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DOMATIC NUMBERS OF CUBE GRAPHS

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Let G be an undirected graph without loops and multiple edges, let $V(G)$ be its vertex set. A subset D of $V(G)$ is called a dominating set in G, if to each vertex $x \in V(G) - D$ there exists a vertex $y \in D$ adjacent to x. A domatic partition of the graph G is a partition of the vertex set of G, all of whose classes are dominating sets in G. The maximal number of classes of a domatic partition of G is called the domatic number [1] of G and denoted by $d(G)$.

The domatic number can be defined in an equivalent way by means of the so-called domatic colouring. A domatic colouring of a graph G is a colouring of the vertices of G with the property that if x is an arbitrary vertex of G, then to each colour different from that of x there exists a vertex of this colour which is adjacent to x. (Two vertices of the same colour may be adjacent.) The maximal number of colours of a domatic colouring of G is called the domatic number of G.

The graph Q_n of the n-dimensional cube (for an arbitrary positive integer n) is an undirected graph whose vertex set is the set of all n-dimensional Boolean vectors (i. e. vectors, all of whose coordinates are equal to 0 or 1) and in which two vertices are adjacent if and only if they differ in exactly one coordinate.

We shall prove one theorem on domatic numbers of the graphs of the n-dimensional cubes.

Theorem. *Let k be a positive integer. Then the graph of the cube of the dimension $2^k - 1$ and the graph of the cube of the dimension 2^k have both the domatic number 2^k.*

Proof. In [1] it was proved that $d(G) \leqq \delta(G) + 1$, where $\delta(G)$ is the minimal degree of a vertex of G. A graph G for which $d(G) = \delta(G) + 1$ is called domatically full. In [2] it was proved that a regular graph can be domatically full with the domatic number d only if d divides the number of vertices of this graph. The graph of the n-dimensional cube has 2^n vertices and is regular of the degree n, hence its domatic number is at most $n+1$ and it can be equal to $n+1$ only if $n+1$ divides 2^n. This is possible only if $n = 2^k - 1$ for some non-negative integer k. We shall consider only positive integers k, because for $k = 0$ we have $n = 0$.

Let $k = 1$; then $n = 1$. The graph Q_1 consists of two vertices joined by an edge and its domatic number is evidently 2. Now we shall proceed by induction according to k. Suppose that the assertion holds for $k=m$, where m is a positive integer. Therefore the graph of the cube of the dimension 2^m-1 has the domatic number 2^m. If we have a domatic colouring of the graph Q_n for an arbitrary n, we can construct a domatic colouring of Q_{n+1} so that the vertex $[v_1, \ldots, v_n, v_{n+1}]$ has the same colour as the vertex $[v_1, \ldots, v_n]$ of Q_n. This implies $d(Q_{n+1}) \geqq d(Q_n)$. In particular, the graph of the cube of the dimension 2^m has the domatic number greater than or equal to that of the graph of the cube of the dimension 2^m-1, namely 2^m. As 2^m+1 does not divide 2^{2^m} and this graph is regular, it cannot be domatically full and its domatic number cannot be greater than 2^m. Therefore its domatic number is 2^m. For the sake of simplicity we denote $2^m=p$.

Consider the graphs Q_{p-1} and Q_p and let a domatic colouring with p colours be given in each of them; the colours will be denoted by $0, 1, \ldots, p-1$. The domatic colouring of Q_p is derived from that of Q_{p-1} in the above described way. If $k=m+1$, then $2^k-1 = 2^{m+1}-1 = 2p-1$. By π_i for $i=0, 1, \ldots, p-1$ we denote the cyclic permutation of the number set $\{0, 1, \ldots, p-1\}$ such that $\pi_i(x) \equiv x+i \pmod p$ for each $x \in \{0, 1, \ldots, p-1\}$. Consider the graph Q_{2p-1}. To each vertex $[v_1, \ldots, v_{2p-1}]$ of Q_{2p-1} we assign a colour in the following way. If $\sum_{i=p}^{2p-1} v_i$ is even, then the vertex $[v_1, \ldots, v_{2p-1}]$ has the colour $\pi_s(r)$, where r is the colour of $[v_1, \ldots, v_{p-1}]$ in Q_{p-1} and s is the colour of $[v_p, \ldots, v_{2p-1}]$ in Q_p. If $\sum_{i=p}^{2p-1} v_i$ is odd, then the vertex $[v_1, \ldots, v_{2p-1}]$ has the colour $\pi_s(r)+p$. Thus we obtain a colouring of the vertices of Q_{2p-1} by the colours $0, 1, \ldots, 2p-1$; we shall prove that it is a domatic colouring.

Let $[v_1, \ldots, v_{2p-1}]$ be a vertex of Q_{2p-1} such that $\sum_{i=p}^{2p-1} v_i$ is even. Then its colour is $\pi_s(r)$, where r and s have the meaning described above. The vertex $[v_1, \ldots, v_{p-1}]$ in Q_{p-1} has the colour r and to each colour $c \in \{0, 1, \ldots, p-1\} - \{r\}$ there exists a vertex $[w_1, \ldots, w_{p-1}]$ of Q_{p-1} adjacent to $[v_1, \ldots, v_{p-1}]$ and having the colour c. Then the vertex $[w_1, \ldots, w_{p-1}, v_p, \ldots, v_{2p-1}]$ is adjacent to $[v_1, \ldots, v_{2p-1}]$ in Q_{2p-1} and its colour is $\pi_s(c)$; when c runs through the whole set $\{0, 1, \ldots, p-1\} - \{r\}$, then $\pi_s(c)$ runs through the whole set $\{0, 1, \ldots, p-1\} - \{\pi_s(r)\}$ and hence to each colour from $\{0, 1, \ldots, p-1\} - \{\pi_s(r)\}$ there exists a vertex in Q_{2p-1} adjacent to $[v_1, \ldots, v_{2p-1}]$ and having this colour. Now let $d \in \{p, \ldots, 2p-1\}$. There exists a vertex $[z_p, \ldots, z_{2p-1}]$ of Q_p adjacent to the vertex $[v_p, \ldots, v_{2p-1}]$ and having the colour $d-p$. (Note that from the construction of the domatic colouring of Q_{2p} it follows that each vertex of Q_{2p} is adjacent to vertices of all colours, no exception being made for its own colour.) As $[v_p, \ldots, v_{2p-1}]$, $[z_p, \ldots, z_{2p-1}[$ are adjacent, we have $|z_i - v_i| = 1$ for exactly one i and $z_j = v_j$ for all $j \neq i$ from the numbers $p, \ldots, 2p-1$. As $\sum_{i=p}^{2p-1} v_i$ is even, $\sum_{i=p}^{2p-1} z_i$ is odd. The vertex $[v_1, \ldots, v_{p-1}, z_p, \ldots, z_{2p-1}]$ has the colour d and is adjacent to $[v_1, \ldots, v_{2p-1}]$. If $\sum_{i=p}^{2p-1} v_i$ is odd, the proof is analogous.

We have proved that our colouring or Q_{2p-1} is domatic and therefore the domatic number of Q_{2p-1} is $2p = 2^{2^k}$. From this domatic colouring we can derive the domatic colouring of Q_{2p} as it was shown above.

In the end we express a conjecture.

Conjecture. *Let Q_n be the graph of the n-dimensional cube, where n is a positive integer such that $n+1$ is not a power of 2. Then $d(Q_n) = n$.*

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ДОМАТИЧЕСКИЕ ЧИСЛА ГРАФОВ КУБОВ

Богдан Зелинка

Резюме

Доматическое число графа G есть максимальное число классов разбиения множества вершин графа G, классы которого являются доминирующими множествами в G. В статье найдено доматическое число графа куба размерности n для $n = 2^k - 1$ и $n = 2^k$, где k есть натуральное число.