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Mathematica Slovaca, Vol. 47 (1997), No. 3, 319--338

Persistent URL: http://dml.cz/dmlcz/129912

Math. Slovaca, **47** (1997), No. 3, 319–338

Mathematica Slovaca

K-THEORY OF ORIENTED GRASSMANN MANIFOLDS

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(*Communicated by Július Korbaš*)

ABSTRACT. The complex K-theory of the oriented Grassmann manifold $\tilde{G}_{n,k}$ is computed. The techniques involve the representation theory of Spin(n), and the Hodgkin spectral sequence. Applications are given to the (stable) parallelizability of $\tilde{G}_{n,k}$, the existence of (weak) almost complex structures on $\tilde{G}_{n,k}$, and the existence of weak almost complex structures on oriented flag manifolds.

1. Introduction

The complex K-theory of $\tilde{G}_{n,k} = SO(n)/SO(k) \times SO(l)$, where $n = k + l$, can, in principle, be calculated for n odd or k even from theorems of Atiyah—Hirzebruch [2] and Pittie [16] (we only consider $k, l > 1$ since the case $\tilde{G}_{n,1} = \tilde{G}_{n,n-1} = S^{n-1}$ is well known). However, detailed results for these cases do not seem to be available and require substantial calculations. The situation where n is even and k odd is more delicate, mainly because here $SO(k) \times SO(l)$ is no longer a subgroup of maximal rank in $SO(n)$, and no results on $K^*(\tilde{G}_{n,k})$ seem to be available in this case. In some sense, this case is also the most interesting, since span $\tilde{G}_{n,k} = 0$ unless n is even and k odd (cf. [11; Theorem 3.1.16]), and we use the Hodgkin spectral sequence to carry out the calculations here. For the real and complex K-theory of the complex Grassmann manifolds $CG_{n,k}$, the reader is referred to [9].

In §2 and §3, the calculation of $K^*(\tilde{G}_{n,k})$ for n even, k odd is carried out. The final result is given as Theorem 3.6. In §4, the slightly easier cases n even, k even, or n odd are completed. The main results are Theorem 4.1 and Theorem 4.3.

AMS Subject Classification (1991): Primary 55N15; Secondary 19L64, 53C15, 57R25.
Key words: Grassmann manifold, K-theory, almost complex structure.

Research supported in part by a grant to the second named author from the Natural Sciences and Engineering Research Council of Canada.

The α-construction (cf. [2], [8], or [17]) is then applied in §5 to identify the (stable) tangent bundle of $\tilde{G}_{n,k}$ in $K(\tilde{G}_{n,k})$, in terms of the (K-theory) Pontrjagin classes p_1, q_1. As a first consequence, we obtain an easy proof of the following.

THEOREM 1.1. *Apart from the spheres $\tilde{G}_{n,n-1} \cong \tilde{G}_{n,1} = S^{n-1}$, the only stably parallelizable $\tilde{G}_{n,k}$ are $\tilde{G}_{4,2}$ and $\tilde{G}_{6,3}$, while the only parallelizable $\tilde{G}_{n,k}$ are $\tilde{G}_{2,1} = S^1$, $\tilde{G}_{4,3} \cong \tilde{G}_{4,1} = S^3$, $\tilde{G}_{8,7} \cong \tilde{G}_{8,1} = S^7$, and $\tilde{G}_{6,3}$.*

We remark that this result was incompletely proved in [15], and a complete proof first given in [19]. R. Kultze [12] has found a proof of the above theorem using (ordinary) Pontrjagin classes. The present proof is purely K-theoretic and very short.

The existence of almost complex structures on $\tilde{G}_{n,k}$ was studied and nearly solved (up to about ten undecided cases) in the work of several authors, notably [4], [5], [18], [21], and [22]. Just as with the parallelizability results, we are again able to apply our techniques to give a short and direct (purely K-theoretic) proof of the following theorem, which completely solves this problem.

THEOREM 1.2. *Apart from the spheres $\tilde{G}_{n,n-1} \cong \tilde{G}_{n,1} = S^{n-1}$, the only oriented Grassmann manifolds admitting a weak almost complex structure are $\tilde{G}_{6,3}$, $\tilde{G}_{n,2}$ for any n. Of these only S^2, S^6, $\tilde{G}_{n,2}$, $n \geq 4$, admit almost complex structures.*

This theorem is readily applied to completely settle the existence of (weak) almost complex structures on oriented flag manifolds, cf. Theorem 5.7.

2. Representation rings and the restriction map

We assume some familiarity with the (complex) representation ring of $\mathrm{Spin}(n)$, and the use of the Hodgkin spectral sequence in computing $K^*(G/H)$, where H is a closed subgroup of a compact connected Lie group G with $\pi_1(G)$ torsion free. A brief summary of these techniques is found in [1], and our notation will also be similar to that used in this reference. For further references on this material, see [6], [8], [10], [17].

Let $n \geq 6$ be even, $3 \leq k \leq n-3$ be odd. Write $n = k + l = 2(s+t+1)$, where $k = 2s+1$, $l = 2t+1$, $s,t \geq 1$. The oriented Grassmann manifold $\tilde{G}_{n,k}$ is precisely the homogeneous space $SO(n)/SO(k) \times SO(l)$. To apply the Hodgkin spectral sequence, we lift this to $\tilde{G}_{n,k} \cong G/H$, using the double covering ϕ: $\mathrm{Spin}(n) \twoheadrightarrow SO(n)$, where

$$G = \mathrm{Spin}(n),$$
$$H = \phi^{-1}(SO(k) \times SO(l)) \subset \mathrm{Spin}(n).$$

It is easy to see that $H \approx \mathrm{Spin}(k) \times_{\mathbb{Z}/2} \mathrm{Spin}(l)$, where $\mathbb{Z}/2$ acts on $\mathrm{Spin}(k) \times \mathrm{Spin}(l)$ by the involution $\theta(x, y) = (-x, -y)$.

The complex representation ring $RG = R\,\mathrm{Spin}(n)$ is conveniently expressed as a polynomial algebra with generators all in the kernel of the augmentation map $\varepsilon\colon RG \twoheadrightarrow \mathbb{Z}$ as $RG = \mathbb{Z}[\mathcal{P}_1, \dots, \mathcal{P}_{s+t-1}, X_n, \delta_n^+]$.

Here $\mathcal{P}_i$ (also p_i, q_i) denotes the ith Pontrjagin class ($\mathcal{P}_0 = 1$), $X_n = \triangle_n^+ - \triangle_n^-$, $\delta_n^+ = \triangle_n^+ - 2^{s+t}$, and $\triangle_n^\pm$ are the standard spinor representations (of degree 2^{s+t}).

Note that θ is not a group homomorphism, but does preserve conjugacy classes. Hence RH is readily calculated as the $\mathbb{Z}/2$ invariant part of $R(\mathrm{Spin}(k) \times \mathrm{Spin}(l)) \approx R\,\mathrm{Spin}(k) \otimes R\,\mathrm{Spin}(l) = \mathbb{Z}[p_1, \dots, p_{s-1}, \delta_k] \otimes \mathbb{Z}[q_1, \dots, q_{t-1}, \delta_l]$, where $\delta_k = \triangle_k - 2^s$, $\delta_l = \triangle_l - 2^t$. The Pontrjagin classes are all induced from corresponding orthogonal representations by the double cover ϕ, and hence are $\mathbb{Z}/2$-invariant. On the other hand, $\triangle_k$, $\triangle_l$, arise from certain Clifford modules and are $\mathbb{Z}/2$-equivariant ($\triangle_k(-x) = -\triangle_k(x)$), so it is clear that RH is generated by the classes $p_1, \dots, p_{s-1}, \triangle_k^2, q_1, \dots, q_{t-1}, \triangle_l^2, \triangle_k\triangle_l$. To obtain a class in $\mathrm{Ker}\,\varepsilon$, write $\delta_{k,l} = \triangle_k\triangle_l - 2^{s+t}$, and we now have the following convenient description of RH.

Lemma 2.1. *$RH \approx \mathbb{Z}[p_1, \dots, p_s, q_1, \dots, q_t, \delta_{k,l}]/\sim$, with the single defining relation*

$$\delta_{k,l}^2 = \sum_{0 \le r < s+t} \Big(4^r \sum_{i+j=r} p_{s-i} q_{t-j}\Big) - 2^{s+t+1}\delta_{k,l}. \tag{1}$$

Proof. Since $p_s = \triangle_k^2 - \sum_{j=1}^{s} 4^j p_{s-j}$ (cf. [1; p. 33]), the subalgebra of $R\,\mathrm{Spin}(k)$ generated by $p_1, \dots, p_{s-1}, \triangle_k^2$ is the same as the polynomial algebra $\mathbb{Z}[p_1, \dots, p_s]$, and similarly, for $R\,\mathrm{Spin}(l)$. Then RH is isomorphic to $\mathbb{Z}[p_1, \dots, p_s, q_1, \dots, q_t, \delta_{k,l}]$ modulo the ideal generated by the minimal polynomial of $\delta_{k,l}$, which is easily seen to be the relation (1) from

$$\begin{aligned}
\delta_{k,l}^2 &= \left(\triangle_k\triangle_l - 2^{s+t}\right)^2 \\
&= \triangle_k^2\triangle_l^2 - 2^{s+t+1}\triangle_k\triangle_l + 4^{s+t} \\
&= \Big(\sum_{0 \le i \le s} 4^i p_{s-i}\Big)\Big(\sum_{0 \le j \le t} 4^j q_{t-j}\Big) + 4^{s+t} - 2^{s+t+1}(\triangle_k\triangle_l - 2^{s+t}) - 2 \cdot 4^{s+t} \\
&= \sum\left\{4^{i+j} p_{s-i} q_{t-j} : 0 \le i \le s,\ 0 \le j \le t,\ i+j < s+t\right\} - 2^{s+t+1}\delta_{k,l}
\end{aligned}$$

(note that the term in the sum where $i = s$, $j = t$ is simply 4^{s+t}). $\square$

Corollary 2.2. *$RH \approx P[\delta_{k,l}]/\sim$, where $P = \mathbb{Z}[p_1, \ldots, p_s, q_1, \ldots, q_t]$, and the relation is again* (1).

Now that RH and RG are known, the next step is to determine the structure of RH as an RG-module, i.e., to calculate $j^\sharp\colon RG \to RH$, where $j\colon H \hookrightarrow G$. To do this, the standard method of restricting to maximal tori is used.

Recall $\phi\colon \mathrm{Spin}(n) \twoheadrightarrow SO(n)$ is the well known double cover. Then, following well-known methods (cf. [10; Ch. 13, §8]) the maximal tori of $\mathrm{Spin}(n)$ and $SO(n)$ are respectively the $s+t+1$ tori T', T, where $(\phi \mid T')\colon T' \twoheadrightarrow T$ is a double cover. Following the same reference (or [1; §4]), we have $RT = \mathbb{Z}[u_1^2, u_1^{-2}, \ldots, u_{s+t+1}^2, u_{s+t+1}^{-2}]$, $RT' = [u_1^2, u_1^{-2}, \ldots, u_{s+t+1}^2, u_{s+t+1}^{-2}, u_1 u_2 \cdot \ldots \cdot u_{s+t+1}]$, with $\phi^\sharp(u_i^2) = u_i^2$.

To find a suitable maximal torus for H, let $\phi|_H = \varphi\colon H \twoheadrightarrow SO(k) \times SO(l)$. As maximal torus for $SO(k) \times SO(l)$ we select $T_0 \subset T$, where relative to a standard basis $e_1, \ldots, e_n \in \mathbb{R}^n$, T_0 fixes e_k, e_{k+1} and preserves all 2-planes $\mathrm{span}\{e_{2j-1}, e_{2j}\}$, $0 < j \le s+t+1$. Thus T_0 consists of diagonal blocks

$$\begin{bmatrix} \cos\theta_j & \sin\theta_j \\ -\sin\theta_j & \cos\theta_j \end{bmatrix}, \qquad 0 < j \le s+t+1,$$

with $\theta_{s+1} = 0$ (corresponding to the fixing of e_k, e_{k+1}). Then take $T_0' = \varphi^{-1}(T_0) \subset T'$ as maximal torus of H (it is a torus since it is a finite cover of T_0, and clearly, has maximal rank). We have

$$\begin{aligned} RT_0 &= \mathbb{Z}\big[u_i^2, u_i^{-2} : 1 \le i \le s+t+1,\ i \ne s+1\big], \\ RT_0' &= \mathbb{Z}\big[u_i^2, u_i^{-2}, u_1 \cdot \ldots \cdot u_s \cdot u_{s+2} \cdot \ldots \cdot u_{s+t+1} : 1 \le i \le s+t+1,\ i \ne s+1\big]. \end{aligned}$$

By considering first the inclusion $T_0 \hookrightarrow T$, then $j\colon T_0' \hookrightarrow T'$, it is now clear that

$$j^\#(u_i^2) = \begin{cases} u_i^2 & \text{for } i \ne s+1, \\ 1 & \text{for } i = s+1. \end{cases}$$

Proposition 2.3. *We have*

(a) $j^\#(\mathcal{P}_r) = \sum_{i+j=r} p_i q_j$, $1 \le r$,

(b) $j^\#(X_n) = 0$,

(c) $j^\#(\delta_n^+) = \delta_{k,l}$.

P r o o f. Identifying representations in the customary way with their restrictions to the maximal torus, and letting σ_r denote the rth symmetric polynomial, for $r > 0$ we have

$$\begin{aligned} \mathcal{P}_r &= \sigma_r(u_1^2 + u_1^{-2} - 2, \ldots, u_{s+t+1}^2 + u_{s+t+1}^{-2} - 2) \\ &= \sigma_r(z_1, \ldots, z_{s+t+1}), \end{aligned}$$

where $z_i = u_i^2 + u_i^{-2} - 2$.

Then

$$\begin{aligned} j^{\#}\mathcal{P}_r &= \sigma_r(z_1, \dots, z_s, 0, z_{s+2}, \dots, z_{s+t+1}) \\ &= \sigma_r(z_1, \dots, \hat{z}_{s+1}, \dots, z_{s+t+1}) \\ &= \sum_{i+j=r} \sigma_i(z_1, \dots, z_s) \cdot \sigma_j(z_{s+2}, \dots, z_{s+t+1}) \\ &= \sum_{i+j=r} p_i q_j , \end{aligned}$$

proving (a).

For (b) and (c), first note these are equivalent to showing $j^{\#}(\triangle_n^+ - \triangle_n^-) = 0$, $j^{\#}(\triangle_n^+) = \triangle_k \triangle_l$.

$$\triangle_n^+ = \sum_{\prod \varepsilon_i = +1} u_1^{\varepsilon_1} \cdot \dots \cdot u_{s+t+1}^{\varepsilon_{s+t+1}} , \qquad \triangle_n^- = \sum_{\prod \varepsilon_i = -1} u_1^{\varepsilon_1} \cdot \dots \cdot u_{s+t+1}^{\varepsilon_{s+t+1}} ,$$

where $\varepsilon_i = \pm 1$, implies

$$\begin{aligned} j^{\#}(\triangle_n^-) = j^{\#}(\triangle_n^+) &= \sum_{\varepsilon_i = \pm 1} u_1^{\varepsilon_1} \cdot \dots \cdot \hat{u}_{s+1}^{\varepsilon_{s+1}} \cdot \dots \cdot u_{s+t+1}^{\varepsilon_{s+t+1}} \\ &= \prod_{1 \le i \le s} (u_i + u_i^{-1}) \cdot \prod_{s+2 \le j \le s+t+1} (u_j + u_j^{-1}) = \triangle_k \triangle_l , \end{aligned}$$

giving the desired results. □

Remark. We will write $\mathcal{P}_i' = j^{\#}(\mathcal{P}_i) \in RH$. Also note that although $R\,\mathrm{Spin}(n) = \mathbb{Z}[\mathcal{P}_1, \dots, \mathcal{P}_{s+t-1}, X_n, \delta_n^+]$, the classes $\mathcal{P}_r$ are in this ring for any $r \ge 0$; thus (a) holds as stated for $r \ge 1$.

In much the same way as in the proof of Lemma 2.1, the relation $\mathcal{P}_{s+t} = \triangle_n^+ \triangle_n^- - \sum_{2 \le k \le s+t+1} 4^{k-1} \mathcal{P}_{s+t+1-k}$, in $R\,\mathrm{Spin}(n)$, implies

$$\Lambda := \mathbb{Z}[\mathcal{P}_1, \dots, \mathcal{P}_{s+t}] \subset R\,\mathrm{Spin}(n) .$$

We also write L for the subalgebra $\mathbb{Z}[\mathcal{P}_1, \dots, \mathcal{P}_{s+t-1}]$ of Λ. From Proposition 2.3(a), $j^{\#} \colon \Lambda \to P \subset RH$, hence RH has a Λ-module (and L-module) structure induced by $j^{\#}$.

Lemma 2.4. *RH is free as a Λ-module.*

P r o o f. From Corollary 2.2, RH is free as a P-module (on generators $1, \delta_{k,l}$), so it will suffice to show that P is Λ-free. For the purpose of this proof (and a few times later), we introduce gradings on these rings by letting

$w(p_i) = w(q_i) = w(\mathcal{P}_i) = i$, $i \geq 1$. Then $j^{\#}\colon \Lambda \to P$ is a homomorphism of w-graded algebras, and we can rewrite 2.3(a) in graded form as

$$1 + \mathcal{P}'_1 + \cdots + \mathcal{P}'_{s+t} = (1 + p_1 + \cdots + p_s)(1 + q_1 + \cdots + q_t).$$

Now consider the fibration $p\colon B\big(U(s) \times U(t)\big) = BU(s) \times BU(t) \to BU(s+t)$ with fibre $CG_{s+t,s} = U(s+t)/U(s) \times U(t)$. The cohomology structure of these spaces is well known, but in any case following [19; Ch. 5, §7], we note first that the bundle admits a cohomology extension of the fibre with $\mathbb{Z}$ coefficients, whence the Leray-Hirsch theorem implies that $H^*\big(BU(s) \times BU(t); \mathbb{Z}\big)$ is a free $H^*\big(BU(s+t); Z\big)$-module via p^*. But apart from notation, Λ is the same as $H^*\big(BU(s+t); \mathbb{Z}\big)$, P the same as $H^*\big(BU(s) \times BU(t); \mathbb{Z}\big)$, and $j^{\#}$ the same as p^*. Hence P is Λ-free. □

COROLLARY 2.5. *RH is free as an L-module.*

3. Calculation of $K^*(\tilde{G}_{n,k})$, n even and k odd

Having determined the structure of RH as an RG-module, we now turn to calculating $\mathrm{Tor}^*_{RG}(RH, \mathbb{Z})$, the E_2 term of the Hodgkin spectral sequence. First we reduce this to a simpler calculation by applying a change of rings theorem.

DEFINITION 3.1. Let $A = RG/\langle \mathcal{P}_1, \dots, \mathcal{P}_{s+t-1}\rangle$, $B = RH/\langle \mathcal{P}'_1, \dots, \mathcal{P}'_{s+t-1}\rangle$.

Notice that B becomes an A-module, the structure induced by $j^{\#}\colon RG \to RH$ (recall $\mathcal{P}'_i = j^{\#}(\mathcal{P}_i)$), and also that A is just the polynomial algebra $A \approx \mathbb{Z}[X_n, \delta_n^+]$.

LEMMA 3.2. $\mathrm{Tor}^*_{RG}(RH, \mathbb{Z}) \approx \mathrm{Tor}^*_A(B, \mathbb{Z})$.

Proof. Following [7; Ch. 16, Theorem 6.1] and the related definitions, we wish to apply the change of rings spectral sequence

$$\mathrm{Tor}_\Omega\big(\mathrm{Tor}_\Phi(M, K), C\big) \implies \mathrm{Tor}_\Gamma(M, C), \tag{2}$$

where $\Phi = L = \mathbb{Z}[\mathcal{P}_1, \dots, \mathcal{P}_{s+t-1}]$, $\Gamma = RG$, $M = RH$, $K = C = \mathbb{Z}$, $\Omega = \Gamma/\!/\varphi$ with $\varphi\colon L \hookrightarrow RG$. We check that

(a) φ is supplemented since the composition $\varepsilon\varphi\colon L \to \mathbb{Z}$ is the same as $\varepsilon\colon L \to \mathbb{Z}$.

(b) $RG = L[X_n, \delta_n^+]$ is clearly L-flat.

(c) φ is normal since RG is commutative.

Now $\Omega = \Gamma/\!/\varphi = \Gamma/\Gamma \cdot (\varphi IL) = RG/\langle IL\rangle$, where IL is the augmentation ideal $\operatorname{Ker}\varepsilon$. Thus

$$\Omega = RG/\langle \mathcal{P}_1, \dots, \mathcal{P}_{s+t-1}\rangle = A.$$

Substituting into (2), we have $\operatorname{Tor}^*_A(\operatorname{Tor}^*_L(RH,\mathbb{Z}),\mathbb{Z}) \Longrightarrow \operatorname{Tor}^*_{RG}(RH,\mathbb{Z})$.

Recall, by 2.5, that RH is L-free. Therefore

$$\operatorname{Tor}^*_L(RH,\mathbb{Z}) = \operatorname{Tor}^o_L(RH,\mathbb{Z}) \approx RH\otimes_L \mathbb{Z} \approx RH/\langle \mathcal{P}'_1,\dots,\mathcal{P}'_{s+t-1}\rangle = B,$$

since L acts trivially on $\mathbb{Z}$. The spectral sequence thus collapses and gives $\operatorname{Tor}^*_A(B,\mathbb{Z}) \approx \operatorname{Tor}^*_{RG}(RH,\mathbb{Z})$. □

Since $A = \mathbb{Z}[X_n, \delta^+_n]$ is a polynomial algebra, the Koszul resolution can be applied to compute $\operatorname{Tor}^*_A(B,\mathbb{Z})$ (cf. [14; p. 205]).

DEFINITION 3.3. Let $H_{s,t} = \mathbb{Z}[p_1,\dots,p_s,q_1,\dots,q_t]/\langle \mathcal{P}'_1,\dots,\mathcal{P}'_{s+t}\rangle$. Note that as a graded algebra via w (as in the proof of 2.4), we can write

$$H_{s,t} = \mathbb{Z}[p_1,\dots,p_s,q_1,\dots,q_t]/\langle (1+p_1+\dots+p_s)(1+q_1+\dots+q_t)-1\rangle .$$

Note, also, that as a graded algebra, $H_{s,t}$ is isomorphic to the integral cohomology ring $H^*(\mathcal{C}G_{s+t,s};\mathbb{Z})$, which is well known to be free as an abelian group.

THEOREM 3.4. $\operatorname{Tor}^*_{RG}(RH,\mathbb{Z}) \approx E^*_{\mathbb{Z}}[X]\otimes H_{s,t}$, *where* $E^*_{\mathbb{Z}}[X]$ *is exterior algebra over* $\mathbb{Z}$ *on the generator* X. *The isomorphism is an isomorphism of graded algebras with* $\deg X = 1$, $\deg p_i = \deg q_j = 0$.

P r o o f. $\operatorname{Tor}^*_{RG}(RH,\mathbb{Z}) \approx \operatorname{Tor}^*_A(B,\mathbb{Z})$ by 3.2, so we compute the latter.

$$B = \mathbb{Z}[p_1,\dots,p_s,q_1,\dots,q_t,\delta_{k,l}] \Big/ \Big\langle \mathcal{P}'_1,\dots,\mathcal{P}'_{s+t-1}, \delta^2_{k,l} + 2^{s+t+1}\delta_{k,l} - \sum_{0\le r<s+t} 4^r \sum_{i+j=r} p_{s-i}q_{t-j}\Big\rangle ,$$

$$A = \mathbb{Z}[X_n,\delta^+_n],$$

and B is an A-module via $\theta\colon A\to B$ (induced by $j^{\#}$), where, by 2.3, $\theta(X_n)=0$ and $\theta(\delta^+_n) = \bar\delta_{k,l}$, writing $\bar\delta_{k,l}$ for the class of $\delta_{k,l}$ in B. The Koszul complex of A is $E^* = \Lambda^*_A[X,D]$, $\deg X = \deg D = 1$, $d(x) = X_n$, $d(D) = \delta^+_n$. Together with $\varepsilon(X_n) = \varepsilon(\delta^+_n) = 0$, this gives $E^* \overset{\varepsilon}{\twoheadrightarrow} \mathbb{Z}$, a free A-resolution of $\mathbb{Z}$. Then

$$\operatorname{Tor}^*_A(B,\mathbb{Z}) = H_*(B\otimes_A E^*) = H_*(\Lambda^*_B[X,D])$$

with $d(X) = \theta(X_n) = 0$, $d(D) = \theta(\delta^+_n) = \bar\delta_{k,l}$. We have thus to compute the homology of the chain complex

$$0 \longrightarrow \underset{X\wedge D}{B} \xrightarrow{d_2} \underset{X,D}{B\oplus B} \xrightarrow{d_1} \underset{1}{B} \longrightarrow 0$$

with $d_1(X) = 0$, $d_2(D) = \bar\delta_{k,l}$, $d_1(X\wedge D) = 0\cdot D - X\cdot\bar\delta_{k,l} = -\bar\delta_{k,l}\cdot X$. Clearly $H_i = 0$, $i\ge 2$, and $H_1 \approx H_0 \approx B/\langle\bar\delta_{k,l}\rangle$, where H_0 is generated by 1, and H_1 by X. Thus

$$\operatorname{Tor}^*_A(B,\mathbb{Z}) \approx E^*_{\mathbb{Z}}[X]\otimes \big(B/\langle\bar\delta_{k,l}\rangle\big).$$

To complete the proof, we simply note

$$\sum_{0\le r<s+t} 4^r \sum_{i+j=r} p_{s-i}q_{t-j} = \sum_{0\le r<s+t} 4^r \mathcal{P}'_{s+t+r} \qquad \text{(by 2.3)}$$
$$= \mathcal{P}'_{s+t} + \sum_{1\le j\le s+t-1} 4^{s+t-j}\mathcal{P}'_j,$$

hence,

$$B/\langle\bar{\delta}_{k,l}\rangle \approx \mathbb{Z}[p_1,\dots,p_s,q_1,\dots,q_t] \Big/ \Big\langle \mathcal{P}'_1,\dots,\mathcal{P}'_{s+t-1}, \sum_{0\le r<s+t} 4^r \sum_{i+j=r} p_{s-i}q_{t-j} \Big\rangle$$
$$\approx \mathbb{Z}[p_1,\dots,p_s,q_1,\dots,q_t]/\langle\mathcal{P}'_1,\dots,\mathcal{P}'_{s+t-1},\mathcal{P}'_{s+t}\rangle = H_{s,t}.$$

□

COROLLARY 3.5. *The Hodgkin spectral sequence for $\tilde{G}_{n,k}$ collapses.*

Proof. Since E_2^p is 0 unless $p = 0,1$, by 3.4, all differentials vanish for dimensional reasons. □

THEOREM 3.6. $K^*(\tilde{G}_{n,k}) \approx E^*_{\mathbb{Z}}[X]\otimes H_{s,t}$, *where* $\deg X = 1$ *and* $\deg H_{s,t} = 0$.

Proof. One applies the techniques of [1; 6.1 – 6.5], which solve the extension problem to reconstruct K^* from $E_\infty = E_2$. □

4. $K^*(\tilde{G}_{n,k})$, n odd or k even

We are now in the situations where H is a subgroup of maximal rank in G, so the answer is given at least in theory by a theorem of Pittie [16] (or see [8; p. 81]):

$$K(\tilde{G}_{n,k}) = K(G/H) \approx RH \otimes_{RG} \mathbb{Z},$$
$$K^1(\tilde{G}_{n,k}) = 0.$$

Here RG acts on RH by restriction and on $\mathbb{Z}$ by dimension. It is also proved there that RH is now a free RG-module, which is easily seen to imply that $K(\tilde{G}_{n,k})$ is a free abelian group (as far as its additive structure) and, in particular, is torsion free.

In spite of this simplification of the work as compared to §2 and §3, it is still non-trivial to determine the precise structure of $K(\tilde{G}_{n,k})$ here, due mainly to algebraic complications. We will first state and prove the result for n even, k even, then handle the n odd case.

Let $n = 2m = k + l$, $m \ge 3$, $k = 2s$, $l = 2t$, $s,t \ge 1$, $u = s + t - 1$. Recall $R\,\mathrm{Spin}(k)$ has the Pontrjagin classes $p_1, p_2, \dots$ as well as $\triangle_s^+$, $\triangle_s^-$, and similarly, $R\,\mathrm{Spin}(l)$ has $q_1, q_2, \dots$, $\triangle_t^+$, $\triangle_t^-$. Defining $H_{s,t}$ exactly as in Definition 3.3, and setting $\triangle^{++} = \triangle_s^+\triangle_t^+$, $\triangle^{+-} = \triangle_s^+\triangle_t^-$, we have the following.

THEOREM 4.1. *$K(\tilde{G}_{2s+2t,2s}) \approx H_{s,t}[(\Delta_s^+)^2, (\Delta_t^+)^2, \Delta^{++}, \Delta^{+-}]/\sim$, where the relations are derived from* (a) – (f) *below, and the torsion freeness of $K(\tilde{G}_{2s+2t,2s})$:*

(a) $2^{s+t-1}\Delta^{++} = g_s(\Delta_t^+)^2 + g_t(\Delta_s^+)^2$,
(b) $2^{s+t-1}\Delta^{+-} = -g_s(\Delta_t^+)^2 + g_t(\Delta_s^+)^2 + f_t g_s$,
(c) $p_s\big((\Delta_t^+)^2 - g_t\big) = 0 = q_t\big((\Delta_s^+)^2 - g_s\big)$,
(d) $(\Delta_s^+)^4 = f_s(\Delta_s^+)^2 - g_s^2$,
(e) $(\Delta_t^+)^4 = f_t(\Delta_t^+)^2 - g_t^2$,
(f) $(\Delta_s^+)^2(\Delta_t^+)^2 = g_t(\Delta_s^+)^2 + g_s(\Delta_t^+)^2 - g_s g_t$,

with $g_s = \sum_{i=1}^{s} 4^{i-1}p_{s-i} \in H_{s,t}$, $f_s = p_s + 2g_s \in H_{s,t}$, *and* $g_t, f_t \in H_{s,t}$ *are defined similarly.*

Remark. It is not necessary to change to augmentation 0 classes $(\delta_s^+)^2$, etc. here since the Koszul resolution will not be needed. We could similarly replace the Pontrjagin classes by exterior powers, but this seems less convenient so is not done. The notation f_s, g_s, f_t, g_t is slightly ambiguous if $s = t$, but the correct interpretation should always be clear.

Proof of Theorem 4.1. As in §2, $\tilde{G}_{n,k} = G/H$, where $G = \mathrm{Spin}(n)$, $H = \mathrm{Spin}(k) \times_{\mathbb{Z}/2} \mathrm{Spin}(l)$, and $RG = \mathbb{Z}[\mathcal{P}_1, \dots, \mathcal{P}_{s+t-2}, \Delta_m^+, \Delta_m^-]$ (recall $m = n/2 = s + t$).

Also, as in §2, we calculate RH as the $\mathbb{Z}/2$-invariant part of

$$R\big(\mathrm{Spin}(k) \times \mathrm{Spin}(l)\big) = \mathbb{Z}[p_1, \dots, p_{s-2}, q_1, \dots, q_{t-2}, \Delta_s^+, \Delta_s^-, \Delta_t^+, \Delta_t^-].$$

Now recall

$$\begin{aligned} p_{s-1} &= \Delta_s^+\Delta_s^- - \sum_{i=2}^{s} 4^{i-1}p_{s-i}, \\ p_s &= (\Delta_s^+ - \Delta_s^-)^2, \end{aligned} \tag{3}$$

and similarly for q_{t-1}, q_t (cf. [1; p. 33]). Proceeding as in §2, it is now clear that RH is generated by $p_1, \dots, p_{s-2}, q_1, \dots, q_{t-2}, (\Delta_s^+)^2, (\Delta_s^-)^2, (\Delta_s^+\Delta_s^-), (\Delta_t^+)^2, (\Delta_t^-)^2, \Delta_t^+\Delta_t^-, \Delta_s^+\Delta_t^+, \Delta_s^+\Delta_t^-, \Delta_s^-\Delta_t^+, \Delta_s^-\Delta_t^-$. Equivalently, using (3) and the notation $\Delta^{\varepsilon\eta} = \Delta_s^\varepsilon\Delta_t^\eta$ for $\varepsilon, \eta \in \{+, -\}$, RH is generated by $p_1, \dots, p_s, q_1, \dots, q_t, (\Delta_s^+)^2, (\Delta_t^+)^2, \Delta^{++}, \Delta^{+-}, \Delta^{-+}, \Delta^{--}$.

Remark. The argument thus far really only holds for $s, t > 1$. However, in the somewhat trivial case $s = 1$ or $t = 1$, it is easy to check directly that the last set of generators given for RH is still correct (recall $\Delta_1^+\Delta_1^- = 1$). A similar statement applies in the proof of Theorem 4.3.

Again using (3),

$$(\triangle_s^-)^2 = f_s - (\triangle_s^+)^2\,,\qquad \triangle_s^+\triangle_s^- = g_s\,, \tag{4}$$

and similarly for $(\triangle_t^-)^2$, $\triangle_t^+\triangle_t^-$.

The subalgebra P of RH generated by $\{p_i, q_j\}$ is clearly a polynomial algebra. Furthermore, the elements $1, (\triangle_s^+)^2, (\triangle_t^+)^2, (\triangle_s^+)^2(\triangle_t^+)^2, \triangle^{++}, \triangle^{+-}$, $\triangle^{-+}, \triangle^{--}$ generate RH as a free P-module, due to the following relations:

$$\begin{array}{rl}
\text{(i)} & (\triangle_s^+)^4 = f_s(\triangle_s^+)^2 - g_s^2\,, \quad (\triangle_t^+)^4 = f_t(\triangle_t^+)^2 - g_t^2\,,\\
\text{(ii)} & (\triangle_s^+)^2\triangle^{+\eta} = f_s\triangle^{+\eta} - g_s\triangle^{-\eta}\,, \quad (\triangle_t^+)^2\triangle^{\varepsilon+} = f_t\triangle^{\varepsilon+} - g_t\triangle^{\varepsilon-}\,,\\
 & \text{where } \varepsilon,\eta \in \{+,-\}\,,\\
\text{(iii)} & (\triangle_s^+)^2\triangle^{-\eta} = g_s\triangle^{+\eta}\,, \quad (\triangle_t^+)^2\triangle^{\varepsilon-} = g_t\triangle^{\varepsilon+}\,,\\
\text{(iv)} & \triangle^{++}\triangle^{+-} = g_t(\triangle_s^+)^2\,,\\
\text{(v)} & \triangle^{++}\triangle^{-+} = g_s(\triangle_t^+)^2\,,\\
\text{(vi)} & \triangle^{++}\triangle^{--} = \triangle^{+-}\triangle^{-+} = g_sg_t\,,\\
\text{(vii)} & \triangle^{--}\triangle^{+-} = g_s\big(f_t - (\triangle_t^+)^2\big)\,,\\
\text{(viii)} & \triangle^{--}\triangle^{-+} = g_t\big(f_s - (\triangle_s^+)^2\big)\,,\\
\text{(ix)} & (\triangle^{++})^2 = (\triangle_s^+)^2(\triangle_t^+)^2\,,\\
\text{(x)} & (\triangle^{+-})^2 = (\triangle_s^+)^2\big(f_t - (\triangle_t^+)^2\big)\,,\\
\text{(xi)} & (\triangle^{-+})^2 = (\triangle_t^+)^2\big(f_s - (\triangle_s^+)^2\big)\,,\\
\text{(xii)} & (\triangle^{--})^2 = \big(f_s - (\triangle_s^+)^2\big)\big(f_t - (\triangle_t^+)^2\big)\,.
\end{array} \tag{5}$$

Most of these relations are very easy to verify. For example, let us prove (i) and (ii), the others are even easier.

$$(\triangle_s^+)^4 = \big((\triangle_s^+)^2 + (\triangle_s^-)^2\big)(\triangle_s^+)^2 - (\triangle_s^+\triangle_s^-)^2 = f_s(\triangle_s^+)^2 - g_s^2\,, \quad \text{using (4)}.$$

$$\begin{aligned}
(\triangle_s^+)^2\triangle^{+\eta} &= \big(f_s - (\triangle_s^-)^2\big)\triangle_s^+\triangle_t^\eta\\
&= f_s\triangle^{+\eta} - \triangle_s^+\triangle_s^-\triangle_s^-\triangle_t^\eta = f_s\triangle^{+\eta} - g_s\triangle^{-\eta}\,, \qquad \text{again using (4)}.
\end{aligned}$$

Turning to the restriction map $j^\#\colon R\,\mathrm{Spin}(n) \to RH$, the fact that H has maximal rank in $G = \mathrm{Spin}(n)$ means they have the same maximal torus. It is then easily calculated that

$$\begin{aligned}
j^\#(\mathcal{P}_r) &= \sum_{i+j=r} p_iq_j\,, \qquad r \ge 1\,,\\
j^\#(\triangle_m^+) &= \triangle^{++} + \triangle^{--}\,,\\
j^\#(\triangle_m^-) &= \triangle^{+-} + \triangle^{-+}\,.
\end{aligned}$$

As an immediate consequence

$$K(\tilde G_{2s+2t,2s}) \approx RH \otimes_{RG} \mathbb{Z} \approx H_{s,t}\big[(\triangle_s^+)^2, (\triangle_t^+)^2, \triangle^{++}, \triangle^{+-}\big]/\!\sim,$$

where the relations comprise all of (5) together with $\triangle^{--} = 2^u - \triangle^{++}$, $\triangle^{-+} = 2^u - \triangle^{+-}$ (recall $u = s + t - 1$).

This is essentially Theorem 4.1, apart from a tedious but fairly routine task of showing that the 22 relations involved reduce to the relations (a) – (f) in Theorem 4.1 (note (d), (e) are identical to (i)). Rather than giving all details of this calculation, we will outline the derivations of (a), (b), (c), (f) and leave it to the reader to either take on faith or to verify that all the 22 relations can in turn be derived from (a) – (f), the relation in $H_{s,t}$, and the torsion freeness of $K(\tilde{G}_{2s+2t,2s})$.

To do this, first recall $(1+p_1+\cdots+p_s)(1+q_1+\cdots+q_t)=1$ is the relation (in the graded sense) in $H_{s,t}$. Then it is not hard to derive the following helpful formulae in $H_{s,t}$:

$$\begin{aligned} f_s g_t + f_t g_s &= 4^u\,, \\ p_s g_t + q_t g_s + 4 g_s g_t &= 4^u\,, \\ f_s f_t + 4 g_s g_t &= 2\cdot 4^u\,, \qquad p_s q_t = 0\,. \end{aligned} \tag{6}$$

Combining (iv), (v), and (vii) gives (substituting for $\triangle^{-+}$, $\triangle^{--}$)

$$\triangle^{++}\triangle^{+-} = g_t(\triangle_s^+)^2 = -g_s(\triangle_t^+)^2 + 2^u\triangle^{++} = g_s\big((\triangle_t^+)^2 - f_t\big) + 2^u\triangle^{+-}\,;$$

(a) and (b) follow at once by solving for $2^u\triangle^{++}$, $2^u\triangle^{+-}$.

Similarly, from (vi) and (ix),

$$\begin{aligned} (\triangle^{++})^2 = (\triangle_s^+)^2(\triangle_t^+)^2 &= 2^u\triangle^{++} - g_s g_t\,, \\ (\triangle_s^+)^2(\triangle_t^+)^2 &= g_s(\triangle_t^+)^2 + g_t(\triangle_s^+)^2 - g_s g_t\,, \qquad \text{giving (f)}\,. \end{aligned}$$

Also

$$\begin{aligned} (\triangle^{++})^2 &= (\triangle_s^+)^2(\triangle_t^+)^2 \\ &= (2^u - \triangle^{--})^2 = 4^u - 2^{u+1}\triangle^{--} + (\triangle^{--})^2 \\ &= 4^u - 2^{u+1}(2^u - \triangle^{++}) + f_s f_t - f_s(\triangle_t^+)^2 - f_t(\triangle_t^-)^2 + (\triangle_t^+)^2(\triangle_t^-)^2 \end{aligned}$$

from (xii), giving

$$\begin{aligned} 0 &= -4^u + 2\big(g_s(\triangle_t^+)^2 + g_t(\triangle_s^+)^2\big) + f_s f_t - f_s(\triangle_t^+)^2 - f_t(\triangle_t^-)^2 - 4^u + f_s f_t \\ &= q_t(\triangle_s^+)^2 + p_s(\triangle_t^+)^2\,. \end{aligned}$$

Similarly, using the relations (x), (vi), (xi), one derives

$$4^u - 2 g_s f_t = -q_t(\triangle_s^+)^2 + p_s(\triangle_t^+)^2\,.$$

These last two formulae can easily be proved to be equivalent to the relations (c), using the fact that $K(\tilde{G}_{n,k})$ is here torsion free. $\square$

Corollary 4.2. *$Q \otimes K(\tilde{G}_{2s+2t,2s}) \approx (Q \otimes H_{s,t})[\alpha, \beta]/\sim$, relations being $p_s\beta = q_t\alpha = \alpha\beta = 0$, $\alpha^2 = p_s(\alpha + g_s)$, $\beta^2 = q_t(\beta + g_t)$. In particular, $Q \otimes H_{s,t}$ is a subalgebra, so $H_{s,t}$ is a subalgebra of $K(\tilde{G}_{2s+2t,2s})$.*

This is evident upon setting $\alpha = (\Delta_s^+)^2 - g_s$, $\beta = (\Delta_t^+)^2 - g_t$.

As an example, $Q \otimes K(\tilde{G}_{4,2}) \approx Q \otimes H_{1,1}[\alpha, \beta]/\sim$ with relations $p_1\beta = q_1\alpha = \alpha\beta = 0$, $\alpha^2 = p_1(\alpha + g_s)$, $\beta^2 = q_1(\beta + g_t)$. One readily finds that as a vector space over Q, $\{1, p_1, \alpha, \beta\}$ is a basis. This corresponds to the fact $\tilde{G}_{4,2} \cong S^2 \times S^2$, and $K(S^2 \times S^2)$ is well known to be $\mathbb{Z}^4$.

Since $\tilde{G}_{n,k} \cong \tilde{G}_{n,n-k}$, for the remaining case with n odd it suffices to consider k even. Let $n = 2s + 2t + 1$, $k = 2s$, $l = 2t + 1$, $s \geq 2$, $t \geq 1$. Then $R\,\mathrm{Spin}(k)$ is as before, while $R\,\mathrm{Spin}(l)$ now has $q_1, \dots, q_{t-1}, \Delta_t$. With $H_{s,t}$, f_s, g_s as before, let $\Delta_{s,t}^+ = \Delta_s^+\Delta_t$, $\Delta_{s,t}^- = \Delta_s^-\Delta_t$, and set $h_t = \sum_{i=0}^{t} 4^i q_{t-i}$.

Theorem 4.3. *$K(\tilde{G}_{2s+2t+1,2s}) \approx H_{s,t}\big[(\Delta_s^+)^2, \Delta_{s,t}^+\big]/\sim$ with relations $(\Delta_s^+)^4 = f_s(\Delta_s^+)^2 - g_s^2$, $2^{s+t}\Delta_{s,t}^+ = h_t\big((\Delta_s^+)^2 + g_s\big)$ and torsion freeness.*

Proof. Proceeding as in the previous case, one concludes that

$$\begin{aligned} RH &= \mathbb{Z}[p_1, \dots, p_s, q_1, \dots, q_t, (\Delta_s^+)^2, \Delta_{s,t}^+, \Delta_{s,t}^-] \\ &\subset \mathbb{Z}[p_1, \dots, p_{s-2}, q_1, \dots, q_{t-1}, \Delta_s^+, \Delta_s^-, \Delta_t] = R\big(\mathrm{Spin}(k) \times \mathrm{Spin}(l)\big). \end{aligned}$$

Again, the subalgebra $\mathcal{P}$ of RH generated by the p_i and q_j is a polynomial algebra. One also has the following formulae in RH:

$$\begin{aligned} (\Delta_s^-)^2 &= f_s - (\Delta_s^+)^2, \\ \Delta_s^+\Delta_s^- &= g_s, \\ \Delta_t^2 &= h_t. \end{aligned} \tag{7}$$

Then $RH \approx P\big[(\Delta_s^+)^2, \Delta_{s,t}^+, \Delta_{s,t}^-\big]/\sim$, with relations

$$\begin{aligned} &\text{(i)} \ (\Delta_s^+)^4 = f_s(\Delta_s^+)^2 - g_s^2, \\ &\text{(ii)} \ (\Delta_{s,t}^+)^2 = h_t(\Delta_s^+)^2, \\ &\text{(iii)} \ \Delta_{s,t}^+\Delta_{s,t}^- = g_s h_t, \\ &\text{(iv)} \ (\Delta_{s,t}^-)^2 = \big(f_s - (\Delta_s^+)^2\big)h_t, \\ &\text{(v)} \ (\Delta_s^+)^2\Delta_{s,t}^- = g_s\Delta_{s,t}^+, \\ &\text{(vi)} \ (\Delta_s^+)^2\Delta_{s,t}^+ = f_s\Delta_{s,t}^+ - g_s\Delta_{s,t}^-. \end{aligned} \tag{8}$$

The proofs of these relations are again quite easy, we remark only that (i) is the same as in the previous case, and for (vi) one first uses $(\Delta_s^+)^2 = f_s - (\Delta_s^-)^2$.

The restriction map satisfies

$$\begin{aligned} j^{\#}(\mathcal{P}_r) &= \sum_{i+j=r} p_i q_j, \\ j^{\#}(\Delta_{s+t}) &= \Delta_{s,t}^+ + \Delta_{s,t}^-. \end{aligned}$$

So clearly, $K(\tilde{G}_{2s+2t+1,2s}) \approx H_{s,t}\big[(\Delta_s^+)^2, \Delta_{s,t}^+\big]/\sim$, where the relations consist of those in (8) together with $\Delta_{s,t}^+ + \Delta_{s,t}^- = 2^v$, $v = s+t$.

This is essentially Theorem 4.3, apart from an algebraic exercise (much easier than in the previous case) showing that the 7 relations reduce to the two stated in the theorem. Again, we spare the reader the details and merely note that the relation $(f_s + 2g_s)h_t = 4^v$ follows from the defining relation in $H_{s,t}$ and is useful in the reduction. □

COROLLARY 4.4. $Q \otimes K(\tilde{G}_{2s+2t+1,2s}) \approx (Q \otimes H_{s,t})[\alpha]/\big(\alpha^2 = p_s(\alpha + g_s)\big)$.

In particular, $Q \otimes H_{s,t}$ is a subalgebra, so $H_{s,t}$ is a subalgebra of $K(\tilde{G}_{2s+2t+1,2s})$.

For the applications, the fact that $H_{s,t}$ is in all cases a subalgebra of $K(\tilde{G}_{n,k})$ will be important. It will also be useful to have a complete set of relations for $K(\tilde{G}_{2s+2t,2s})$ (without the extra assumption of torsion freeness). This is easily done by removing Δ^{-+}, Δ^{--} from all twenty relations in (5), using $\Delta^{-+} = 2^u - \Delta^{+-}$, $\Delta^{--} = 2^u - \Delta^{++}$, then discarding any redundant relations. We simply state one such set of relations without proof in the next result; again it is tedious but routine to verify that these relations are equivalent to those in (5).

PROPOSITION 4.5. *$K(\tilde{G}_{2s+2t,2s}) \approx H_{s,t}\big[(\Delta_s^+)^2, (\Delta_t^+)^2, \Delta^{++}, \Delta^{+-}\big]/\sim$, where the relations are* (a) – (f) *in Theorem* 4.1, *and:*

$$f_s\Delta^{++} + 2g_s\Delta^{+-} = 2^u\big((\Delta_s^+)^2 + g_s\big) = f_s\Delta^{+-} + 2g_s\Delta^{++},$$

$$f_t\Delta^{++} - 2g_t\Delta^{+-} = 2^u\big((\Delta_t^+)^2 - g_t\big),$$
$$f_t\Delta^{+-} - 2g_t\Delta^{++} = 2^u\big(-(\Delta_t^+)^2 + f_t - g_t\big),$$

$$(\Delta_s^+)^2\Delta^{++} = 2^u(\Delta_s^+)^2 - g_s\Delta^{+-},$$
$$(\Delta_s^+)^2\Delta^{+-} = 2^u(\Delta_s^+)^2 - g_s\Delta^{++},$$

$$(\Delta_t^+)^2\Delta^{++} = 2^u(\Delta_t^+)^2 + g_t\Delta^{+-} - 2^u g_t,$$
$$(\Delta_t^+)^2\Delta^{+-} = g_t\Delta^{++},$$

$$\Delta^{++}\Delta^{+-} = g_t\Delta^{++},$$
$$(\Delta^{++})^2 = g_t(\Delta_s^+)^2 + g_s(\Delta_t^+)^2 - g_s g_t,$$

$$(\Delta^{+-})^2 = (f_t - g_t)(\Delta_s^+)^2 - g_s(\Delta_t^+)^2 + g_s g_t.$$

5. The tangent bundle of $\tilde{G}_{n,k}$

Let γ be the universal oriented k-plane bundle over $\tilde{G}_{n,k}$, $\gamma^\perp$ its orthogonal complement (an oriented l-plane bundle), and τ the tangent bundle. It is well known (cf. [13]) that $\gamma \oplus \gamma^\perp \approx n\varepsilon$ and $\gamma \otimes \gamma^\perp \approx \tau$, where $n\varepsilon$ is the rank n trivial bundle.

Let λ_k^1 be the first exterior power representation on $\mathrm{Spin}(k)$, which can be thought of as id: $SO(k) \to SO(k)$ induced up to $\mathrm{Spin}(k)$, so $\lambda_k^1 = \phi$: $\mathrm{Spin}(k) \twoheadrightarrow SO(k)$. It is clear that the α-construction on λ_k^1 is just $[\gamma] \in KO(\tilde{G}_{n,k})$, and similarly, $\alpha(\lambda_l^1) = [\gamma^\perp]$. Thus

$$\alpha(\lambda_k^1 \otimes \lambda_l^1) = [\gamma \otimes \gamma^\perp] = [\tau] \in KO(\tilde{G}_{n,k}).$$

As usual, c: $KO \to K$ denotes the complexification homomorphism.

LEMMA 5.1. $[c\tau] = p_1q_1 + kq_1 + lp_1 + kl \in H_{s,t} \subset K(\tilde{G}_{n,k})$.

Proof. We simply use the above formula, giving $[c\tau] = \alpha(c\lambda_k^1 \otimes c\lambda_l^1)$, the relations $c\lambda_k^1 = p_1 + k$, $c\lambda_l^1 = q_1 + l$, and the fact that α is a homomorphism of rings. □

PROPOSITION 5.2. *For k, $n - k \geq 2$, $\tilde{G}_{n,k}$ stably parallelizable implies $(n,k) = (6,3)$ or $(4,2)$.*

Proof. Taking advantage of the w grading of $H_{s,t}$, as remarked after 3.3, we see that $c\tau$ is trivial if and only if $p_1q_1 = 0$ and $kq_1 + lp_1 = (l-k)p_1 = 0$. The relations in $H_{s,t}$ show $p_1q_1 = 0$ only when $s = t = 1$, while $(l-k)p_1 = 0$ implies $k = l$. This leaves $k = l = 2$ or $k = l = 3$ as the only possibilities. □

THEOREM 1.1. *Apart from the spheres $\tilde{G}_{n,n-1} \cong \tilde{G}_{n,1} = S^{n-1}$, the only stably parallelizable $\tilde{G}_{n,k}$ are $\tilde{G}_{4,2}$ and $\tilde{G}_{6,3}$, while the only parallelizable $\tilde{G}_{n,k}$ are $\tilde{G}_{2,1} = S^1$, $\tilde{G}_{4,3} \cong \tilde{G}_{4,1} = S^3$, $\tilde{G}_{8,7} \cong \tilde{G}_{8,1} = S^7$, and $\tilde{G}_{6,3}$.*

Proof. Excluding the well-known results for spheres, Proposition 5.2 shows only $\tilde{G}_{6,3}$ and $\tilde{G}_{4,2}$ can be stably parallelizable. Now $\tilde{G}_{4,2} = S^2 \times S^2$ is obviously stably parallelizable but not parallelizable (indeed, has non-zero euler characteristic, hence span zero). For a proof that $\tilde{G}_{6,3}$ is parallelizable, cf. [15] or [19]. □

We now turn to the non-existence proofs for weak almost complex structures on $\tilde{G}_{n,k}$. The knowledge of complex conjugation within $K(\tilde{G}_{n,k})$ will be essential, and this is accomplished by the next two lemmas.

LEMMA 5.3. *Let $\Gamma\colon R\,\mathrm{Spin}(2m) \to R\,\mathrm{Spin}(2m)$ denote the ring automorphism induced by complex conjugation. Then*

$$
\begin{aligned}
\Gamma(p_i) &= p_i\,, \qquad 0 \le i \le m\,,\\
\Gamma(\Delta_m^+) &= \begin{cases} \Delta_m^+\,, & m \text{ even},\\ \Delta_m^-\,, & m \text{ odd},\end{cases}\\
\Gamma(\Delta_m^-) &= \begin{cases} \Delta_m^-\,, & m \text{ even},\\ \Delta_m^+\,, & m \text{ odd}.\end{cases}
\end{aligned}
\tag{9}
$$

Proof. In the maximal torus, $\Gamma(u_i) = \bar{u}_i = u_i^{-1}$, and the formulae in the lemma follow immediately from the formulae for p_i, Δ_m^+, Δ_m^- in term of the u_i (cf. proof of Proposition 2.3). □

The next result is an easy consequence of Lemma 5.3 and the definitions of the various generating classes, and the proof is omitted. We also use an obvious short-hand notation, e.g., (9) will be written $\Gamma(\Delta_m^{\varepsilon}) = \Delta_m^{\varepsilon'}$, where $\varepsilon, \varepsilon' \in \{+,-\}$, and $\varepsilon' = \varepsilon$ for m even, $\varepsilon' \ne \varepsilon$ for m odd.

LEMMA 5.4. *In $K^*(\tilde{G}_{n,k})$ one has the following formulae (when the class in question is defined, and using the above notations).*

(i) $\Gamma(p_i) = p_i$, $\Gamma(q_i) = q_i$,
(ii) $\Gamma(\Delta_s^{\varepsilon})^2 = (\Delta_s^{\varepsilon'})^2$, *where* $\varepsilon' = \varepsilon$, s *even, and* $\varepsilon' \ne \varepsilon$, s *odd,*
(iii) $\Gamma(\Delta^{\varepsilon\eta}) = \Delta^{\varepsilon'\eta'}$, *where* ε' *is as in* (ii), *and* $\eta' = \eta$ *for* t *even,* $\eta' \ne \eta$ *for* t *odd,*
(iv) $\Gamma(\Delta_{s,t}^{\varepsilon}) = \Delta_{s,t}^{\varepsilon'}$, ε' *again as in* (ii).

Before proving the main theorem on weak almost complex structures, let us recall that for any complex vector bundle ξ one has $c\rho(\xi) \approx \xi + \Gamma\xi$, where ρ is realification. Also for any real vector bundle α one has $\rho c(\alpha) \approx 2\alpha$, in particular, twice any real vector bundle admits a complex structure. Finally, for the case $\tilde{G}_{2s+2t,2s}$ (that appears most delicate), we will use $K(\tilde{G}_{2s+2t,2s}) \otimes (\mathbb{Z}/2)$ and the following lemma.

LEMMA 5.5. *Let $\overline{H}_{s,t} = (\mathbb{Z}/2)[p_1, \dots, p_s, q_1, \dots, q_t]/\!\sim$, the relations being*

$$(1 + p_1 + \dots + p_s)(1 + q_1 + \dots + q_t) = 1\,, \qquad p_s q_{t-1} = p_{s-1} q_t = 0\,.$$

Then $K(\tilde{G}_{2s+2t,2s}) \otimes (\mathbb{Z}/2) \approx \overline{H}_{s,t}\big[(\Delta_s^+)^2, (\Delta_t^+)^2, \Delta^{++}, \Delta^{+-}\big]/\!\sim$ *with relations:*

$$\begin{aligned}
0 \quad &= p_{s-1}(\Delta_t^+)^2 + q_{t-1}(\Delta_s^+)^2 = p_s(\Delta_t^+)^2 = q_t(\Delta_s^+)^2 = p_s\Delta^{++} \\
&= p_s\Delta^{+-} = q_t\Delta^{++} = q_t\Delta^{+-}\,, \\
(\Delta_s^+)^4 \quad &= p_s(\Delta_s^+)^2 + p_{s-1}^2\,, \\
(\Delta_t^+)^4 \quad &= q_t(\Delta_t^+)^2 + q_{t-1}^2\,, \\
(\Delta^{++})^2 \quad &= (\Delta_s^+)^2(\Delta_t^+)^2 = (\Delta^{+-})^2 = p_{s-1}q_{t-1}\,, \\
\Delta^{++}\Delta^{+-} &= q_{t-1}(\Delta_s^+)^2\,, \\
(\Delta_s^+)^2\Delta^{+-} &= p_{s-1}\Delta^{++}\,, \\
(\Delta_s^+)^2\Delta^{++} &= p_{s-1}\Delta^{+-}\,, \\
(\Delta_t^+)^2\Delta^{+-} &= q_{t-1}\Delta^{++}\,, \\
(\Delta_t^+)^2\Delta^{++} &= q_{t-1}\Delta^{+-}\,.
\end{aligned}$$

P r o o f . In Proposition 4.5, we have a complete set of relations for $K(\tilde{G}_{2s+2t,2s})$ as the quotient P/J of the polynomial algebra

$$P = \mathbb{Z}\big[p_1, \dots, p_s, q_1, \dots, q_t, (\Delta_s^+)^2, (\Delta_t^+)^2, \Delta^{++}, \Delta^{+-}\big]$$

by the ideal $J = \langle F_1, \dots, F_{19}\rangle$, where the F_i are the relations in 4.5 together with the (graded) relation in $H_{s,t}$. Let

$$\pi\colon P \to P \otimes (\mathbb{Z}/2) = (\mathbb{Z}/2)\big[p_1, \dots, p_s, q_1, \dots, q_t, (\Delta_s^+)^2, (\Delta_t^+)^2, \Delta^{++}, \Delta^{+-}\big]$$

be the usual epimorphism, and let J_2 be the ideal in $P \otimes (\mathbb{Z}/2)$ generated by $\pi F_1, \dots, \pi F_{19}$. Using the result of a problem in [3; p. 31, #2] (taking $A = P$, $\mathfrak{I} = J$, $M = P \otimes (\mathbb{Z}/2)$ in the notation used there), we see at once that

$$K(\tilde{G}_{2s+2t,2s}) \otimes (\mathbb{Z}/2) \approx P \otimes (\mathbb{Z}/2)/\langle \pi F_1, \dots, \pi F_{19}\rangle\,.$$

Thus we simply take the mod 2 reductions π of all the relations in 4.5 to get the relations giving $K(\tilde{G}_{2s+2t,2s}) \otimes (\mathbb{Z}/2)$. Noting that $\pi f_s = p_s$, $\pi g_s = p_{s-1}$, $\pi f_t = q_t$, $\pi g_t = q_{t-1}$, and also that taking π of relations (a), (b) immediately gives $p_{s-1}q_t = 0$ (so also $p_s q_{t-1} = 0$), while no further relations occur in $H_{s,t}$ itself, the remainder of the proof is just the routine rewriting of the πF_i. $\square$

THEOREM 5.6. *Let $n = k+l \geq 7$, $k, l \geq 3$. Then $\tilde{G}_{n,k}$ is not almost complex.*

P r o o f . Suppose $\tilde{G}_{n,k}$ is weakly almost complex, i.e.,

$$[\tau] - kl = \rho(\xi) \in KO(\tilde{G}_{n,k})$$

for some $\xi \in K(\tilde{G}_{n,k})$. Using Lemma 5.1 for $c[\tau]$, taking c gives

$$p_1 q_1 = (k-l)p_1 + \xi + \Gamma\xi . \tag{10}$$

The remainder of the proof depends on the parities of s, t, where $k = 2s$ or $2s+1$ and $l = 2t$ or $2t+1$. However, the idea in each case is generally to find $\Gamma\xi$, using Lemma 5.4, and show that (10) then leads to a contradiction in the graded subalgebra $H_{s,t}$ (or $\overline{H}_{s,t}$). The fact that $\tilde{G}_{n,k} \cong \tilde{G}_{n,l}$, which renders certain cases redundant, is used without mention below.

Case 1. For k, l even $K(\tilde{G}_{2s+2t,2s}) = H_{s,t}[(\Delta_s^+)^2, (\Delta_t^+)^2, \Delta^{++}, \Delta^{+-}]/\sim$ as given by Theorem 4.1. Then

$$\xi = a_0 + a_1(\Delta_s^+)^2 + a_2(\Delta_t^+)^2 + a_3\Delta^{++} + a_4\Delta^{+-} , \qquad a_i \in H_{s,t} .$$

To compute $\Gamma\xi$, various subcases are necessary.

Case (1a). Let s, t both be even. Note $s,t \geq 2$, and, from Lemma 5.4, $\Gamma\xi = \xi$, so (10) reduces to $p_1 q_1 = 2(s-t) + 2\xi$. In $K(\tilde{G}_{2s+2t,2s}) \otimes (\mathbb{Z}/2)$, this gives $p_1 q_1 = 0$, and, from Lemma 5.5, this is impossible for $s,t \geq 2$.

Case (1b). Let s, t both be odd, $s,t \geq 3$. Here

$$\Gamma\xi = a_0 + a_1(\Delta_s^-)^2 + a_2(\Delta_t^-)^2 + a_3\Delta^{--} + a_4\Delta^{-+} .$$

Using (4), (10), and $\Delta^{++} + \Delta^{--} = 2^u = \Delta^{+-} + \Delta^{-+}$, where $u = s+t-1 \geq 5$, we find

$$p_1 q_1 = 2\left[(s-t)p_1 + a_0 + a_1 g_s + a_2 g_t + 2^{u-1}(a_3 + a_4)\right] + a_1 p_s + a_2 q_t .$$

Now passing to $K(\tilde{G}_{2s+2t,2s}) \otimes (\mathbb{Z}/2)$ and looking at grading 2 in $\bar{H}_{s,t}$ (note $s,t \geq 3$, and the relations in $\bar{H}_{s,t}$ are homogeneous, so do not alter grading), we find $p_1 q_1 = 0$, which is the same contradiction as in (1a).

Case (1c). Let s be even, t odd, so $s \geq 2$, $t \geq 3$. Write $t' = t-1 \geq 2$. The usual inclusion $i = \tilde{G}_{2s+2t',2s} \hookrightarrow \tilde{G}_{2s+2t,2s}$ satisfies

$$i^{\#}(\tau_{2s+2t,2s}) \approx \tau_{2s+2t',2s} \oplus 2\gamma ,$$

where γ is the universal $2s$-bundle over $\tilde{G}_{2s+2t',2s}$. Since 2γ is almost complex (as remarked before the statement of the theorem), we see that $i^{\#}(\tau_{2s+2t,2s})$ is weakly almost complex if and only if $\tau_{2s+2t',2s}$ is weakly almost complex. Using Case (1a) it follows that neither bundle is weakly almost complex, hence also $\tau_{2s+2t,2s}$ is not weakly almost complex.

Case 2. Let $k = 2s$, $l = 2t+1$. Here $K(\tilde{G}_{2s+2t+1,2s}) \approx H_{s,t}[(\Delta_s^+)^2, \Delta_{s,t}^+]/\sim$ as given in Theorem 4.3. Then $\xi = a_0 + a_1(\Delta_s^+)^2 + a_2\Delta_{s,t}^+$, $a_i \in H_{s,t}$, and $s \geq 2$, $t \geq 1$.

Case (2a). For s even $\Gamma\xi = \xi$, and we have

$$p_1 q_1 = (2s - 2t - 1)p_1 + 2a_0 + 2a_1(\Delta_s^+)^2 + 2a_2\Delta_{s,t}^+ .$$

Multiplying through by 2^{v-1}, where $v = s + t$, and using the relation on $2^v\Delta_{s,t}^+$ in Theorem 4.3, we find

$$2^{v-1}p_1q_1 = 2^{v-1}(2s - 2t - 1)p_1 + 2^v a_0 + a_2 h_t g_s + (2^v a_1 + a_2 h_t)(\Delta_s^+)^2 .$$

Now Corollary 4.4 implies that $\{1, \alpha\}$ forms a basis for $Q \otimes K\big(\tilde{G}_{2s+2t+1,2s}\big)$ as a $(Q \otimes H_{s,t})$-module. Since $\alpha = (\Delta_s^+)^2 - g_s$, $\{1, (\Delta_s^+)^2\}$ is also a basis, and hence, the submodule of $K(\tilde{G}_{2s+2t+1,2s})$ generated by $\{1, (\Delta_s^+)^2\}$ is free as an $H_{s,t}$-module. In particular, from the last equation,

$$2^{v-1}p_1q_1 = 2^{v-1}(2s - 2t - 1)p_1 + 2^v a_0 + a_2 h_t g_s .$$

Noting that $v = s + t \geq 3$, we find in gradings 0, 1 that

$$(h_t g_s)^{(0)} = 4^{v-1} = 2^{2v-2}, \qquad (h_t g_s)^{(1)} = 0 .$$

Hence, in grading 1,

$$0 = 2^{v-1}(2s - 2t - 1)p_1 + 2^v a_0^{(1)} + 2^{2v-2}a_2^{(1)} ,$$
$$0 = (2s - 2t - 1)p_1 + 2a_0^{(1)} + 2 \cdot 2^{v-2}a_2^{(1)} , \qquad \text{or} \qquad p_1 = 2b \ \text{in} \ H_{s,t}^{(1)} .$$

This relation is impossible in $H_{s,t}^{(1)}$, which is the free abelian group generated by p_1.

Case (2b). For s odd, $s \geq 3$, let $v = s + t \geq 4$. We find

$$p_1q_1 = (2s - 2t - 1)p_1 + 2a_0 + a_1 p_s + 2(a_1 g_s + 2^{v-1}a_2) ,$$

again impossible in grading 1.

Case 3. Let $k = 2s + 1$, $l = 2t + 1$ both be odd. Without loss of generality, we may suppose $1 \leq s \leq t$ and $t \geq 2$. By Theorem 3.6, we have $\xi = a_0 \in H_{s,t}$, $\Gamma\xi = \xi$, giving $p_1q_1 = 2(s - t)p_1 + 2a_0$, again a contradiction. □

Remark. Case (1c) could also be handled by the K-theory, but with somewhat more work than the "inclusion" method used above.

Combining Theorem 5.6 with the well-known results about almost complex structures on $\tilde{G}_{n,k}$ for $k = 1, 2$ (cf. [5], [21]), Theorem 1.2 is immediate.

THEOREM 5.7. *Let $n_1 \geq \cdots \geq n_s \geq 1$ be integers, $s \geq 3$, and let $n = n_1 + \cdots + n_s$. Let M denote the oriented flag manifold $\widetilde{G}(n_1, \ldots, n_s) = SO(n)/\big(SO(n_1) \times \cdots \times SO(n_s)\big)$. Then M admits a weak almost complex structure if and only if at least one of the following holds:*

(i) n_1 *arbitrary,* $n_2 \leq 2$;
(ii) $n_1 \leq 3$.

Proof. The necessity follows from the observation that M is fibred by $\widetilde{G}_{n_1+n_2,n_1} = \widetilde{G}(n_1, n_2)$, which is not weakly almost complex by Theorem 1.2 if $n_1 \geq 4$ and $n_2 \geq 3$.

The sufficiency can be deduced using the description of the tangent and normal bundles of M as in [13], and the following facts:

(a) $\lambda^2(\xi) \approx \varepsilon_{\mathbb{R}}$, $\lambda^2(\eta) \approx \eta$ if ξ and η are orientable real vector bundles with rank $\xi = 2$, rank $\eta = 3$,

(b) any oriented rank 2 real vector bundle admits a complex structure,

(c) for any two real vector bundles ξ and η, $\xi \otimes_{\mathbb{R}} \eta$ admits a complex structure if ξ admits a complex structure.

We omit the details. □

Remark. Recent work of Tang Zi-Zhou (cf. [22]) shows that the (unoriented) Grassmann manifold $G_{n,k}$, $2 \leq k \leq n/2$, admits a weak almost complex structure if and only if $n = 2k = 4$ or 6. It is known that $G_{4,2}$ does not admit any almost complex structure (cf.[18; Lemma 3.1]).

Acknowledgement

The authors wish to thank J. Korbaš and the referee for valuable comments that clarified and improved this paper.

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Received September 27, 1993
Revised November 17, 1994

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