Mathematica Slovaca

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Mathematica Slovaca, Vol. 51 (2001), No. 5, 529--545

Persistent URL: http://dml.cz/dmlcz/129588

Math. Slovaca, **51** (2001), No. 5, 529–545

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NONLOCAL CAUCHY PROBLEMS ON SEMI-INFINITE INTERVALS FOR NEUTRAL FUNCTIONAL DIFFERENTIAL AND INTEGRODIFFERENTIAL INCLUSIONS IN BANACH SPACES

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(*Communicated by Milan Medveď*)

ABSTRACT. In this paper we investigate the existence of mild solutions on infinite intervals to initial value problems for neutral functional differential and integrodifferential inclusions in Banach spaces with nonlocal conditions. We shall rely on a fixed point theorem due to Ma, which is an extension of Schaefer's theorem on locally convex topological spaces.

1. Introduction

In this paper we prove the existence of mild solutions, defined on a semi-infinite interval, for neutral functional differential and integrodifferential inclusions with nonlocal conditions. In Section 3, we study the neutral functional differential inclusion of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[y(t) - f(t, y_t)\big] \in Ay(t) + F(t, y_t)\,, \qquad \text{a.e.} \quad t \in J := [0, \infty)\,, \tag{1.1}$$

$$y(t) + \big(\xi(y_{t_1}, \dots, y_{t_p})\big)(t) = \phi(t)\,, \qquad t \in [-r, 0]\,, \tag{1.2}$$

where A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t)$ in E, $F\colon J \times C(J_0, E) \to 2^E$ ($J_0 := [-r, 0]$) is a bounded, closed, convex valued multivalued map, $f\colon J \times C(J_0, E) \to E$, $\phi \in C(J_0, E)$, $0 < t_1 < \dots < t_p < \infty$, $p \in \mathbb{N}$, $\xi\colon \big[C(J_0, E)\big]^p \to C(J_0, E)$, and E, a real Banach space with the norm $|\cdot|$.

2000 Mathematics Subject Classification: Primary 34A60, 34G20, 34K40, 34K99.
Keywords: neutral functional differential and integrodifferential inclusion, mild solution, evolution, nonlocal conditions, fixed point.

For any continuous function y defined on the interval $[-r,\infty)$, and for any $t \in J$, we denote by y_t the element of $C(J_0,E)$ defined by

$$y_t(\theta) = y(t+\theta), \qquad \theta \in J_0 .$$

Here, $y_t(\cdot)$ represents the history of the state from time $t-r$ up to the present time t.

Section 4 is devoted to the study of the existence of mild solutions for a neutral functional integrodifferential inclusion of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}[y(t) - f(t,y_t)] \in Ay(t) + \int_0^t K(t,s)F(s,y_s)\,\mathrm{d}s, \qquad t \in J = [0,\infty), \tag{1.3}$$

$$y(t) + \big(\xi(y_{t_1},\dots,y_{t_p})\big)(t) = \phi(t), \quad t \in [-r,0], \tag{1.4}$$

where A, F, f, g, ϕ are as in problem (1.1)–(1.2) and $K\colon D \to \mathbb{R}$, $D = \{(t,s) \in J \times J : t \geq s\}$.

Equations of the types (1.1)–(1.2) or (1.3)–(1.4) arise in many areas of applied mathematics and such equations have received much attention in recent years. We refer to the books of Erbe, Kong and Zhang [14], Hale [15] and Henderson [16], and the references cited therein.

Motivated by the recent papers of Hernandez and Henriquez [17], [18] and Byszewski and Akca [9], the authors in [7] have studied existence results on compact intervals for neutral functional differential inclusions with nonlocal conditions. Here, we extend these results to infinite intervals. The method we are going to use is to reduce the existence of solutions to problems (1.1)–(1.2) and (1.3)–(1.4) to the search for fixed points of a suitable multivalued map on the Fréchet space $C([-r,\infty),E)$. In order to prove the existence of fixed points, we shall rely on a fixed point theorem due to Ma [22], which is an extension of Schaefer's theorem ([30]) to locally convex topological spaces.

For recent results on nonlocal IVP we refer to the papers of Balachandran and Chandrasekaran [1], Byszewski [8], [9], Dauer and Balachandran [11], Lin and Liu [21], Ntouyas [24], Ntouyas and Tsamatos [25]–[27] and Benchohra and Ntouyas [2]–[6].

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper.

J_m is the compact real interval $[0,m]$ $(m \in \mathbb{N})$.

$C(J,E)$ is the linear metric Fréchet space of continuous functions from J into E with the metric (see Dugundji and Granas [13])

$$d(y,z) = \sum_{m=0}^{\infty} \frac{2^{-m}\|y-z\|_m}{1+\|y-z\|_m} \qquad \text{for each} \quad y,z \in C(J,E),$$

where

$$\|y\|_m := \sup\{|y(t)| : \ t \in J_m\}.$$

A measurable function $y\colon J \to E$ is *Bochner integrable* if and only if $|y|$ is Lebesgue integrable. For properties of the Bochner integral we refer to Yosida [31].

$L^1(J,E)$ denotes the linear space of equivalence classes of measurable functions $y\colon J \to E$ such that $\int_0^\infty |y(s)|\, \mathrm{d}s < \infty$.

U_p denotes the neighbourhood of 0 in $C(J,E)$ defined by

$$U_p := \{y \in C(J,E) : \ \|y\|_m \le p \text{ for each } m \in \mathbb{N}\}.$$

The convergence in $C(J,E)$ is the *uniform convergence on compact intervals*, i.e. $y_j \to y$ in $C(J,E)$ if and only if for each $m \in \mathbb{N}$, $\|y_j - y\|_m \to 0$ in $C(J_m, E)$ as $j \to \infty$.

$M \subseteq C(J,E)$ is a *bounded set* if and only if there exists a positive function $\varphi \in C(J,\mathbb{R})$ such that

$$|y(t)| \le \varphi(t) \qquad \text{for all} \quad t \in J, \ \ y \in M.$$

A set $M \subseteq C(J,E)$ is *compact* if and only if for each $m \in \mathbb{N}$, M is a compact set in the Banach space $\big(C(J_m,E), \|\cdot\|_m\big)$.

Let $(X, |\cdot|)$ be a Banach space. A multivalued map $G\colon X \to 2^X$ is *convex* (*closed*) *valued* if $G(x)$ is convex (closed) for all $x \in X$.

G is *bounded* on bounded sets if $G(B) = \bigcup_{x\in B} G(x)$ is bounded in X for any bounded set B of X (i.e. $\sup_{x\in B}\{\sup\{|y| : \ y \in G(x)\}\} < \infty$).

G is called *upper semicontinuous* (*u.s.c.*) on X if for each $x_* \in X$, the set $G(x_*)$ is a nonempty, closed subset of X, and if for each open set B of X containing $G(x_*)$, there exists an open neighbourhood V of x_* such that $G(V) \subseteq B$.

G is said to be *completely continuous* if $G(B)$ is relatively compact for every bounded subset $B \subseteq X$.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$).

G has a *fixed point* if there is $x \in X$ such that $x \in G(x)$.

In the following, $BCC(X)$ denotes the set of all nonempty bounded, closed and convex subsets of X.

A multivalued map $G\colon J \to BCC(E)$ is said to be *measurable* if for each $x \in E$, the function $Y\colon J \to \mathbb{R}$ defined by

$$Y(t) = d\big(x, G(t)\big) = \inf\big\{|x - z| : z \in G(t)\big\}$$

belongs to $L^1(J, \mathbb{R})$. For more details on multivalued maps, see the books of Deimling [12] and Hu and Papageorgiou [19].

Our existence results will be proved using the following fixed point result.

Lemma 2.1. ([22]) *Let X be a locally convex space and $N\colon X \to 2^X$ be a compact convex valued, u.s.c. multivalued map such that for every closed neighbourhood V_p of 0, $N(V_p)$ is a relatively compact set for each $p \in \mathbb{N}$. If the set*

$$\Omega := \{y \in X : \ \lambda y \in Ny \ \text{for some} \ \lambda > 1\}$$

is bounded, then N has a fixed point.

3. Existence results for neutral functional differential inclusions

In order to define the concept of mild solution for (1.1) – (1.2), by analogy with the abstract Cauchy problem

$$y'(t) = Ay(t) + h(t), \quad y(0) = a,$$

whose properties are well known ([29]), we associate (1.1) – (1.2) to the integral equation

$$\begin{aligned} y(t) = T(t)\big[\phi(0) - \big(\xi(y_{t_1}, \dots, y_{t_p})\big)(0) - f(0, \phi)\big] + f(t, y_t) \\ + \int_0^t AT(t-s) f(s, y_s)\, \mathrm{d}s + \int_0^t T(t-s) g(s)\, \mathrm{d}s, \qquad t \in J, \end{aligned} \tag{3.1}$$

where

$$g \in S_{F,y} = \big\{g \in L^1(J, E) : \ g(t) \in F(t, y_t) \ \text{for a.e.} \ t \in J\big\}.$$

Definition 3.1. A function $y\colon (-r, \infty) \to E$ is called a *mild solution of the Cauchy problem* (1.1) – (1.2) if $y(t) + \big(\xi(y_{t_1}, \dots, y_{t_p})\big)(t) = \phi(t)$, $t \in [-r, 0]$, the restriction of $y(\cdot)$ to the interval $[0, \infty)$ is continuous and for each $t \geq 0$ the function $AT(t-s) f(s, y_s)$, $s \in [0, t)$, is integrable and the integral equation (3.1) is satisfied.

Assume that:

(H1) A is the infinitesimal generator of a compact semigroup of bounded linear operators $T(t)$ in E such that

$$|T(t)| \leq M_1 \quad \text{for some } M_1 \geq 1 \quad \text{and} \quad |AT(t)| \leq M_2\,, \quad M_2 \geq 0\,, \qquad t \in J\,;$$

(H2) there exists constants $0 \leq c_1 < 1$ and $c_2 \geq 0$ such that

$$|f(t,u)| \leq c_1\|u\| + c_2\,, \qquad t \in J\,, \quad u \in C(J_0,E)\,;$$

(H3) ξ is completely continuous and there exists a constant Q such that:

$$\big\|\big(\xi(y_{t_1},\dots,y_{t_p})\big)(t)\big\| \leq Q \qquad \text{for} \quad y \in C\big([-r,\infty),E\big)\,;$$

(H4) $F\colon J\times C(J_0,E) \to BCC(E)$; $(t,u) \mapsto F(t,u)$ is measurable with respect to t for each $u \in C(J_0,E)$, u.s.c. with respect to u for each $t \in J$, and for each fixed $u \in C(J_0,E)$ the set

$$S_{F,u} = \big\{g \in L^1(J,E):\ g(t) \in F(t,u) \ \text{for a.e. } t \in J\big\}$$

is nonempty;

(H5) $\|F(t,u)\| := \sup\big\{|v| :\ v \in F(t,u)\big\} \leq p(t)\psi\big(\|u\|\big)$ for almost all $t \in J$ and all $u \in C(J_0,E)$, where $p \in L^1(J,\mathbb{R}_+)$ and $\psi\colon \mathbb{R}_+ \to (0,\infty)$ is continuous and increasing with

$$\int\limits_{c_m}^{\infty} \frac{\mathrm{d}\tau}{\psi(\tau)} = \infty\,,$$

where

$$c_m = \frac{1}{1-c_1}\{[M_1(1+c_1)\|\phi\| + Q + c_2] + c_2 + M_2c_2m\}\,;$$

(H6) the function f is completely continuous, and for any bounded set $B \subseteq C\big([-r,\ \infty),E\big)$ the set $\big\{t \to f(t,y_t) :\ y \in B\big\}$ is equicontinuous in $C(J,E)$.

Remark 3.2.

(i) If $\dim E < \infty$, then for each $u \in C(J_0,E)$, $S_{F,u} \neq \emptyset$ (see Lasota and Opial [20]).

(ii) $S_{F,u}$ is nonempty if and only if the function $Y\colon J \to \mathbb{R}$ defined by

$$Y(t) := \inf\big\{|v| :\ v \in F(t,u)\big\}$$

belongs to $L^1(J,\mathbb{R})$ (see Papageorgiou [28]).

The following lemma is crucial in the proof of our existence results.

Lemma 3.3. ([20]) *Let I be a compact real interval and X be a Banach space. Let F be a multivalued map satisfying* (H4) *and let Γ be a linear continuous mapping from $L^1(I,X)$ to $C(I,X)$. Then the operator*

$$\Gamma \circ S_F\colon C(I,X) \to BCC(C(I,X)), \qquad y \mapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F,y})$$

is a closed graph operator in $C(I,X) \times C(I,X)$.

Now, we are in a position to state and prove our main theorem for this section.

Theorem 3.4. *Assume that hypotheses* (H1)-(H6) *hold. Then IVP* (1.1)-(1.2) *has at least one mild solution on $[-r,\infty)$.*

Proof. Let $C := C\big([-r,\infty),E\big)$ be the Fréchet space of continuous functions from $[-r,\infty)$ into E endowed with the seminorms

$$\|y\|_{r,m} := \sup\big\{|y(t)| :\ t \in [-r,m]\big\} \qquad \text{for} \quad y \in C .$$

Transform the problem into a fixed point problem. Consider the multivalued map $N\colon C \to 2^C$ defined by:

$$N(y) := \left\{ h \in C :\ h(t) = \begin{cases} \phi(t)-(\xi(y_{t_1},\dots,y_{t_p}))(t) & \text{if } t\in J_0 \\ T(t)[\phi(0)-(\xi(y_{t_1},\dots,y_{t_p}))(0)-f(0,\phi)]+f(t,y_t) & \\ \quad +\int_0^t AT(t-s)f(s,y_s)\,\mathrm{d}s+\int_0^t T(t-s)g(s)\,\mathrm{d}s & \text{if } t\in J \end{cases} \right\}$$

where

$$g \in F_{F,y} = \big\{g \in L^1(J,E) :\ g(t) \in F(t,y_t) \ \text{for a.e.}\ t \in J\big\} .$$

Remark 3.5. It is clear that the fixed points of N are mild solutions to (1.1)-(1.2).

We will show that N has a fixed point. We first shall show that $N(V_q)$ is relatively compact for each neighbourhood V_q of $0 \in C\big([-r,\infty),E\big)$ with $q \in \mathbb{N}$ and the multivalued map N has bounded, closed and convex values and it is u.s.c. The proof will be given in several steps.

Step 1: $N(y)$ is convex for each $y \in C$.
This step is obvious. However, for completeness, we give the proof. If h_1, h_2 belong to $N(y)$, then there exist $g_1, g_2 \in S_{F,y}$ such that for each $t \in J$ we have

$$h_i(t) = T(t)\big[\phi(0) - \big(\xi(y_{t_1},\dots,y_{t_p})\big)(0) - f(0,\phi)\big] + f(t,y_t) + \int_0^t AT(t-s)f(s,y_s)\,\mathrm{d}s + \int_0^t T(t-s)g_i(s)\,\mathrm{d}s , \qquad i = 1,2 .$$

Let $0 \le k \le 1$. Then for each $t \in J$ we have

$$\begin{aligned}
&\big(kh_1 + (1-k)h_2\big)(t) \\
&= T(t)\big[\phi(0) - \big(\xi(y_{t_1}, \dots, y_{t_p})\big)(0) - f(0,\phi)\big] + f(t, y_t) + \int_0^t AT(t-s) f(s, y_s)\, ds \\
&\qquad + \int_0^t T(t-s)\big[kg_1(s) + (1-k)g_2(s)\big]\, ds .
\end{aligned}$$

Since $S_{F,y}$ is convex (because F has convex values), we have

$$kh_1 + (1-k)h_2 \in N(y)$$

completing the proof of Step 1.

We will next prove that N is a completely continuous operator. Using (H3) and (H6) it suffices to show that the operator $N_1 : C \to 2^C$ defined by:

$$N_1(y) := \left\{ h \in C : \ h(t) = \int_0^t AT(t-s) f\big(s, y_s\big)\, ds + \int_0^t T(t-s) g(s)\, ds \ \text{ if } t \in J \right\},$$

where $g \in F_{F,y}$, is completely continuous.

Step 2: N_1 maps bounded sets into bounded sets in C.
Indeed, it is enough to show that there exists a positive constant $\tilde{\ell}$ such that for each $h \in N_1(y)$, $y \in B_q = \{ y \in C : \ \|y\|_m \le q \}$ one has $\|h\|_{r,m} \le \tilde{\ell}$.

If $h \in N_1(y)$, then there exists $g \in S_{F,y}$ such that for each $t \in J$ we have

$$h(t) = \int_0^t AT(t-s) f(s, y_s)\, ds + \int_0^t T(t-s) g(s)\, ds .$$

By (H1), (H2), (H4) and (H5) we have for each $t \in J_m$

$$\begin{aligned}
|h(t)| &\le \int_0^t |AT(t-s) f(s, y_s)|\, ds + \int_0^t \|T(t-s) g(s)\|\, ds \\
&\le M_2 m (c_1 q + c_2) + M_1 \sup_{y \in [0,q]} \psi(y) \left(\int_0^t p(s)\, ds \right) \\
&\le M_2 m (c_1 q + c_2) + M_1 \sup_{y \in [0,q]} \psi(y) \left(\int_0^m p(s)\, ds \right) =: \ell_m .
\end{aligned}$$

Then for each $h \in N(B_q)$ we have

$$\|h\|_{r,m} \le \tilde{\ell}_m = \max\{ \|\phi\|, \ell_m \} .$$

Step 3: For each $q \in \mathbb{N}$, $N_1(V_q)$ is equicontinuous for V_q, a neighbourhood of 0 in C.

Let $\tau_1, \tau_2 \in J_m$, $0 < \tau_1 < \tau_2$ and V_q be a neighbourhood of 0 in C for $q \in \mathbb{N}$.

For each $y \in V_q$ and $h \in N_1(y)$, there exists $g \in S_{F,y}$ such that

$$h(t) = \int_0^t AT(t-s)f(s, y_s)\,\mathrm{d}s + \int_0^t T(t-s)g(s)\,\mathrm{d}s\,, \qquad t \in J\,.$$

Thus

$$\begin{aligned}
&|h(\tau_2) - h(\tau_1)| \\
&\le \left| \int_0^{\tau_1} [AT(\tau_2 - s) - AT(\tau_1 - s)] f(s, y_s)\,\mathrm{d}s \right| + \left| \int_{\tau_1}^{\tau_2} AT(\tau_2 - s) f(s, y_s)\,\mathrm{d}s \right| \\
&\qquad + \left\| \int_0^{\tau_1} [T(\tau_2 - s) - T(\tau_1 - s)] g(s)\,\mathrm{d}s \right\| + \left\| \int_{\tau_1}^{\tau_2} T(\tau_2 - s) g(s)\,\mathrm{d}s \right\| \\
&\le \int_0^{\tau_1} \big|A\big[T(\tau_2 - s) - T(\tau_1 - s)\big]\big| (c_1 q + c_2)\,\mathrm{d}s + \int_{\tau_1}^{\tau_2} |AT(\tau_2 - s)|(c_1 q + c_2)\,\mathrm{d}s \\
&\qquad + \int_0^{\tau_1} |T(\tau_2 - s) - T(\tau_1 - s)| \|g(s)\|\,\mathrm{d}s + \int_{\tau_1}^{\tau_2} |T(t_2 - s)| \|g(s)\|\,\mathrm{d}s\,.
\end{aligned}$$

As $\tau_2 \to \tau_1$, the right-hand side of the above inequality tends to zero.

As a consequence of Step 2, Step 3, (H3) and (H6), together with the fact that $T(t)$ is compact and the definition of the metric of the Fréchet space C, we can conclude that $N_1(V_q)$ is relatively compact in C.

Step 4: N has a closed graph.

Let $y_n \to y_*$, $h_n \in N(y_n)$, and $h_n \to h_*$. We shall prove that $h_* \in N(y_*)$.

$h_n \in N(y_n)$ means that there exists $g_n \in S_{F,y_n}$ such that

$$\begin{aligned}
h_n(t) = T(t)\big[\phi(0) - \big(\xi\big((y_n)_{t_1}, \dots, (y_n)_{t_p}\big)\big)(0) - f(0,\phi)\big] + f(t, y_{nt}) \\
+ \int_0^t AT(t-s) f(t, y_{ns})\,\mathrm{d}s + \int_0^t T(t-s) g_n(s)\,\mathrm{d}s\,, \qquad t \in J\,.
\end{aligned}$$

We have to prove that there exists $g_* \in S_{F,y_*}$ such that

$$h_*(t) = T(t)\big[\phi(0) - \big(\xi\big((y_*)_{t_1}, \dots, (y_*)_{t_p}\big)\big)(0) - f(0,\phi)\big] + f(t, y_{*t}) + \int_0^t AT(t-s) f(t, y_{*s})\, ds + \int_0^t T(t-s) g_*(s)\, ds\,, \qquad t \in J\,. \tag{3.2}$$

The idea is then to use the facts that

(i) $h_n \to h_*$;
(ii) $h_n - T(t)\big[\phi(0) - \big(\xi\big((y_n)_{t_1}, \dots, (y_n)_{t_p}\big)\big)(0) - f(0,\phi)\big] - f(t, y_{nt}) - \int_0^t AT(t-s) f(s, y_{ns})\, ds \in \Gamma(S_{F,y_n})$,

where $\Gamma\colon L^1(J,E) \to C(J,E)$ is defined by $(\Gamma g)(t) := \int_0^t T(t-s) g(s)\, ds$.

If $\Gamma \circ S_F$ is a closed graph operator, we would be done. But, we do not know whether $\Gamma \circ S_F$ is a closed graph operator. So, we cut the functions y_n, $h_n - T(t)\big[\phi(0)\big(\xi\big((y_n)_{t_1}, \dots, (y_n)_{t_p}\big)\big)(0) - f(0,\phi)\big] - f(t, y_{nt}) - \int_0^t AT(t-s) f(s, y_{ns})\, ds$, g_n and we consider them defined on the interval $[k, k+1]$ for any $k \in \mathbb{N} \cup \{0\}$. Then, using Lemma 3.3, in this case we are able to show that (3.2) is true on the compact interval $[k, k+1]$, i.e.

$$h_*(t)\big|_{[k,\,k+1]} = T(t)\big[\phi(0) - \big(\xi\big((y_*)_{t_1}, \dots, (y_*)_{t_p}\big)\big)(0) - f(0,\phi)\big] - f(t, y_{*t}) - \int_0^t AT(t-s) f(s, y_{*s})\, ds + \int_0^t T(t-s) g_*^k(s)\, ds$$

for a suitable L^1-selection g_*^k of $F\big(t, y_*(t)\big)$ on the interval $[k, k+1]$.

At this point, we can past the functions g_*^k obtaining the selection g_* defined by

$$g_*(t) = g_*^k(t) \qquad \text{for} \quad t \in [k, k+1)\,.$$

We obtain then that g_* is a L^1-selection and (3.2) is satisfied.

We give now the details.

Since f is continuous, we have

$$\Big\|\Big(h_n - T(t)\big[\phi(0) - \big(\xi\big((y_n)_{t_1},\dots,(y_n)_{t_p}\big)\big)(0) - f(0,\phi)\big] - f(t,y_{nt}) - \int_0^t AT(t-s)f(t,y_{ns})\,\mathrm{d}s\Big)$$
$$- \Big(h_* - T(t)\big[\phi(0) - \big(\xi\big((y_*)_{t_1},\dots,(y_*)_{t_p}\big)\big)(0) - f(0,\phi)\big] - f(t,y_{*t}) - \int_0^t AT(t-s)f(t,y_{*s})\,\mathrm{d}s\Big)\Big\|_\infty \longrightarrow 0,$$

as $n \to \infty$.

Consider the linear continuous operator

$$\Gamma\colon L^1(J,E) \longrightarrow C(J,E),$$
$$g \longmapsto \Gamma(g)(t) = \int_0^t T(t-s)g(s)\,\mathrm{d}s.$$

From Lemma 3.3, it follows that $\Gamma \circ S_F$ is a closed graph operator.

Moreover, we have that

$$h_n(t) - T(t)\big[\phi(0) - \big(\xi\big((y_n)_{t_1},\dots,(y_n)_{t_p}\big)\big)(0) - f(0,\phi)\big] - f(t,y_{nt}) - \int_0^t AT(t-s)f(t,y_{ns})\,\mathrm{d}s \in \Gamma(S_{F,y_n}).$$

Since $y_n \to y_*$, it follows from Lemma 3.3 that

$$h_*(t) - T(t)\big[\phi(0) - \big(\xi\big((y_*)_{t_1},\dots,(y_*)_{t_p}\big)\big)(0) - f(0,\phi)\big] - f(t,y_{*t}) - \int_0^t AT(t-s)f(t,y_{*s})\,\mathrm{d}s = \int_0^t T(t-s)g_*(s)\,\mathrm{d}s$$

for some $g_* \in S_{F,y_*}$.

Therefore, $N(V_q)$ is relatively compact for each neighbourhood V_q of $0 \in C([-r,\infty),E)$ with $q \in \mathbb{N}$ and the multivalued map N has bounded, closed and convex values and it is u.s.c. In order to prove that N has a fixed point, we need one more step.

Step 5: The set $\Omega := \{y \in C : \ \lambda y \in N(y) \ \text{for some} \ \lambda > 1\}$ is bounded.
Let $y \in \Omega$. Then $\lambda y \in N(y)$ for some $\lambda > 1$. Thus there exists $g \in S_{F,y}$ such that

$$y(t) = \lambda^{-1}T(t)\big[\phi(0) - \big(\xi(y_{t_1}, \dots, y_{t_p})\big)(0) - f(0,\phi)\big] + \lambda^{-1}f(t, y_t) + \lambda^{-1}\int_0^t AT(t-s)f(s,y_s)\,\mathrm{d}s + \lambda^{-1}\int_0^t T(t-s)g(s)\,\mathrm{d}s, \qquad t \in J.$$

This implies by (H1), (H2), (H4) and (H5) that for each $t \in J_m$ we have

$$\|y(t)\|_m \le M_1\big[\|\phi\| + Q + c_1\|\phi\| + c_2\big] + c_1\|y_t\| + c_2 + M_2\int_0^t \big(c_1\|y_s\| + c_2\big)\,\mathrm{d}s + M_1\int_0^t p(s)\psi\big(\|y_s\|\big)\,\mathrm{d}s.$$

We consider the function μ defined by

$$\mu(t) = \sup\big\{|y(s)| : \ -r \le s \le t\big\}, \qquad 0 \le t \le m.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in J_m$, by the previous inequality we have for $t \in J_m$

$$\begin{aligned}\mu(t) &\le M_1\big[\|\phi\| + Q + c_1\|\phi\| + c_2\big] + c_1\mu(t) + c_2 + M_2\int_0^{t^*}\big(c_1\mu(s) + c_2\big)\,\mathrm{d}s + M_1\int_0^{t^*} p(s)\psi\big(\mu(s)\big)\,\mathrm{d}s\\ &\le M_1\big[\|\phi\| + Q + c_1\|\phi\| + c_2\big] + c_1\mu(t) + c_2 + M_2c_1\int_0^t \mu(s)\,\mathrm{d}s + M_2c_2m + M_1\int_0^t p(s)\psi(\mu(s))\,\mathrm{d}s.\end{aligned}$$

Thus

$$\mu(t) \le \frac{1}{1-c_1}\Big\{M_1\big[(1+c_1)\|\phi\| + Q + c_2\big] + c_2 + M_2c_2m + M_2c_1\int_0^t \mu(s)\,\mathrm{d}s + M_1\int_0^t p(s)\psi\big(\mu(s)\big)\,\mathrm{d}s\Big\}, \qquad t \in J_m.$$

If $t^* \in J_0$, then $\mu(t) \le \|\phi\| + Q$ and the previous inequality holds since $M_1 \ge 1$.

Let us take the right-hand side of the above inequality as $v(t)$. Then we have

$$c = v(0) = \frac{1}{1-c_1}\Big\{M_1\big[(1+c_1)\|\phi\| + Q + c_2\big] + c_2 + M_2c_2m\Big\},$$

$$\mu(t) \le v(t), \qquad t \in J_m,$$

and

$$v'(t) = \frac{1}{1-c_1} M_2 c_1 \mu(t) + \frac{1}{1-c_1} M_1 p(t) \psi(\mu(t)) , \qquad t \in J_m .$$

Using the nondecreasing character of ψ we get

$$\begin{aligned} v'(t) &\le \frac{1}{1-c_1} M_2 c_1 v(t) + \frac{1}{1-c_1} M_1 p(t) \psi(v(t)) \\ &\le \widehat{m}(t) \big[v(t) + \psi(v(t)) \big] , \qquad t \in J_m , \end{aligned}$$

where

$$\widehat{m}(t) = \max \left\{ \tfrac{1}{1-c_1} M_2 c_1 , \ \tfrac{1}{1-c_1} M_1 p(t) \right\} .$$

This implies that for each $t \in J_m$

$$\int_{v(0)}^{v(t)} \frac{du}{u + \psi(u)} \le \int_0^t \widehat{m}(s) \, ds \le \int_0^m \widehat{m}(s) \, ds < \infty .$$

Now, the hypotheses on ψ imply (see [10]) that

$$\int_{v(0)}^{\infty} \frac{du}{u + \psi(u)} = \infty ,$$

thus there exists a constant L_m such that $v(t) \le L_m$, $t \in J_m$, and hence $\mu(t) \le L_m$, $t \in J_m$. Since for every $t \in J_m$, $\|y_t\| \le \mu(t)$, we have

$$\|y\|_{r,m} := \sup\{ |y(t)| : \ -r \le t \le m \} \le L_m ,$$

where L_m depends on m and on the functions p and ψ. This shows that Ω is bounded.

Set $X := C$. As a consequence of Lemma 2.1 we deduce that N has a fixed point which is a solution of (1.1)–(1.2). □

4. Existence results for neutral functional integrodifferential inclusions

In this section we consider the solvability of IVP (1.3)–(1.4).

Let us list the following hypotheses:

(H7) for each $t \in J_m$, $K(t,s)$ is measurable on $[0,t]$ and

$$K(t) = \operatorname{ess\,sup}\{ |K(t,s)| : \ 0 \le s \le t \}$$

is bounded on J;

(H8) the map $t \mapsto K_t$ is continuous from J to $L^\infty(J_m, \mathbb{R})$; here $K_t(s) = K(t,s)$;

(H9) $\|F(t,u)\| := \sup\{|v| : v \in F(t,u)\} \le p(t)\psi(\|u\|)$ for almost all $t \in J$ and all $u \in C(J_0, E)$, where $p \in L^1(J, \mathbb{R}_+)$ and $\psi\colon \mathbb{R}_+ \to (0,\infty)$ is continuous and increasing with

$$\int_{\bar{c}_m}^{\infty} \frac{\mathrm{d}\tau}{\psi(\tau)} = \infty,$$

where

$$\bar{c}_m = \frac{1}{1-c_1}\Big\{M_1\big[(1+c_1)\|\phi\| + Q + c_2\big] + c_2 + M_2 c_2 m\Big\}.$$

We define a mild solution for problem (1.3)–(1.4) by the integral equation

$$y(t) = T(t)\big[\phi(0) - \big(\xi(y_{t_1}, \dots, y_{t_p})\big)(0) - f(0,\phi)\big] + f(t, y_t) + \int_0^t AT(t-s)f(s,y_s)\,\mathrm{d}s + \int_0^t T(t-s)\int_0^s K(s,u)g(u)\,\mathrm{d}u\,\mathrm{d}s, \qquad t \in J, \tag{4.1}$$

where

$$g \in S_{F,y} = \big\{g \in L^1(J,E) : g(t) \in F(t,y_t) \text{ for a.e. } t \in J\big\}.$$

Definition 4.1. A function $y\colon (-r,\infty) \to E$ is called a *mild solution of the Cauchy problem* (1.3)–(1.4) if $y(t) + \big(\xi(y_{t_1}, \dots, y_{t_p})\big)(t) = \phi(t)$, $t \in [-r,0]$, the restriction of $y(\cdot)$ to the interval $[0,\infty)$ is continuous and for each $t \ge 0$ the function $AT(t-s)f(s,y_s)$, $s \in [0,t)$, is integrable and the integral equation (4.1) is satisfied.

Now, we are able to state and prove our main theorem.

Theorem 4.2. *Assume that hypotheses* (H1)–(H4), (H6)–(H9) *are satisfied. Then IVP* (1.3)–(1.4) *has at least one mild solution on* $[-r,\infty)$.

Proof. Let $C := C\big([-r,\infty), E\big)$ be the Fréchet space of continuous functions from $[-r,\infty)$ into E endowed with the seminorms

$$\|y\|_{r,m} := \sup\{|y(t)| : t \in [-r,m]\} \qquad \text{for} \quad y \in C.$$

Transform the problem into a fixed point problem. Consider the multivalued map, $\tilde{N}\colon C \to 2^C$ defined by:

$$\tilde{N}(y) := \left\{ h \in C : h(t) = \begin{cases} \phi(t) - (\xi(y_{t_1}, \dots, y_{t_p}))(t) & \text{if } t \in J_0, \\ T(t)[\phi(0) - (\xi(y_{t_1}, \dots, y_{t_p}))(0) - f(0,\phi)] \\ \quad + f(t,y_t) + \int_0^t AT(t-s)f(s,y_s)\,\mathrm{d}s & \text{if } t \in J, \\ \quad + \int_0^t T(t-s)\int_0^s K(s,u)g(u)\,\mathrm{d}u\,\mathrm{d}s \end{cases} \right\}$$

where $g \in S_{F,y}$.

Remark 4.3. It is clear that the fixed points of $\tilde{N}$ are solutions to (1.3) (1.4).

As in Theorem 3.4, we can show (with obvious modifications) that $\widetilde{N}$ is a completely continuous multivalued map, u.s.c. with convex closed values, and therefore a condensing map.

Here we repeat only the proof that the set

$$\Omega := \{y \in C : \ \lambda y \in \widetilde{N}(y) \ \text{for some} \ \lambda > 1\}$$

is bounded.

Let $y \in \Omega$. Then $\lambda y \in \widetilde{N}(y)$ for some $\lambda > 1$. Thus there exists $g \in S_{F,y}$ such that

$$\begin{aligned} y(t) = {} & \lambda^{-1} T(t)\big[\phi(0) - \big(\xi(y_{t_1}, \dots, y_{t_p})\big)(0) - f(0,\phi)\big] + \lambda^{-1} f(t, y_t) \\ & + \lambda^{-1} \int_0^t AT(t-s) f(s, y_s)\, ds + \lambda^{-1} \int_0^t T(t-s) \int_0^s K(s,u) g(u)\, du\, ds, \\ & \qquad t \in J. \end{aligned}$$

This implies by (H1), (H2), (H4), (H7)-(H9) that for each $t \in J_m$ we have

$$\begin{aligned} \|y(t)\|_m \le {} & M_1 [\|\phi\| + Q + c_1 \|\phi\| + c_2] + c_1 \|y_t\| + c_2 \\ & + M_2 \int_0^t (c_1 \|y_s\| + c_2)\, ds + M_1 \bigg\| \int_0^t \int_0^s K(s,u) g(u)\, du\, ds \bigg\| \\ \le {} & M_1 [\|\phi\| + Q + c_1 \|\phi\| + c_2] + c_1 \|y_t\| + c_2 \\ & + M_2 \int_0^t (c_1 \|y_s\| + c_2)\, ds + M_1 \int_0^t \int_0^s |K(s,u)| p(u) \psi(\|y_u\|)\, du\, ds \\ \le {} & M_1 [\|\phi\| + Q + c_1 \|\phi\| + c_2] + c_1 \|y_t\| + c_2 \\ & + M_2 \int_0^t (c_1 \|y_s\| + c_2)\, ds + M_1 m \sup_{t \in J_m} K(t) \int_0^t p(s) \psi(\|y_s|)\, ds. \end{aligned}$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : \ -r \le s \le t\}, \qquad 0 \le t \le m.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in J_m$, by the previous inequality we have for $t \in J_m$

$$\begin{aligned}\mu(t) &\le M_1\left[\|\phi\| + Q + c_1\|\phi\| + c_2\right] + c_1\mu(t) + c_2\\ &\quad + M_2c_1\int_0^{t^*}\mu(s)\,\mathrm{d}s + M_2c_2m + M_1m\sup_{t\in J_m}K(t)\int_0^{t^*}p(s)\psi(\mu(s))\,\mathrm{d}s\\ &\le M_1\left[\|\phi\| + Q + c_1\|\phi\| + c_2\right] + c_1\mu(t) + c_2\\ &\quad + M_2c_1\int_0^{t}\mu(s)\,\mathrm{d}s + M_2c_2m + M_1m\sup_{t\in J_m}K(t)\int_0^{t}p(s)\psi(\mu(s))\,\mathrm{d}s\,.\end{aligned}$$

Thus

$$\begin{aligned}\mu(t) &\le \frac{1}{1-c_1}\Big\{M_1\left[(1+c_1)\|\phi\| + Q + c_2\right] + c_2 + M_2c_2m\\ &\quad + M_2c_1\int_0^{t}\mu(s)\,\mathrm{d}s + M_1b\sup_{t\in J_m}K(t)\int_0^{t}p(s)\psi(\mu(s))\,\mathrm{d}s\Big\}, \qquad t\in J_m\,.\end{aligned}$$

If $t^* \in J_0$, then $\mu(t) \le \|\phi\| + Q$ and the previous inequality holds since $M_1 \ge 1$.

Let us take the right-hand side of the above inequality as $v(t)$. Then we have

$$c = v(0) = \frac{1}{1-c_1}\Big\{M_1\left[(1+c_1)\|\phi\| + Q + c_2\right] + c_2 + M_2c_2m\Big\},$$
$$\mu(t) \le v(t)\,, \qquad t\in J_m\,,$$

and

$$v'(t) = \frac{1}{1-c_1}M_2c_1\mu(t) + \frac{1}{1-c_1}M_1b\sup_{t\in J_m}K(t)p(t)\psi(\mu(t))\,, \qquad t\in J_m\,.$$

Using the nondecreasing character of ψ we get

$$\begin{aligned}v'(t) &\le \frac{1}{1-c_1}M_2c_1v(t) + \frac{1}{1-c_1}M_1m\sup_{t\in J_m}K(t)p(t)\psi(v(t))\\ &\le \overline{m}(t)\left[v(t) + \psi(v(t))\right], \qquad t\in J_m\,,\end{aligned}$$

where

$$\overline{m}(t) = \max\left\{\frac{1}{1-c_1}M_2c_1, \frac{1}{1-c_1}M_1m\sup_{t\in J_m}K(t)\right\}.$$

Now, the hypotheses on ψ imply (see [10]) that

$$\int_{v(0)}^{\infty}\frac{\mathrm{d}u}{u+\psi(u)} = \infty\,,$$

thus there exists a constant $\overline{L}_m$ such that $v(t) \leq \overline{L}_m$, $t \in J_m$, and hence $\mu(t) \leq \overline{L}_m$, $t \in J_m$. Since for every $t \in J_m$, $\|y_t\| \leq \mu(t)$, we have

$$\|y\|_{r,m} := \sup\{|y(t)| : -r \leq t \leq m\} \leq \overline{L}_m,$$

where $\overline{L}_m$ depends on m and on the functions p and ψ. This shows that Ω is bounded.

Set $X := C$. As a consequence of Lemma 2,1 we deduce that $\widetilde{N}$ has a fixed point, which is a solution of (1.3) – (1.4). □

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Received May 4, 2000

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