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ON 4-CONNECTED, PLANAR 4-ALMOST PANCYCLIC GRAPHS

MARIÁN TRENKLER

A graph G with v vertices is *pancyclic* if G has cycles of all lengths m, where $3 \leqslant m \leqslant v$. If G contains cycles of all lengths except for k, then G is called a *k-almost pancyclic graph.*

Conjecture (Bondy [2]). *A planar hamiltonian graph in which every vertex has the valency at least* 4 *is pancyclic.*

Choudum [3] and Malkevitch [4] gave several counter-examples to this conjecture, which are not 4-connected. Since Tutte [6] proved that every 4-connected planar graph is hamiltonian, there arose the problem whether the Conjecture is true if the conditions concerning the valency of the vertices are replaced by a stronger condition so that G be 4-connected.

Malkevitch constructed a 4-connected planar 4-almost pancyclic graph with 30 vertices. This introduces the problem for which number of vertices there exists a 4-connected 4-almost pancyclic graph. At the Czechoslovak Conference on Graph Theory held in May 1985 Jacoš and Ninčák described the construction of such graphs for all $30 \leqslant v \equiv 0 \pmod 3$ except for $v = 33$ and all $v \geqslant 50$.

We shall prove the following.

Theorem. *A* 4*-connected planar* 4*-almost pancyclic graph with v vertices exists if and only if*

a) $30 \leqslant v \equiv 0 \pmod 3$, $v \neq 33$ *or*

b) $46 \leqslant v \equiv 1 \pmod 3$ *or*

c) $44 \leqslant v \equiv 2 \pmod 3$, $v \neq 47$.

In the first part we describe the construction of 4-connected planar 4-almost pancyclic graphs for every required number of vertices. In the second part we prove the non-existence of such graphs for other values of v.

Let G be a planar graph imbedded in a plane with v vertices and h edges and p faces. Denote by v_i the number of its vertices of degree i (i-valent vertices) and by p_i the number of i-gonal faces (i-gons).

Lemma 1. *Every* 3*-connected planar graph G satisfies the conditions*

a) $\sum_{k \geqslant 3} (6 - k) p_k + 2 \sum_{k \geqslant 3} (3 - k) v_k = 12$, *and*

b) $\sum_{k \geqslant 3} (4 - k)(p_k + v_k) = 8.$

These conditions follow immediately from Euler's theorem for planar graphs.

Lemma 2. *A planar 4-edge-connected graph and 3-connected graph H containing exactly p_k k-gons and k-valent vertices for all k exists iff there exists a 4-connected 4-valent planar graph $\mathscr{M}(H)$ containing exactly p_k k-gons for all k.*

Proof. A medial graph $\mathscr{M}(H)$ of a planar graph H is formed in the following way (see [5, p. 47]): To each edge of H there corresponds a vertex of $\mathscr{M}(H)$ and two vertices of $\mathscr{M}(H)$ are joined by an edge if they correspond to two edges incident with the same face of H and have a common vertex. (In Fig. 1 H is depicted by full lines and $\mathscr{M}(H)$ by dashed lines.) A medial graph of any planar graph is 4-valent.

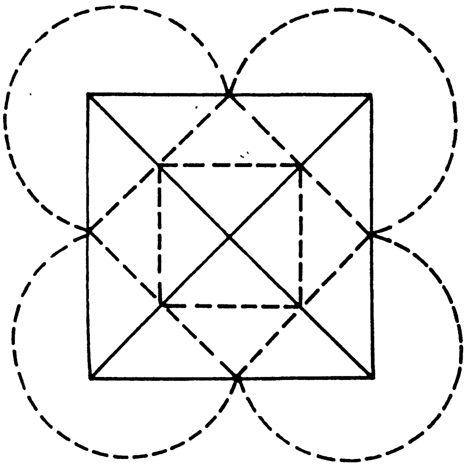

Fig. 1

It is easy to show that every 4-valent planar graph is a medial graph of a planar graph because all faces of the 4-valent planar graph are regularly colourable by two colours. The faces of one colour correspond to the vertices of H and the others to the faces of H.

Note that there exist two mutually dual graphs with the same medial graph $\mathscr{M}(H)$. From Lemma 1 it follows that at least one of them has at least four 3-valent vertices. This graph H will be called the *antemedial* graph to $\mathscr{M}(H)$.

The construction of a 4-connected, planar 4-almost pancyclic graph G with v vertices.

a) $v = 30, 36, 39, 42, 45$

The graph G is a medial graph of the dodecahedron for $v = 30$, or a medial graph of the starting graphs in Figure 2. For $v = 36$ it is depicted by full lines and for $v = 39$ by full and dot-and-dashed lines and for $v = 42$ or 45 by full and dot-and-dashed and one or two dashed lines, respectively. The corresponding medial graph contains cycles of all lengths, except for cycles of length four. If the depicted graph has the cycle C of length k, then its medial graph has cycles of all lengths m for $k \leqslant m \leqslant \min\{2k, v\}$ passing through vertices added to edges of cycle C and to edges having one common vertex with C.

b) $48 \leqslant v \equiv 0 \pmod 3$

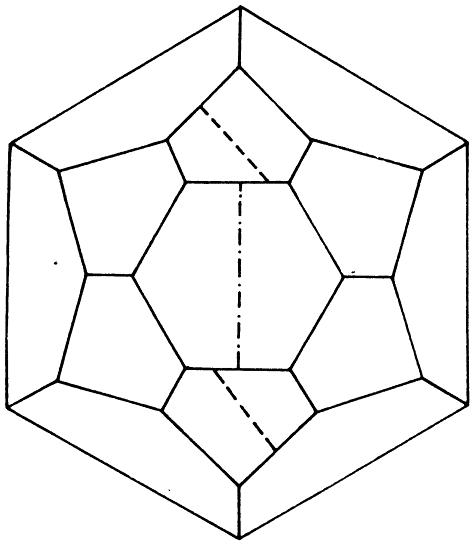

Fig. 2

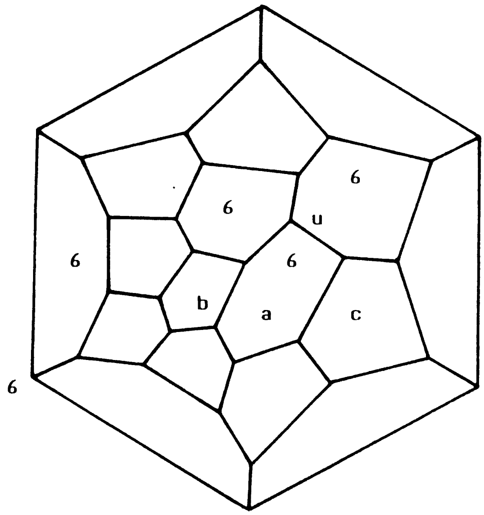

Fig. 3a

For $v = 48$ the graph G is a medial graph of the starting graph in Figure 3a. If $v = 48 + 3s$, for $1 \leqslant s \equiv 0 \pmod 2$, then to the faces indicated by a, b we add alternately s edges. In the case $s = 4$, the order (indicated by numbers) of adding edges is showed by dashed lines in Figure 3b. If $s \equiv 1 \pmod 2$, we add analogously $(s - 1)$ edges to the faces a, b and the last one is added to the face c such that the faces incident with the vertex u (indicated in Fig. 3a) will be 6-gonal.

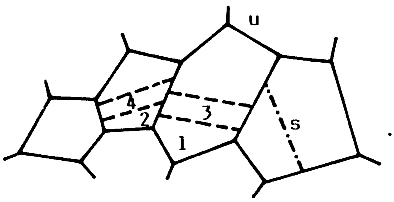

Fig. 3b

c) $46 \leqslant v \equiv 1 \pmod 3$

Using the construction described in b) we construct a 4-almost pancyclic graph G^*, with $v + 2$ vertices. In G^* there exist two triangles having common edges only with k-gons, $k \geqslant 6$. By identification the vertices of one of these triangles (see Fig. 6, where the inverse transformation is depicted) we obtain the graph G with exactly one 6-valent vertex and $(v - 1)$ 4-valent vertices without cycles of length four.

d) $44 \leqslant v \equiv 2 \pmod 3$, $v \neq 47$

Using the construction described in b) we construct a 4-almost pancyclic graph with $v + 4$ vertices. As in c) we use two triangles non-incident with 5-gons to form two 6-valent vertices.

In the next part we prove the non-existence of a 4-almost pancyclic graph G with exactly p_i i-gons and v_i i-valent vertices, whose every vertex is of degree $\geqslant 4$ (not necessarily 4-connected) with v vertices for

$$1 \leqslant v \leqslant 29, \quad 31 \leqslant v \leqslant 35, \quad v = 37, 38, 40, 41, 43, 47.$$

The necessary condition for G to be a planar 4-almost pancyclic graph is that G must contain neither two triangles with a common edge nor a quadrangle, i.e.

$$3p_3 \leqslant \sum_{5 \leqslant k} kp_k .$$

We designate the edges of the triangles of G as t-edges and the edges non-incident with triangles d-edges. By d we denote the number of d-edges. Evidently every vertex of odd degree is incident with at least one d-edge. From this follows.

Lemma 3. *The subgraph of a planar 4-almost pancyclic graph with $v_3 = 0$ formed from d-edges consists of paths joining couples of vertices of odd degree and possibly edge-disjoint cycles.*

A consequence of Lemma 3 is that in a planar 4-almost pancyclic graph G with all vertices of even degree the d-edges must from cycles of length at least five.

Lemma 4. *The necessary condition for G with $v_3 = 0$ to be a planar 4-almost pancyclic is*

$$\sum_{5 \leqslant k} (k-3) p_k = v - 6 + \tfrac{1}{2} \sum_{5 \leqslant k} (4-k) v_k .$$

Proof. In the identity

$$\sum_{3 \leqslant k} k p_k = \sum_{4 \leqslant k} k v_k$$

we substitute from relation b) of Lemma 1, obtaining

$$3\left[8 + \sum_{4 \leqslant k} (k-4)(p_k + v_k)\right] + \sum_{5 \leqslant k} k p_k = \sum_{4 \leqslant k} k v_k$$

$$24 + 4 \sum_{5 \leqslant k} (k-3) p_k = -2 \sum_{4 \leqslant k} k v_k + 12 \sum_{4 \leqslant k} v_k$$

$$\sum_{5 \leqslant k} (k-3) p_k = v - 6 + \tfrac{1}{2} \sum_{4 \leqslant k} (4-k) v_k .$$

The necessary conditions from Lemmas 3 and 4 are satisfied by the following 16 cases in Table 1.

Table 1

Case	v	v_5	v_6	p_3	p_5	p_6	p_7	d	Remark
1	33	0	0	22	12	1	0	0	
2	40	0	0	25	17	0	0	5	$\mathbf{M}_1$
3	43	0	0	27	17	1	0	5	$\mathbf{M}_1$
4	47	0	0	29	19	1	0	7	
5	39	2	0	26	16	0	0	1	$\mathbf{M}_2$
6	41	2	0	27	17	0	0	2	$\mathbf{M}_2$
7	43	2	0	28	18	0	0	3	$\mathbf{M}_2$
8	47	2	0	30	20	0	0	5	$\mathbf{M}_2$
9	47	2	0	31	17	2	0	2	$\mathbf{M}_2$
10	47	2	0	31	18	0	1	2	$\mathbf{M}_2$
11	37	0	1	25	15	0	0	0	$\mathbf{M}_3$
12	40	0	1	27	15	1	0	0	$\mathbf{M}_3$
13	43	0	1	29	15	2	0	0	$\mathbf{M}_3$
14	43	0	1	29	16	0	1	0	$\mathbf{M}_3$
15	47	0	1	30	20	0	0	5	$\mathbf{M}_1$, $\mathbf{M}_3$
16	47	0	2	32	18	1	0	0	$\mathbf{M}_3$, $\mathbf{M}_3$

In case 4 the corresponding graph G does not exist because otherwise it would have at least one 7-gon.

We suppose that in all the other cases the corresponding graph G exists. Using three modifications $\mathbf{M}_i$, i = 1, 2, 3, in the order described in the remark in Table 1 we transform these graphs into 4-valent planar 4-almost pancyclic graphs having no d-edges. The antemedial graph of such a graph is 3-valent. In [1] there are described all 3-valent planar graphs having no triangles and quadrangles with less than 20 faces and none of them has the configurations of faces arising by such modifications. For example, in case 2 the antemedial graph to the modified graph after $\mathbf{M}_1$ must contain one 5-gon having the common edges only with five 6-gons and all the other faces must be 5-gons. From this there follows the non-existence of planar 4-almost pancyclic graphs which are formed by these modifications and have less than 20 faces of degree $\geqslant 5$, except for the cases 4, 8, 15, 16.

Modification $\mathbf{M}_1$: In every edge of a cycle C of length s formed only from d-edges we choose one new vertex. Adding s new edges (dashed lines in Fig. 4)

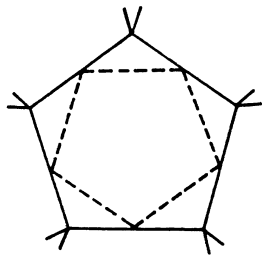

Fig. 4

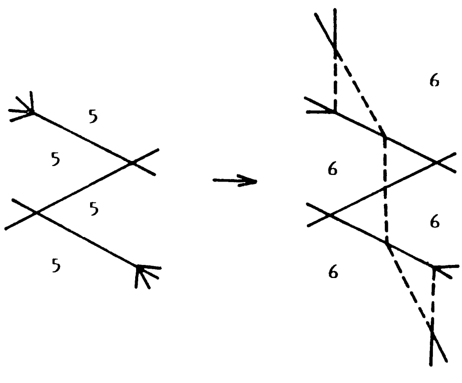

Fig. 5

joining the chosen vertices to the incident d-edges we obtain a graph with at least s faces of degree $\geqslant 6$. The corresponding medial graph has at least one k-gon, $k \geqslant 5$, incident only with h-gons, $h \geqslant 6$.

Modification $\mathbf{M}_2$: We divide every 5-valent vertex into two vertices joined by a new edge. We add to every d-edge one new vertex and connect these vertices by new edges as shown for the case 8 in Figure 5 (all new edges are indicated by dashed lines).

Modification $\mathbf{M}_3$: The 6-valent vertex (Fig. 6) is replaced by a triangle. (This is an opposite operation to the construction step used in c).)

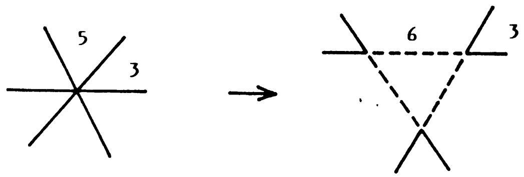

Fig. 6

Remark. Modification $\mathbf{M}_3$ may be used because no edge incident with 6-valent vertices is a d-edge.

The non-existence of the corresponding graph in case 8 follows from the fact that the antemedial graph to the modified graph after $\mathbf{M}_2$ must contain seven 6-gons in configuration showed in Figure 7 and all the other faces are 5-gons. This is impossible from the unambiguity of the construction.

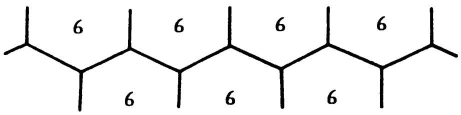

Fig. 7

Analogously we can show the non-existence of the corresponding graph in cases 15 and 16.

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ОБ 4-СВЯЗНЫХ ПЛАНАРНЫХ 4-ПОЧТИ ПАНЦИКЛИЧЕСКИХ ГРАФАХ

Marián Trenkler

Резюме

Планарный 4-связный 4-почти панциклический граф G (граф G, содержающий циклы всех длин кроме 4) с v вершинами существует тогда и только тогда, когда

$$\begin{aligned} &30 \leqslant v \equiv 0 \pmod 3,\ v \neq 33 \quad \text{или} \\ &46 \leqslant v \equiv 1 \pmod 3, \qquad\quad \text{или} \\ &44 \leqslant v \equiv 2 \pmod 3,\ v \neq 47. \end{aligned}$$