Mathematica Slovaca

Igor Kossaczký

Green's relations and regular elements of transformation semigroups

Mathematica Slovaca, Vol. 37 (1987), No. 2, 159--168

Persistent URL: http://dml.cz/dmlcz/129599

Math. Slovaca 37, 1987, No. 2, 159—168

GREEN'S RELATIONS AND REGULAR ELEMENTS OF TRANSFORMATION SEMIGROUPS

IGOR KOSSACZKÝ

The characterization of Green's relations of the semigroup $\mathscr{T}(X)$ of all selfmaps of an arbitrary set X is wel known (see [2]). It is also known for the semigroup of all linear mappings of a linear space. In paper [8] K. A. Zareckij gave a characterization of Green's relations and regular elements of the semigroup $\mathscr{B}(\Omega)$ of all binary relations on a set Ω. The purpose of this paper is to give necessary and sufficient conditions of $\mathscr{L}$, $\mathscr{J}$, $\mathscr{D}$-equivalence and regularity of arbitrary elements of semigroups which belong to a certain class of subsemigroups of $\mathscr{T}(X)$. This class contains all regular subsemigroups of $\mathscr{T}(X)$, but not only those (see Example 2). If $X = 2^{\Omega}$, then this class contains a semigroup isomorphic to the semigroup $\mathscr{B}(\Omega)$. In paper [6] K. D. Magill gave necessary and sufficient conditions of $\mathscr{L}$, $\mathscr{R}$, $\mathscr{J}$, $\mathscr{D}$-equivalence of two regular elements of an arbitrary subsemigroup of $\mathscr{T}(X)$. It is stated there [6, p. 1487] that two regular elements f, g of an arbitrary subsemigroup S of $\mathscr{T}(X)$ are $\mathscr{D}$-equivalent if and only if there exists a one-to-one map φ from the range of f onto the range of g such that both φ^{-1} and φ are restrictions of certain elements of S. We shall prove (Theorem 1) that if S belong to the class mentioned above, then this equivalence holds for arbitrary elements of S, not necessarily regular.

We shall use the following notation. The element into which the element $a \in X$ is mapped by the mapping $f \in \mathscr{T}(X)$ will be written in the form of a product af. The product of mappings $f, g \in \mathscr{T}(X)$ will be denoted by fg. Thus for any $f, g \in \mathscr{T}(X)$ and $a \in X$ we have $a(fg) = (af)g$.

Let A be a subset of X and $f \in \mathscr{T}(X)$, then $Af = \{af;\ a \in A\}$, $f|A$ denotes the restriction of the mapping f on the set A. If f is an idempotent and $Xf = A$, then f is said to be a projection on the set A.

Let S be a semigroup and $a \in S$, then $L(a)$, $R(a)$, $J(a)$ are the left, right, two-sided ideal envelopes of a, respectively. Green's relations will be denoted by $\mathscr{L}$, $\mathscr{R}$, $\mathscr{J}$, $\mathscr{D}$. S^1 will denote a semigroup equal to S if S has an identity and it is equal to S with an externally added identity (the identity mapping if $S \subset \mathscr{T}(X)$) otherwiese. S^* is the semigroup of all right transformations of S^1 corresponding to the elements of S. Note that S^* is isomorphic to S.

Lemma 1. *Let X be a set, S be a subsemigroup of $\mathcal{T}(X)$ and $f, g \in S$. Then the following hold:*

(i) *If $f \in L(g)$, then $Xf \subset Xg$.*

(ii) *If $f \mathcal{L} g$, then $Xf = Xg$.*

(iii) *If $f \mathcal{J} g$, then there are $a, b \in S^1$ such that $Xf \subset Xgb$ and $Xg \subset Xfa$.*

(iv) *If $f \mathcal{D} g$, then there are $a, b \in S^1$ such that $Xf = Xgb$, $Xg = Xfa$ and both $a|(Xf)$, $b|(Xg)$ are bijections and $(a|(Xf))^{-1} = b|(Xg)$.*

(v) *If f is regular, then there exists a projection on the set Xf in S.*

Proof. (i)—(iii) follow immediately from definitions.

(iv) If $f \mathcal{D} g$, then there exists $h \in S$ such that $f \mathcal{L} h$ and $h \mathcal{R} g$. Thus by (ii) it follows that $Xf = Xh$. Since $g \mathcal{R} h$ it follows that there are $a, b \in S$ such that $gb = h$ and $ha = g$, thus $gba = g$ and $hab = h$. Hence for every $\alpha \in Xg$ we have $\alpha ba = \alpha$, similarly for every $\alpha \in Xh = Xf$ we have $\alpha ab = \alpha$. It is also clear that $Xf = Xh = Xgb$ and $Xg = Xha = Xfa$.

(v) If f is regular, then there is an idempotent $i \in S$ such that $i \mathcal{L} f$ (see [2, Lemma 1.13]). Thus by (ii) we have $Xf = Xi$. It follows that i is a projection on the set Xf.

We are going to find conditions for the validity of the converses to (i) through (v). It is possible to prove that the validity of the converse to (i) implies the validity of the converses to the other ones.

Definition 1. *Let X be a set and S be a subsemigroup of $\mathcal{T}(X)$. S is said to be an $\mathcal{L}$-subsemigroup of $T(X)$ if for every $f, g \in S$, $X_f \subset Xg$ implies $f \in L(g)$.*

Lemma 2. *Let S be a semigroup. Then S^* is an $\mathcal{L}$-subsemigroup of $T(S^1)$.*

Proof. Suppose that $f^*, g^* \in S^*$ are right translations corresponding to elements $f, g \in S$ and $S^1 f^* \subset S^1 g^*$. Since $S^1 f^* = L(f)$ and $S^1 g^* = L(g)$ it follows that $f \in L(g)$, thus $f^* \in L(g^*)$.

Lemma 3. *Let S be a regular subsemigroup of $T(X)$. Then S is an $\mathcal{L}$-subsemigroup of $T(X)$.*

Proof. Let $f, g \in S$ and $Xf \subset Xg$. According to the Axiom of Choice there exists $t \in \mathcal{T}(X)$ such that $f = tg$. Since S is a regular semigroup there is $\bar{g} \in S$ such that $g = g\bar{g}g$. Thus we have $f = tg = tg\bar{g}g = f\bar{g}g \in L(g)$.

I. I. Valuce proved that the converse to (i) is true for the semigroup of all endomorphisms of any free universal algebra over an equational class (see [7]). Essential here is the following property of free generators.

Deffinition 2. *Let S be a subsemigroup of $\mathcal{T}(X)$. S is said to be a V-subsemigroup of $\mathcal{T}(X)$ if there exists a subset $A \subset X$ with the following property: For each mapping $j: A \to X$ there exists exactly one element $s \in S$ such that $s|A = j$.*

We shall say that such a set A is a set of V-generators of the semigroup S.

Remark. It is easy to see that the semigroup of all endomorphism of an arbitrary universal algebra X, which is free over some class, is a V-subsemigroup of $\mathscr{T}(X)$. Any set of free generators of the algebra is clearly a set of V-generators of its endomorphism semigroup.

Lemma 4. *Let S be a V-subsemigroup of $\mathscr{T}(X)$*, then it is *an $\mathscr{L}$-subsemigroup of $\mathscr{T}(X)$.*

Proof. Suppose that $A \subset X$ is a set of V-generators of S. Let $f, g \in S$ and $Xf \subset Xg$. According to the Axiom of Choice there exists $t \in \mathscr{T}(X)$ such that $f = tg$. Thus there is $s \in S$ such that

$$f|A = (tg)|A = (t|A)g = (s|A)g = (sg)|A.$$

Since $sg \in S$ and there is exactly one element of S coinciding with $f \in S$ on the set A it follows that $f = sg \in L(g)$.

Theorem 1. *Let S be an $\mathscr{L}$-subsemigroup of $\mathscr{T}(X)$ and $f, g \in S$. Then the following statements hold*:

(i) *$f \in L(g)$ if and only if $Xf \subset Xg$.*

(ii) *$f\mathscr{L}g$ if and only if $Xf = Xg$.*

(iii) *$f\mathscr{J}g$ if and only if there are $a, b \in S^1$ such that $Xf \subset Xgb$ and $Xg \subset Xfa$.*

(iv) *$f\mathscr{D}g$ if and only if there are $a, b \in S^1$ such that $Xf = Xgb$, $Xg = Xfa$, both $a|(Xf)$ and $b|(Xg)$ are bijections and $(a|(Xf))^{-1} = b|(Xg)$.*

(v) *f is regular if and only if there exists a projection on the set Xf in S.*

Proof. The "only if" parts. follow from Lemma 1. The statements (i)—(iii) follow from the definition of an $\mathscr{L}$-subsemigroup.

(iv) Let us denote $h = gb$. For every $a \in X$ we have

$$aha = agba = ag(b|(Xg))a = ag(b|(Xg))(a|(Xf)) = ag,$$

thus $ha = g$. Hence $h\mathscr{R}g$. Since $Xf = Xgb = Xh$ it follows by (ii) that $h\mathscr{L}f$, thus $f\mathscr{D}g$.

(v) Let $i \in S$ be a projection on the set Xf. According to (ii), $f\mathscr{L}i$. Thus f is a regular element.

Corollary 1. *Let S be an $\mathscr{L}$-subsemigroup of $\mathscr{T}(X)$, $f, g \in S$ and Xf be a finite set. Then $f\mathscr{D}g$ if and only if $f\mathscr{J}g$.*

Proof. If $f\mathscr{J}g$, then there are $a, b \in S^1$ such that $Xf \subset Xgb$ and $Xg \subset Xfa$. Thus Xf and Xg have the same cardinal number and $Xf = Xgb$, $Xg = Xfa$. It is clear that $a|(Xf)$ is a bijection on Xg and $b|(Xg)$ is a bijection on Xf. Let us denote $h = ab$. Thus $h|(Xf) = a|(Xf)b|(Xg)$ is a bijection from Xf onto Xf. Since Xf is a finite set it follows that there is an integer n such that $[h|(Xf)]^n$ is the identity mapping on the set Xf. Denote $a' = h^{n-1}a$ and $b' = b$. Hence we have $Xfa' = Xg$, $Xgb' = Xf$, $(a'|(Xf))^{-1} = b'|(Xg)$. According to (iv) of Theorem 1, $f\mathscr{D}g$.

Corollary 2. *Let S be a semigroup and f, $g \in S$, then the following holds. $f\mathcal{D}g$ if and only if there are a, $b \in S^1$ such that:*

$$\begin{aligned} a^*\colon\ L(f) \to L(g);\ \ xa^* = xa \\ b^*\colon\ L(g) \to L(f);\ \ xb^* = xb \end{aligned}$$

are bijections such that $(a^*)^{-1} = b^*$.

Proof. It follows immediately from (iv) of Theorem 1 and Lemma 2.

Corollary 3. *Let S be a semigroup, f, $g \in S$ and $L(f)$ be a finite set. Then $f\mathcal{D}g$ if and only if $f\mathcal{J}g$.*

Proof. It follows immediately from Corollary 1 and Lemma 2.

Corollary 4. *Let X be a universal algebra such that the semigroup $End\,(X)$ of all endomorphisms of X is regular. Then for every f, $g \in End\,(X)$ the subalgebra Xf is isomorphic to the subalgebra Xg if and only if $f\mathcal{D}g$.*

Proof. The "only if" part follows from (iv) of lemma 1. Let h: $Xf \to Xg$ be an isomorphism of algebras. Since f, g are regular it follows by (v) of Lemma 1 that there are $i, j \in End\,(X)$ such that i is projection on the set Xf and j is a projection on the set Xg. Denote $a = ih$ and $b = jh^{-1}$, it is clear that $a, b \in End\,(X)$. Thus $Xf = Xgh^{-1} = Xgjh^{-1} = Xgb$ and $Xg = Xfh = Xfih = Xfa$. since $a|(Xf) = h$ and $b|(Xg) = h^{-1}$ it follows by (iv) of Theorem 1 that $f\mathcal{D}g$.

The endomorphism semigroup of a finitely generated abelian group

We are going to describe all finitely generated abelian groups G such that the endomorphism semigroup End (G) is an $\mathcal{L}$-subsemigroup of $\mathcal{T}(G)$.

Let $G = \bigoplus_{i=1}^{n} G_i$ be a direct sum of abelian groups. Let us denote a projection from G onto G_i by π_i. We shall use a correspondence between the endomorphism semigroup End (G) and a certain semigroup of matrices. Let $\mathbf{A}$ be a matrix, the element in the i^{th} row and the j^{th} column of $\mathbf{A}$ will be denoted by A_{ij}. Consider the set M of all $n \times n$ matrices $\mathbf{A}$ such that A_{ij} is a homomorphism from G_i to G_j. The set M with respect to the operation of multiplication of matrices forms a semigroup isomorphic to End (G). Let $f \in$ End (G), the matrix corresponding to the endomorphism f will also be denoted by $\mathbf{f}$. This matrix has the following property: $\underset{ij}{f} = (f\pi_j)|G_i$.

We shall use the following notation. If G is a group and k is a positive integer, then $\bigoplus_k G = \bigoplus_{i=1}^{k} G_i$ where $G_i = G$ for each i, if $a \in G$, then $ka = \bigoplus_{i=1}^{k} a_i$ where $a_i = a$ for each i and $kG = \{ka;\ a \in G\}$.

Lemma 5. *Let $f\colon G \to H$ be a homomorphism of abelian groups. If G is indecomposable and H is not trivial, then f has no left inverse or $G \cong H$.*

Proof. Suppose that $gf = 1_H$ for some $g\colon H \to G$. Then fg is an idempotent. Since G is indecomposable $fg = 1_G$ or $fg = 0$. Since H is not trivial we have $fg = 1_G$, thus $G \cong H$.

Theorem 2. *Let G be a finitely generated abelian group. End (G) is an $\mathscr{L}$-subsemigroup of $\mathscr{T}(G)$ if and only if $G \cong \bigoplus_n Z$ or $G \cong \bigoplus_{i=1}^{n} \bigoplus_{k_i} Z_{n_i}$, where n_i are powers of prime numbers p_i such that $p_i \neq p_j$ if $i \neq j$.*

Proof. The "if" part. If $G \cong \bigoplus_n Z$, then it is a free abelian group. Thus according to Lemma 4 End (G) is an $\mathscr{L}$-subsemigroup of $\mathscr{T}(G)$.

Let $G = \bigoplus_{i=1}^{n} G_i$, where $G_i \cong \bigoplus_{k_i} Z_{n_i}$ and n_i are powers of different prime numbers. It is easy to see, that for every $i \neq j$ and $f \in \text{End}(G)$, we have $f_{ij} = 0$. Let $f, g \in \text{End}(G)$ and $Gf \subset Gg$, thus for every i, $Gf\pi_i \subset Gg\pi_i$, hence we have:

$$G_if_{ii} = G_if\pi_i = \bigoplus_{j=1}^{n} G_jf\pi_i = Gf\pi_i \subset Gg\pi_i = \bigoplus_{j=1}^{n} G_jg\pi_i = G_ig\pi_i = G_ig_{ii}.$$

Note that G_i is a free group over a certain equational class, thus by Lemma 4 End (G_i) is an $\mathscr{L}$-subsemigroup of $\mathscr{T}(G_i)$. It implies that there exists $h_i \in \text{End}(G_i)$ such that $f_{ii} = h_ig_{ii}$. Hence there is $h \in \text{End}(G)$ such that $h_{ii} = h_i$ for each i. Thus:

$$(hg)_{ii} = \bigoplus_{j=1}^{n} h_{ij}g_{ji} = h_{ii}g_{ii} = h_ig_{ii} = f_{ii}$$

for every i. It implies that $hg = f$.

The "only if" part. Every finitely generated abelian group G is a direct sum of cyclic groups G_i such that $G_i \cong Z$ or the order of G_i is a power of a prime number (see [5]). Suppose $G = \bigoplus_{i=1}^{n} G_i$ and there are integers $a, b \in \{1, 2, \ldots n\}$ such that $G_a \cong Z_{p^m}$ and $G_b \cong Z_{p^r}$ where $r > m$ or $G_b \cong Z$. It is easy to see that there exists a homomorphism Φ from G_b onto G_a. Define $f, g \in \text{End}(G)$ in the following way:

$f_{ij} = 0$ if $i \neq b$ or $j \neq a$ and $f_{ba} = \Phi$,

$g_{ij} = 0$ if $i \neq a$ or $j \neq a$ and $g_{aa} = 1_a$ the identity mapping on G_a Clearly $Gf = G_a = Gg$, thus if End (G) is an $\mathscr{L}$-subsemigroup of $T(G)$, then by Theorem 1 there is $h \in \text{End}(G)$ such that $hf = g$. Hence we have

$$1_a = g_{aa} = (hf)_{aa} = \bigoplus_{j=1}^{n} h_{aj}f_{ja} = h_{ab}\Phi,$$

but by Lemma 5 it is impossible. Thus End(G) is not an $\mathscr{L}$-subsemigroup of $\mathscr{T}(G)$.

Using this result it is possible to prove the following well known statement (see [4]).

Theorem 3. *Let G be a finitely generated abelian group.* End(G) *is a regular semigroup if and only if* $G = \bigoplus_{i=1}^{n} Z_{p_i}$ *where p_i are prime numbers.*

Proof. The "only if" part. Suppose that $G = \bigoplus_{j=1}^{n} G_j$ where G_j are cyclic groups and there is $i \in \{1, 2, \dots n\}$ such that $G_i \cong Z$ or $G_i \cong Z_{p^m}$ where $m \geqq 2$. We shall show that End(G) cannot be regular. Let us suppose that End(G) is regular. Let Φ: $G_i \to G_i$ be a homomorphism such that for every $a \in G_i$ $a\Phi = ka$ where $k = 2$ if $G_i \cong Z$ and $k = p$ if $G_i \cong Z_{p^m}$. Denote $f = \pi_i \Phi$, clearly $f \in$ End(G). It follows by Lemma 1 that there is $a \in$ End(G) such that $Ga = Gf = kG_i$ and $a|(kG_i)$ is an identity mapping. If $G_i \cong Z$, then $G_i = G_i a \subset Ga = 2G_i$, but it is a contradiction. Let $G_i \cong Z_{p^m}$, suppose that $G_i \neq G_i a$. The subgroup pG_i is the biggest proper subgroup of G_i, thus $G_i a \subset pG_i$. It follows that

$$kG_i = (kG_i)a = (pG_i)a \subset p^2 G_i = k^2 G_i,$$

but it is impossible. Suppose that $G_i = G_i a$. It implies that $G_i = G_i a \subset Ga = kG_i$, but it is also impossible.

The "if" part. Let $G = \bigoplus_{i=1}^{n} Z_{p_i}$. It follows by Theorem 2 that End(G) is an $\mathscr{L}$-subsemigroup of $\mathscr{T}(G)$. Let $f \in$ End(G). Since Z_{p_i} has no proper subgroups it follows that there is a subset $I \subset \{1, 2, \dots n\}$ such that $Gf = \bigoplus_{i \in I} Z_{p_i}$. Denote $a = \bigoplus_{i \in I} \pi_i$. Clearly, $a \in$ End(G) is a projection on the set Gf. It follows by (v) of Theorem 1 that f is regular.

It follows by Lemma 3 and Lemma 4 that if S is either regular or a V-subsemigroup of $\mathscr{T}(X)$, then it is an $\mathscr{L}$-subsemigroup of $\mathscr{T}(X)$. We shall show that there exists a group G such that End(G) is not an $\mathscr{L}$-subsemigroup of $\mathscr{T}(G)$. There exists also a group G such that End(G) is an $\mathscr{L}$-subsemigroup of $\mathscr{T}(G)$, but it is neither regular nor a V-subsemigroup of $\mathscr{T}(G)$. There exists also G such that End(G) is not regular, but it is a V-subsemigroup of $\mathscr{T}(G)$ and on the other hand there is G such that End(G) is not a V-subsemigroup of $\mathscr{T}(G)$, but it is regular.

Example 1. Let $G = Z_2 \oplus Z_4$. End(G) is not an $\mathscr{L}$-subsemigroup of $\mathscr{T}(G)$.

Example 2. Let $G = Z_2 \oplus Z_2 \oplus Z_9$. End$(G)$ is an $\mathscr{L}$-subsemigroup of

$\mathscr{T}(G)$, but it is not regular. We shall show that it is not a V-subsemigroup of $\mathscr{T}(G)$ either. Elements of G will be written in the form of triplets (a, b, c) where $a, b \in Z_2$ and $c \in Z_9$. The subgroup generated by an element $\alpha \in G$ will be denoted by $[\alpha]$. Let us suppose that End(G) is a V-subsemigroup of $\mathscr{T}(G)$ and $A \subset G$ is a set of V-generators. For every $\alpha \in A$ and $t \in \mathscr{T}(G)$ the order of αt is a divisor of the order of α, it implies that the order of α is 18. Thus $[\alpha]$ is equal to $[(0, 1, 1)]$ or $[(1, 0, 1)]$ or $[(1, 1, 1)]$. Let us suppose that $\alpha, \beta \in A$ and $\alpha \neq \beta$. Thus there is $f \in$ End(G) such that $\alpha f = (0,0,0)$ and $\beta f = \beta$. It follows that $\xi f = (0,0,0)$ if $\xi \in [\alpha]$ and $\xi f = \xi$ if $\xi \in [\beta]$. Thus $[\alpha] \cap [\beta] = (0,0,0)$. Since

$[(0, 1, 1)] \cap [(1, 0, 1)] \neq (0, 0, 0)$,

$[(0, 1, 1)] \cap [(1, 1, 1)] \neq (0, 0, 0)$,

$[(1, 0, 1)] \cap [(1, 1, 1)] \neq (0, 0, 0)$ it follows that A has no more than one element. Let $A = \{\alpha\}$. Let us define $f, g \in$ End(G) in the following way:

$$\begin{aligned} (a, b, c)g &= (a, 0, 0) && \text{if } [\alpha] = [(0, 1, 1)] \\ (a, b, c)g &= (0, b, 0) && \text{if } [\alpha] = [(1, 0, 1)] \\ (a, b, c)g &= (a + b, 0, 0) && \text{if } [\alpha] = [(1, 1, 1)] \end{aligned}$$

and $\xi f = (0,0,0)$ for each $\xi \in G$. Clearly $f \neq g$, but $\alpha f = (0,0,0) = \alpha g$, thus A is not a set of V-generators.

Example 3. End(Z_4) is a V-subsemigroup of $\mathscr{T}(Z_4)$, but it is not regular.

Example 4. Let $G = Z_2 \oplus Z_6$. End(G) is regular, but it is possible to prove in a way similar to that in Example 3 that it is not a V-subsemigroup of $\mathscr{T}(G)$.

The semigroup of binary relations

In paper [8] K. A. Zareckij characterized the Green's $\mathscr{D}$-relation and regular elements of the semigroup $\mathscr{B}(\Omega)$ of all binary reletions on a set Ω. We can obtain the same result in another way, using Theorem 1.

$\mathscr{B}(\Omega)$ is isomorphic to a subsemigroup of $\mathscr{T}(2^\Omega)$. Indeed, one can define the mapping Φ from $\mathscr{B}(\Omega)$ to $\mathscr{T}(2^\Omega)$ such that the relation $r \in \mathscr{B}(\Omega)$ will be mapped by Φ into the mapping $r^* \in \mathscr{T}(2^\Omega)$ such that for each $\alpha \in 2^\Omega$, $\alpha r^* = \{x \in \Omega;\ \exists\, a \in \alpha;\ arx\}$. Clearly Φ is a one-to-one mapping. For every $r, q \in \mathscr{B}(\Omega)$ and each $\alpha \in 2^\Omega$ we have $\alpha(rq)^* = \{x \in \Omega;\ \exists\, a \in \alpha;\ arqx\} = \{x \in \Omega;\ \exists\, a \in \alpha$ and $b \in \Omega;\ arb$ and $bqx\} = \{x \in \Omega;\ \exists\, b \in \alpha r^*;\ bqx\} = (\alpha r^*)q^*$. Hence Φ is a homomorphism of semigroups.

Each $r^* \in \Phi(\mathscr{B}(\Omega))$ preserves arbitrary set unions. Indeed, for every $r \in \mathscr{B}(\Omega)$ and each system $\alpha_i \in 2^\Omega$; $i \in I$ we have:

$$\left(\bigcup_{i \in I} \alpha_i\right) r^* = \left\{x \in \Omega;\ \exists\, a \in \bigcup_{i \in I} \alpha_i;\ arx\right\} = \bigcup_{i \in I} \{x \in \Omega;\ \exists\, a \in \alpha_i;\ arx\} = \bigcup_{i \in I} (\alpha_i r^*).$$

It also follows that for every $r \in \mathscr{B}(\Omega)$ *the set* $2^{\Omega}r^*$ *forms a complete subsemilattice of the semilattice* $(2^{\Omega}, \cup, \emptyset)$.

Lemma 6. $\Phi(\mathscr{B}(\Omega))$ *is an* $\mathscr{L}$*-subsemigroup of* $\mathscr{T}(2^{\Omega})$.

Proof. In virtue of Lemma 4 it is sufficient to prove that $\Phi(\mathscr{B}(\Omega))$ is a V-subsemigroup of $\mathscr{T}(2^{\Omega})$. Let $A \subset 2^{\Omega}$ be a system of all single element subset of Ω. Let $t: A \to 2^{\Omega}$, define a relation $r \in \mathscr{B}(\Omega)$ such that xry if and only if $y \in \{x\}t$. Clearly $\{a\}r^* = \{x \in \Omega; arx\} = \{a\}t$ for each $a \in \Omega$. Thus $t = r^*|A$. Suppose that there are $r, q \in \mathscr{B}(\Omega)$ such that $r^*|A = q^*|A$ hence for every $a \in \Omega$ we have $\{x \in \Omega; arx\} = \{a\}r^* = \{a\}q^* = \{x \in \Omega; aqx\}$, thus $r = q$. It implies that A is a set of V-generators of $\Phi(\mathscr{B}(\Omega))$. Thus $\Phi(\mathscr{B}(\Omega))$ is a V-subsemigroup of $\mathscr{T}(2^{\Omega})$.

Lemma 7. *Let* $(A, \cup)$ *be a complete subsemilattice of the semilattice* $(2^{\Omega}, \cup)$ *containing the empty set. if a mapping* $f: A \to 2^{\Omega}$ *preserves arbitrary set unions, then there is a relation* $r \in \mathscr{B}(\Omega)$ *such that* $r^*|A = f$.

Proof. Define the set $I_x = \{\alpha \in A; x \in \alpha\}$ for every $x \in \Omega$ and the relation $r \in \mathscr{B}(\Omega)$ such that for each pair $x, y \in \Omega$, xry if and only if $x \in U = \bigcup_{\alpha \in A} \alpha$ (it means $I_x \neq \emptyset$) and $y \in \bigcap_{\alpha \in I_x} \alpha f$. Let $\alpha \in A$ and $\alpha \neq \emptyset$, hence we have:

$$\alpha r^* = \bigcup_{a \in \alpha} \{x \in \Omega, arx\} = \bigcup_{a \in U \cap \alpha} \left(\bigcap_{\beta \in I_a} \beta f\right) = \bigcup_{a \in \alpha} \left(\bigcap_{\beta \in I_a} \beta f\right) \subset \bigcup_{a \in \alpha} \alpha f = \alpha f.$$

Suppose that there is $m \in \alpha f$ such that $m \notin \alpha r^*$. Thus there is no $a \in \alpha$ such that arm. Hence for every $a \in \alpha$ there is $\beta_a \in I_a$ such that $m \notin \beta_a f$. Denote $\beta = \bigcup_{a \in \alpha} \beta_a$. It is clear that $\alpha \subset B$. Since f preserves the union, it preserves the inclusion as well. Thus $\alpha f \subset \beta f$. Hence $m \notin \bigcup_{a \in \alpha} (\beta_a f) = \beta f \supset \alpha f$, but it is a contradiction, thus $\alpha f = \alpha r^*$. At last, clearly, $\emptyset r^* = \emptyset = \emptyset f$.

If $(A, \cup)$ is a complete subsemilattice of $(2^{\Omega}, \cup)$ containing the empty set, then one can define an operation $\wedge$ in the natural way, $\bigwedge_{i \in I} \alpha_i = \cap \left\{\beta \in A; \beta \subset \bigcap_{i \in I} \alpha_i\right\}$. $(A, \cup, \wedge)$ is a complete lattice, but not necessarily a sublattice of the lattice $(2^{\Omega}, \cup, \cap)$.

Definition 3. (See [1]) *A complete lattice L is said to be completely distributive if for every system* $\alpha_{ij} \in L$, $i \in I$ *and* $j \in J_i$ *the following equation is true*:

$$\bigvee_{i \in I} \left(\bigwedge_{j \in J_i} \alpha_{ij}\right) = \bigwedge_{\varphi \in \Delta} \left(\bigvee_{i \in I} \alpha_{i, \varphi(i)}\right), \text{ where } \Delta = \left\{\varphi: I \to \bigcup_{i \in I} J_i;\ \varphi(i) \in J_i\right\}.$$

Lemma 8. *Let* $(A, \cup)$ *be a complete subsemilattice of* $(2^{\Omega}, \cup)$ *be a containing the empty set. The lattice* $(A, \cup, \wedge)$ *is completely distributive if and only if there exists a relation* $r \in \mathscr{B}(\Omega)$ *such that* $2^{\Omega}r^* = A$ *and* $r^*|A = 1_A$.

Proof. The "if" part. Let $a_j \in A$, $j \in J$, it is clear that $\bigwedge_{j \in J} a_j \subset \bigcap_{j \in J} a_j$, thus $\left(\bigwedge_{j \in J} a_j\right) r^* \subset \left(\bigcap_{j \in J} a_j\right) r^*$. Since $\left(\bigcap_{j \in J} a_j\right) r^* \in A$ and $\left(\bigcap_{j \in J} a_j\right) r^* \subset a_j r^*$ it follows that $\left(\bigcap_{j \in J} a_j\right) r^* \subset \bigwedge_{j \in J} (a_j r^*) = \bigwedge_{j \in J} a_j = \left(\bigwedge_{j \in J} a_j\right) r^*$, thus $\left(\bigcap_{j \in J} a_j\right) r^* = \left(\bigwedge_{j \in J} a_j\right) r^*$.

Hence, for each system a_{ij}, $i \in I$ and $j \in J_i$ we have:

$$\bigcup_{i \in I} \left(\bigwedge_{j \in J_i} a_{ij}\right) = \left[\bigcup_{i \in I} \left(\bigwedge_{j \in J_i} a_{ij}\right)\right] r^* = \bigcup_{j \in I} \left[\left(\bigwedge_{j \in J_i} a_{ij}\right) r^*\right] = \bigcup_{i \in I} \left[\left(\bigcap_{j \in J_i} a_{ij}\right) r^*\right] =$$
$$= \left[\bigcup_{i \in I} \left(\bigcap_{j \in J_i} a_{ij}\right)\right] r^* = \left[\bigcap_{\varphi \in \Delta} \left(\bigcup_{i \in I} a_{i, \varphi(i)}\right)\right] r^* = \left[\bigwedge_{\varphi \in \Delta} \left(\bigcup_{i \in I} a_{i, \varphi(i)}\right)\right] r^* =$$
$$= \bigwedge_{\varphi \in \Delta} \left(\bigcup_{i \in I} a_{i, \varphi(i)}\right).$$

The "only if" part. Define the set $I_x = \{\alpha \in A; x \in \alpha\}$ for every $x \in \Omega$, and relation $r \in \mathscr{B}(\Omega)$ such that for every $x, y \in \Omega$ xry if and only if $x \in U = \bigcup_{\alpha \in A} \alpha$ and $y \in \bigwedge_{\alpha \in I_x} \alpha$. Clearly, $\emptyset r^* = \emptyset$. Let $a \in 2^{\Omega}$, $a \neq \emptyset$, thus we have $ar^* =$
$= \bigcup_{a \in a} \{x \in \Omega; arx\} = \bigcup_{a \in U \cap a} \left(\bigwedge_{\beta \in I_a} \beta\right) \in A.$

Let $\alpha \in A$, $\alpha \neq \emptyset$, denote $J = \{j \colon \alpha \to A$ such that $j(a) \in I_a\}$. Hence we have

$$\alpha r^* = \bigcup_{a \in U \cap \alpha} \left(\bigwedge_{\beta \in I_a} \beta\right) = \bigcup_{a \in \alpha} \left(\bigwedge_{\beta \in I_a} \beta\right) = \bigwedge_{j \in J} \left(\bigcup_{a \in \alpha} j(a)\right) \supset \bigwedge_{j \in J} \alpha = \alpha.$$

On the other hand, since $\alpha \in I_a$ for each $a \in \alpha$, we have

$$\alpha r^* = \bigcup_{a \in U \cap \alpha} \left(\bigwedge_{\beta \in I_a} \beta\right) = \bigcup_{a \in \alpha} \left(\bigwedge_{\beta \in I_a} \beta\right) \subset \bigcup_{a \in \alpha} \alpha = \alpha. \text{ Thus } \alpha r^* = \alpha.$$

The following statements proved by K. A. Zareckij are immediate consequences of Theorem 1 and Lemma 6, Lemma 7, Lemma 8.

Theorem 4. (Zareckij [8, Theorem 2.8]) *Let* $r, g \in \mathscr{B}(\Omega)$, *then* $r\mathscr{D}q$ *if and only if* $(2^{\Omega}r^*, \cup)$ *and* $(2^{\Omega}q^*, \cup)$ *are isomorphic semilattices.*

Theorem 5. (Zareckij [8, Theorem 3.2]) *Let* $r \in \mathscr{B}(\Omega)$, *then* r *is regular if and only if the lattice* $(2^{\Omega}r^*, \cup, \wedge)$ *is completely distributive.*

REFERENCES

[1] BIRKHOFF, G.: Теория структур. Москва, ИЛ, 1952

[2] CLIFFORD, A. H. – PRESTON, G. B.: The Algebraic Theory of Semigroups. Volume I. Amer. Math. Soc., Providence, R. I. 1964.

[3] FUCHS, L.: Abelian Groups. New York, 1960.

[4] FUCHS, L.: Бесконечные абелевы группы. Том 2. Москва, 1977.

[5] KUROŠ, A. G.: Kapitoly z obecné algebry. Praha, 1968.

[6] MAGILL, K. D. – SUBBIAH, S.: Green's relations for regular elements of semigroups of endomorphisms. Can. J. Math., 26, 1974, 1484–1497.

[7] ВАЛУЦЭ, И. И.: Левые идеалы полугруппы эндоморфизмов свободной универсальной алгебры. Мат. Сборник, 62, 1963, 371–384.

[8] ЗАРЕЦКИЙ, К. А.: Полугруппа бинарных отношений. Мат. сборник, 61, 1963, 291–305.

Received June 26, 1985

Katedra matematiky
Elektrotechnickej fakulty SVŠT
Mlynská dolina blok A
812 19 Bratislava

ОТНОШЕНИЯ ГРИНА И РЕГУЛЯРНЫЕ ЭЛЕМЕНТЫ ПОЛУГРУПП ПРЕОБРАЗОВАНИЙ

Igor Kossaczky

Резюме

Пусть $\mathscr{T}(X)$-полугруппа всех преобразований множества X. Подполугруппа $S \subset \mathscr{T}(X)$ называется $\mathscr{L}$-подполугруппой $\mathscr{T}(X)$, если для произвольных $f, g \in S$, таких что $Xf \subset Xg$, существует $h \in S$ так, что $f = hg$. В работе I) характеризуются $\mathscr{L}$, $\mathscr{R}$, $\mathscr{J}$-отношения Грина и резулярные элементы $\mathscr{L}$-подполугрупп, II) Даны примеры $\mathscr{L}$-подполугрупп, полугруппа эндоморфизмов свободной универсальной алгебры, полугруппа бинарных отношений ...