseteq [0,1]^n$. Let $B_{\mathcal{S}}$ denote the set of Schauder hats of $\mathcal{S}$. Let us identify* Free_n *with the MV-algebra M of all McNaughton functions over the n-cube as in* [1; 9.1.5]. *Let $M|_W$ denote the MV-algebra of restrictions to W of the McNaughton functions of M. We then have:*

(i) *For every vertex $v \in \mathcal{S}$, $h_v \in M|_W$;*
(ii) *A function $f \in M|_W$ belongs to the monoid* $\operatorname{mon}(B_{\mathcal{S}})$ *generated by $B_{\mathcal{S}}$ in $M|_W$ iff it is linear over each simplex of $\mathcal{S}$;*
(iii) *$B_{\mathcal{S}}$ is a basis in $M|_W$;*
(iv) *The MV-algebra generated by $B_{\mathcal{S}}$ in $M|_W$ coincides with $M|_W$.*

Proof.

(i) This follows from a routine argument, to the effect that $\mathcal{S}$ can be extended to a unimodular triangulation of the whole n-cube.

(ii) This is an immediate consequence of the unimodularity of $\mathcal{S}$.

(iii) This follows immediately from Definition 1.1.

(iv) Let $f \in M|_W$. The same argument of [5; 1.2] yields a unimodular triangulation $\mathcal{F}$ over W such that f is linear over each simplex of $\mathcal{F}$. A further argument ([1; 9.2]) using the De Concini-Procesi Lemma ([2; Lemma 2.3]) on elimination of points of indeterminacy in toric varieties yields a unimodular triangulation $\mathcal{U}$ such that every simplex of $\mathcal{F}$ is a union of simplexes of $\mathcal{U}$ and, in addition,

$$B_{\mathcal{U}} \preceq^* B_{\mathcal{S}}. \tag{4}$$

(see [9; p. 569] for an elementary MV-algebraic proof of the De Concini-Procesi Lemma). Since by (ii) f belongs to $\operatorname{mon}(B_{\mathcal{F}})$, it follows that f belongs to $\operatorname{mon}(B_{\mathcal{U}})$, whence a fortiori f belongs to the MV-algebra generated by $B_{\mathcal{U}}$. By (4), $B_{\mathcal{U}}$ and $B_{\mathcal{S}}$ generate the same MV-algebra. Since f is arbitrary, we have the desired conclusion. □

Remark. The set $B_{\mathcal{S}}$ determined by the unimodular triangulation $\mathcal{S}$ over W is said to be a *Schauder basis* of $M|_W$. For $n \geq 2$, an automorphism α of M may transform a Schauder basis $B_{\mathcal{S}}$ into a set $\alpha(B_{\mathcal{S}}) \subseteq M$ which no longer is a Schauder basis. However, direct inspection shows that $\alpha(B_{\mathcal{S}})$ is still a basis of M. Definition 1.1 and Lemma 2.1(iii) show that bases are an "invariant" generalization of Schauder bases.[1]

P r o o f o f P r o p o s i t i o n 1.4 . By McNaughton theorem ([1; 9.1.5]) we can safely identify Free_n with the MV-algebra M of McNaughton functions over the n-cube $[0,1]^n$. We similarly identify the free MV-algebra Free_u with the MV-algebra N of McNaughton functions over the u-cube, and we choose the projection functions $\pi_1, \dots, \pi_u \colon [0,1]^u \to [0,1]$ as the free generators of N.

Let the homomorphism

$$\eta \colon N \to M \tag{5}$$

be the canonical extension of the map $\pi_i \mapsto b_i$ $(i = 1, \dots, u)$. Then η is onto M, because B generates M. Let the transformation $\vec{b} \colon [0,1]^n \to [0,1]^u$ be defined by

$$\vec{b} \colon z \mapsto \big(b_1(z), \dots, b_u(z)\big) . \tag{6}$$

Denote by X the range of $\vec{b}$, and observe that X is a compact subset of the u-cube. Actually, X is the union of finitely many simplexes with rational vertices. Further $\vec{b}$ is injective, for otherwise (the functions b_i in) B would not separate points in the n-cube, and hence also the MV-algebra M generated by B would not separate points, a contradiction. We then see that $\vec{b}$ is a homeomorphism of the n-cube onto X, in symbols,

$$\vec{b} \colon [0,1]^n \cong X . \tag{7}$$

The homomorphism $N \ni f \mapsto f \circ \vec{b} \in M$ agrees with η on the π_i's; thus

$$\eta(f) = f \circ \vec{b} \qquad \text{for all} \quad f \in N . \tag{8}$$

Let $N\big|_X$ denote the MV-algebra of restrictions to X of the McNaughton functions of N. Define the homomorphism $\theta \colon N\big|_X \to M$ by

$$\theta \colon g \mapsto g \circ \vec{b} . \tag{9}$$

Letting $\chi \colon f \mapsto f\big|_X$ be the restriction homomorphism, by (8) we can write

$$\eta = \theta \circ \chi . \tag{10}$$

Direct inspection shows that θ is surjective (because so is η) and is injective: indeed, if $g \in N\big|_X$ is nonzero at $y \in X$, then by (7) (9), $\theta(g)$ is nonzero at $\vec{b}^{-1}(y)$. Therefore, we have an isomorphism

$$\theta \colon N\big|_X \cong M . \tag{11}$$

[1]See [10] for nontrivial automorphisms of Free_n, already in the case $n = 2$. An invariant notion of basis was first introduced by the first author in his Ph D thesis. The present definition was introduced in C. Manara's Ph D thesis. The equivalence of the two definitions is essentially proved in [3], in the framework of lattice-ordered groups.

By (9) every element $b_i \in B$ is mapped by θ^{-1} to the restriction to X of the ith coordinate function of N, in symbols,

$$\theta^{-1}\colon b_i \mapsto b_i' = \pi_i|_X\,. \tag{12}$$

Letting $B' = \theta^{-1}(B) = \{b_1', \dots, b_u'\}$ from (11) it follows that B' is a basis in $N|_X$ with the same multiplicities $m_1, \dots, m_u$ as B. The clusters of B' are the θ^{-1}-images of the clusters in B.

Focusing now attention on the maximal spectral spaces of M and of $N|_X$, by (11) we also have a (canonical, dual) homeomorphism

$$\tilde{\theta}\colon \mathcal{M}(M) \cong \mathcal{M}(N|_X)\,. \tag{13}$$

Specifically, for each maximal ideal I of M,

$$\tilde{\theta}(I) = \{\theta^{-1}(f) :\ f \in I\}\,. \tag{14}$$

We shall need a more concrete representation of $\tilde{\theta}$. To this purpose let us recall ([4; 8.1], see also [1; 3.4.7]) the canonical homeomorphisms $\mu\colon [0,1]^n \cong \mathcal{M}(M)$ and $\nu\colon [0,1]^u \cong \mathcal{M}(N)$ given by $\mu(z) = \{f \in M :\ f(z) = 0\}$ and $\nu(y) = \{g \in N :\ g(y) = 0\}$. One has a similar homeomorphism $\nu'\colon X \cong \mathcal{M}(N|_X)$ given by $\nu'(y) = \{g \in N|_X :\ g(y) = 0\}$. Recalling (7), the composite map $\nu' \circ \vec{b} \circ \mu^{-1}$ yields a homeomorphism of $\mathcal{M}(M)$ onto $\mathcal{M}(N|_X)$, and a moment's reflection using (14) shows that

$$\tilde{\theta} = \nu' \circ \vec{b} \circ \mu^{-1}\,. \tag{15}$$

To increase readability it is convenient to assume that μ and ν' are identity functions; via the identifications

$$[0,1]^n = \mathcal{M}(M)\,, \qquad X = \mathcal{M}(N|_X) \tag{16}$$

the quotient map at a maximal ideal boils down to evaluation at its corresponding point. Then (15) becomes

$$\tilde{\theta} = \vec{b}\,. \tag{17}$$

The one-set 1_C of any cluster C of B is tacitly identified via μ with the closed subset of $[0,1]^n$ given by $\Big\{z \in [0,1]^n :\ \sum_{b_i \in C} m_i b_i(z) = 1\Big\}$. Similarly, for any cluster C' in B' we can write

$$1_{C'} = \Big\{x \in X :\ \sum_{b_i' \in C'} m_i b_i'(x) = 1\Big\}\,. \tag{18}$$

Let $(G,1)$ be the lattice-ordered abelian group with order-unit 1 such that $N|_X = \Gamma(G,1)$. Direct inspection shows that $(G,1)$ is the lattice-ordered group of real-valued functions over X generated by $N|_X$, with the con tant 1 as th strong unit. From our assumption about B, recalling [4; 3.2, 3 3] and 12), it follows that the *sum* (in G) of the functions $m_i b'_i$ is constantly equal to 1 over X, in symbols,

$$m_1 b'_1(x) + \cdots + m_u b'_u(x) = m_1\pi_1(x) + \cdots + m_u\pi_u(x) = 1 \qquad \text{for all} \quad x \in X . \tag{19}$$

Thus X is contained in the affine hyperplane L given by

$$L = \{(x_1, \ldots, x_u) \in \mathbb{R}^u : \ m_1 x_1 + \cdots + m_u x_u \quad 1\} . \tag{20}$$

Claim 1. *Let $C = \{b_{i_1}, \ldots, b_{i_r}\}$ be a cluster of B. Let $1_{C'} \subset [0,1]$ denot the one-set of C. Then the one-set $\vec{b}(1_{C'})$ of $C' = \theta^{\ 1}(C)$ coincides with th set $\{x \in X : \ m_{i_1} x_{i_1} + \cdots + m_{i_r} x_{i_r} = 1\}$.*

As a matter of fact, from (15) (16) we have

$$\vec{b}(1_{C'}) - \{x \in X : \ m_{i_1}\pi_{i_1}(x) \quad \cdots \oplus m_{i_r}\pi_{i_r}(x) - 1\} .$$

On the other hand, by (19) (20) we can write

$$m_{i_1} x_{i_1} \oplus \cdots \quad m_{i_r} x_{i_r} = m_{i_1} x_{i_1} + \cdots + m_{i_r} x_{i_r}$$

all over X.

Claim 2. *Let $e_1, \ldots, e_u$ be the standard basis vectors of $\mathbb{R}^u$. For each $i - 1, \ldots, u$, let the 1-cluster C_i defined by $C_i - \{b_i\}$. Let $1_{C'}$ denote its one-set Then the one-set $\vec{b}(1_{C_i})$ of the 1-cluster $C' - \theta^{\ 1}(C_i)$ coincides with $\{e \quad m \}$ Thus the point e_i/m_i lies in X.*

Indeed, by our identification (17) the one-set of $\{b_i\}$ is a singleton $\{z\}$ in the n-cube. By Claim 1, $\vec{b}(z)$ is the only point $x \in X \subseteq L$ where π_i takes value $1/m_i$, namely $x = e_i/m_i$.

Claim 3. *Let $r = 2, 3, \ldots, u$. Then for every r-cluster $C \quad \{b_{i_1}, \ldots, b \ \}$ in B, the one-set $\vec{b}(1_C)$ of the 1-cluster $C' = \theta^{\ 1}(C)$ coincides with the convex hull*

$$[e_{i_1}/m_{i_1}, \ldots, e_{i_r}/m_{i_r}]$$

of the vectors $e_{i_1}/m_{i_1}, \ldots, e_{i_r}/m_{i_r}$. Thus in particular $[e_{i_1}/m_i \ , \ldots, e_i \quad m \ \subseteq X$.

The proof is by induction on r.

Basis.

Suppose $\{b_i, b_j\}$ forms a 2-cluster C of B. By Claim 1, $\vec{b}(1_{C'})$ is the set Y $\{x \in X : \ m_i b'_i + m_j b'_j = 1\}$. By (20), Y is a subset of the closed segment

$[e_i/m_i, e_j/m_j]$. By Claim 2, both vectors e_i/m_i and e_j/m_j belong to Y. If Y were a *proper* subset of $[e_i/m_i, e_j/m_j]$, then it would not be connected; since Y is homeomorphic to 1_C, the latter, too, would not be connected, thus contradicting the definition of B.

Induction step. Let $W = [e_{i_1}/m_{j_1}, \dots, e_{i_{r+1}}/m_{j_{r+1}}]$. Let $P = \{b_{i_1}, \dots, b_{i_{r+1}}\}$ be a $(r+1)$-cluster of B. *A fortiori*, every subset $Q = \{b_{j_1}, \dots, b_{j_r}\}$ of P is a cluster of B. By induction hypothesis, the $\vec{b}$-image of the one-set 1_Q is the r-simplex $[e_{j_1}/m_{j_1}, \dots, e_{j_r}/m_{j_r}]$. Thus the $\vec{b}$-image of the one-set 1_P is a suitable subset $Y \subseteq W$ containing the union of all $(r-1)$-dimensional faces of W. Suppose Y is a *proper* subset of W (*absurdum hypothesis*). Write Y as $W \setminus U$ for a suitable nonempty subset U of the relative interior of W. One then verifies that the *singular homology* groups of $W \setminus U$ and W are not isomorphic: W is shrinkable to a point, while $W \setminus U$ is not. See [8] for the appropriate computations. It follows that Y, as well as its homeomorphic copy 1_P, are *not* homeomorphic to the r-disk D^r, thus contradicting the definition of B. Claim 3 is settled.

To conclude the proof, for every $x \in X$ let $b'_{i_1}, \dots, b'_{i_t}$ be the subset of B' given by those elements which are nonzero at x. Then $m_{i_1} b'_{i_1}(x) + \dots + m_{i_t} b'_{i_t}(x) = 1$ and $b'_{i_1}, \dots, b'_{i_t}$ form a t-cluster of B'. It follows that X is the union of the one-sets of all clusters C' of B'; this is the same as the union of the $\vec{b}$-images of the one-sets of all clusters C of B. Let T_C denote the $\vec{b}$-image of one-set 1_C of C, in symbols,

$$T_C = \vec{b}(1_C) = 1_{C'} \,. \tag{21}$$

By Claim 3, T_C is a simplex in the u-cube. Further inspection of the above construction shows that *any two simplexes T_{C_1} and T_{C_2} intersect in a common face.* Therefore, X is the *support* of the simplicial complex $\mathcal{S}$ determined by the simplexes T_C, letting C range over clusters of B. The vertices of (simplexes of) $\mathcal{S}$ are given by one-sets $\{e_1/m_1\}, \dots, \{e_u/m_u\}$ of the 1-clusters of B'. Each $\{e_j/m_j\}$ correspond via $\vec{b}$ to the one-set of $\{b_j\}$. Direct inspection using Claims 1 3 shows that $\mathcal{S}$ is *unimodular*. By (21), its simplexes $T_1, \dots, T_m$ are in $1-1$ correspondence with the clusters of B.

Each projection $\pi_i|_X$ is linear over X, hence in particular $\pi_i|_X$ is linear over each simplex $T \in \mathcal{S}$. Further, each $\pi_i|_X$ attains its maximum value $1/m_i$ at the only point e_i/m_i in the one-set of the 1-cluster $\{\pi_i|_X\}$, and vanishes at all other vertices. Thus, B' is a *Schauder* basis of $N|_X$. We have shown that *B is an isomorphic copy of a Schauder basis B'.*

Binary starring of B' at any 2-cluster $\{b'_i, b'_j\}$ yields a new Schauder basis D'. (Compare with [1; 9.2].) The isomorphism θ between $N|_X$ and M transforms the Schauder basis D' into a basis $D \preceq^* B$, as required. $\square$

Remark. It is instructive to explicitly give the multiplicities and the clusters of D, for these are the exact counterparts of the multiplicities and clusters of D'. Thus, the multiplicities m'_i and m'_j of $b_i^{\downarrow}$ and $b_j^{\downarrow}$ respectively coincide with m_i and m_j; the multiplicity of $b^{\wedge}$ is $m_i + m_j$. The remaining multiplicities are unchanged. The clusters of D are obtained as follows:

(1) add the 1-cluster $\{b^{\wedge}\}$;
(2) replace 1-cluster $\{b_j\}$ by $\{b_j^{\downarrow}\}$; more generally, replace every cluster C containing b_j but not b_i by the cluster $C' = (C \setminus \{b_j\}) \cup \{b_j^{\downarrow}\}$;
(3) replace the 1-cluster $\{b_i\}$ by $\{b_i^{\downarrow}\}$; more generally, replace every cluster C containing b_i but not b_j by the cluster $C' = (C \setminus \{b_i\}) \cup \{b_i^{\downarrow}\}$;
(4) replace the 2-cluster $\{b_i, b_j\}$ by the two 2-clusters $\{b^{\wedge}, b_j^{\downarrow}\}$ and $\{b_i^{\downarrow}, b^{\wedge}\}$; more generally, replace every cluster C containing $\{b_i, b_j\}$ by the two clusters $C' = (C \setminus \{b_i, b_j\}) \cup \{b^{\wedge}, b_j^{\downarrow}\}$ and $C'' = (C \setminus \{b_i, b_j\}) \cup \{b_i^{\downarrow}, b^{\wedge}\}$.
(5) leave unchanged all other clusters of B.

Proof of Theorem 1.5. Let $S = \{g_1, \dots, g_n\}$ be a free generating set of Free_n. Let $\beta\colon g_i \mapsto \xi_i$, where $\xi_i\colon [0,1]^n \to [0,1]$ is the ith canonical projection (we reserve the notation π_j for projections of the u-cube). Canonically extend β to the homomorphism

$$\tilde{}\ \colon \mathrm{Free}_n \to \mathcal{C}([0,1]^n)\,.$$

Then $\tilde{}$ is an isomorphism of Free_n onto the MV-algebra M of McNaughton functions over the n-cube [1; 9.1.5]. Let $\varsigma_{S,\beta}\colon \mathrm{Free}_n \to [0,1]$ be defined by

$$\varsigma_{S,\beta}(f) = \int_{[0,1]^n} \tilde{f}\,. \tag{22}$$

Direct inspection shows that $\varsigma_{S,\beta}$ is a state of Free_n. For the verification that $\varsigma_{S,\beta}$ satisfies (1) we can safely identify Free_n and M, and also assume that S coincides with the set of projection functions, whence β is the identity map. Let $B = \{b_1, \dots, b_u\}$ be an arbitrary basis in M.

CLAIM 1. *There exists a* Schauder *basis* $D \preceq^* B$ *in* M.

As a matter of fact, let us write N instead of Free_u, the latter being identified with the MV-algebra of McNaughton functions over the u-cube. The proof of Proposition 1.4 yields a closed set X in the u-cube, which is the support of a *unimodular* simplicial complex $\mathcal{S}$, whose elements are certain simplexes $T_1, \dots, T_m$; these simplexes are in $1-1$ correspondence with the one-sets of clusters of B. B is the isomorphic copy of a certain Schauder basis $B' = B_{\mathcal{S}}$

of $N|_X$ for some closed subset X of the u-cube. X coincides with the range of the transformation

$$\vec{b}\colon [0,1]^n \ni x \mapsto \big(b_1(x), \dots, b_u(x)\big) \in [0,1]^u .$$

The Schauder hats $B'_1, \dots, B'_u$ of $B_{\mathcal{S}}$ are the restrictions to X of the projection functions $\pi_1, \dots, \pi_u$. The maximum value $\max b'_i = 1/m_i$ is attained by b'_i at the point $x_i = e_i/m_i \in X$ corresponding via $\vec{b}$ to the one-set of the 1-cluster $\{b_i\}$. The isomorphism θ sends each $\pi_u|_X$ into b_i. The map $\vec{b}$ is a homeomorphism of $[0,1]^n$ onto X, and is also identified with the dual homeomorphism $\tilde{\theta}\colon \mathcal{M}(M) \cong \mathcal{M}(N|_X)$.

Let $\mathcal{T}$ be a unimodular triangulation of the n-cube such that each b_i is linear over each simplex $T \in \mathcal{T}$. Existence of $\mathcal{T}$ is ensured by a routine argument [1; Proof of 9.1.2]. Then $\vec{b}$ transforms $\mathcal{T}$ into a unimodular triangulation $\vec{b}(\mathcal{T})$ over X. Unimodularity follows from $\vec{b}$ being the dual of the isomorphism θ. Using the De Concini-Procesi theorem as in [1; 9.2.3] there is a unimodular triangulation $\mathcal{U}$ of X such that every simplex of $\mathcal{U}$ is a union of simplexes of $\vec{b}(\mathcal{T})$ and, crucially,

$$B_{\mathcal{U}} \preceq^* B' = B_{\mathcal{S}} .$$

Since $\vec{b}^{\,-1}$ is linear over each simplex of $\vec{b}(\mathcal{T})$, *a fortiori* it will be linear over each simplex of $\mathcal{U}$. Thus the image $\mathcal{W} = \vec{b}^{-1}(\mathcal{U})$ is a unimodular triangulation of the n-cube; every element h of $B_{\mathcal{W}} = \theta(B_{\mathcal{U}})$ is linear over each simplex of $\mathcal{W}$, because

$$\big(\forall f \in N|_X\big)\big(\theta(f) = f \circ \vec{b}\,\big) .$$

We have found a Schauder basis $D = B_{\mathcal{W}} \preceq^* B$ in M, and our first claim is settled.

Claim 2. *Let D be as in Claim 1. Then for every $E \preceq^* D$ and $h \in E$ we have*

$$\varsigma(h) = \frac{\max h}{(n+1)!} \sum_{C \in E(h)} \prod_{k \in C} \max k , \tag{23}$$

where $E(h)$ is as in the statement of the main theorem.

As a matter of fact, E is automatically a Schauder basis in M. The linearity domains of the hats of E determine a unimodular triangulation $\mathcal{V}$ such that $E = B_{\mathcal{V}}$. Given the Schauder hat $h \in B_{\mathcal{V}}$, let $v_h \in [0,1]^n$ be the only point where h attains its maximum value. We can write

$$h(v_h) = \max h = 1/\operatorname{den}(v_h) . \tag{24}$$

Let A be the closure of the set $\{x \in [0,1]^n : h(x) > 0\}$. Then $\varsigma_{S,\beta}(h)$ is the volume $\operatorname{vol}(P)$ of an $(n+1)$-dimensional pyramid P with base A, and whose lateral faces are given by the graph of $h|_A$. Let $A^1, \ldots, A^m \subseteq A$ be the list of all n-dimensional simplexes of $\mathcal{V}$ having v_h among their vertices. For each $t = 1, \ldots, m$ let P^t be the *rectangular* pyramid of height $\max h$ and base A^t. Then $\mathrm{vol}(P)$ is the sum of the volumes $\mathrm{vol}(P^t)$ of the P^t's. Each A^t is an n-dimensional simplex; say that the vertices of A^t are given by $v_h, v_1^t, \ldots, v_n^t$, in symbols,

$$A^t = [v_h, v_1^t, \ldots, v_n^t]\,. \tag{25}$$

Just as v_h is the maximum point of h, all $v_1^t, \ldots, v_n^t$ are the maximum points of their corresponding Schauder hats $h_1^t, \ldots, h_n^t$ of $B_{\mathcal{V}}$. We can write

$$h_1^t(v_1^t) = \max h_1^t = 1/\mathrm{den}(v_1^t), \ldots, h_n^t(v_n^t) = \max h_n^t = 1/\mathrm{den}(v_n^t)\,. \tag{26}$$

Let S^t be the $(n+1)$-simplex given by

$$S^t = [0, (v_h, 1), (v_1^t, 1), \ldots, (v_n^t, 1)]\,. \tag{27}$$

Then S^t is an $(n+1)$-dimensional pyramid of unit height and base Z^t, where

$$Z^t = [(v_h, 1), (v_1^t, 1), \ldots, (v_n^t, 1)]\,. \tag{28}$$

S^t is contained in the $(n+1)$-dimensional parallelepiped $R^t \subseteq \mathbb{R}^{n+1}$ determined by the vectors $\{(v_h, 1), (v_1^t, 1), \ldots, (v_n^t, 1)\}$. R^t is in turn included in the parallelepiped Q^t determined by the homogeneous correspondents (as given by (3)) $\mathbf{v}_h, \mathbf{v}_1^t, \ldots, \mathbf{v}_n^t$ of the vectors $v_h, v_1^t, \ldots, v_n^t$. The assumed unimodularity of $\mathcal{V}$ is to the effect that Q^t has unit volume. Now the vector $(v_h, 1)$ is obtained dividing $\mathbf{v}_h$ by $\mathrm{den}(v_h)$ (recalling that $\mathrm{den}(v_h)$ coincides with the last coordinate of $\mathbf{v}_h$, and also with $1/\max h$). Similarly, by (26)

$$(v_1^t, 1) = \max h_1^t \cdot \mathbf{v}_1^t\,, \quad \ldots\,, \quad (v_n^t, 1) = \max h_n^t \cdot \mathbf{v}_n^t\,. \tag{29}$$

It follows that

$$\mathrm{vol}(R^t) = \max h \cdot \max h_1^t \cdots \max h_n^t\,.$$

Elementary geometry shows that $\mathrm{vol}(S^t) = \mathrm{vol}(R^t)/(n+1)!$; since by (25) and (28) the bases A^t and Z^t of the two pyramids S^t and P^t have equal area, their volumes are proportional to their respective heights 1 and $\max h$. Thus

$$\mathrm{vol}(P^t) = \max h \cdot \mathrm{vol}(S^t) = \max h \cdot \frac{\max h \cdot \max h_1^t \cdots \max h_n^t}{(n+1)!}\,.$$

Recalling that $\mathrm{vol}(P) = \sum_{t=1}^{m} \mathrm{vol}(P^t)$, we have proved (23), thus settling our second claim.

Claim 3. *The state $\varsigma_{S,\beta}$ is uniquely determined by* (1).

As a matter of fact, suppose a state $\sigma\colon M \to [0,1]$ satisfies (1), with the intent of proving $\sigma = \varsigma_{S,\beta}$. By way of contradiction suppose $\sigma(f) \neq \varsigma_{S,\beta}(f)$ for some $f \in M$. By the De Concini-Procesi lemma together with Lemma 2.1(ii) there exists a *Schauder* basis $B_f = \{l_1, \ldots, l_v\}$ for M such that f is a linear combination of the l_i's with integer coefficients ≥ 0, in symbols, $f \in \operatorname{mon}(B_f)$. By hypothesis there is $D \preceq^* B_f$ such that, for all $E \preceq^* D$ and all $h \in E$, $\sigma(h)$ is as in (23). Note that D, as well as any such E, are automatically Schauder bases. Thus σ coincides with $\varsigma_{S,\beta}$ over all elements of any basis $E \preceq^* D$. Again by Lemma 2.1(ii), f is a linear combination of the hats of E with integer coefficients ≥ 0, in symbols, $f \in \operatorname{mon}(E)$. Since σ is additive, we infer $\sigma(f) = \varsigma_{S,\beta}(f)$, which is a contradiction. Our third claim is settled.

We have proved the uniqueness of $\varsigma_{S,\beta}$. Different choices of S and β result in a state still satisfying (1). Thus we can unambiguously write ς instead of $\varsigma_{S,\beta}$. It follows that ς is invariant under automorphisms. Recalling the elementary properties of the integral and the definition of McNaughton function, one immediately verifies that ς also has the remaining properties. □

PROBLEM. *Prove or disprove that the state ς of Theorem 1.5 satisfies*

$$\varsigma(h) = \frac{\max h}{(n+1)!} \sum_{C \in E(h)} \prod_{k \in C} \max k \,,$$

for every basis E of Free_n *and every $h \in E$.*

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