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MODULAR MEDIAN ALGEBRAS

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J. Hashimoto [5] shoved that bounded modular lattices can be characterized by the ternary operation $(abc)=(a\wedge(b\vee c))\vee(b\wedge c)$ and exhibited a system of identities characterizing such a ternary algebra. Another system of axioms was given by M. Kolibiar and T. Marcisová [9]. (In fact it is not clear whether the systems of axioms in [5] and [9] are equivalent.) We shall adopt for algebras in [9] the term modular median algebras (m.m. algebras). The modular median algebras corresponding to distributive lattices are characterized by the identity $(xyz)=(yxz)$ [9]. These algebras were studied by several authors (e.g. G. Birkhoff and S. A. Kiss [1], M. Sholander [10], H. J. Bandelt and J. Hedlíková [2]; in the last paper a large list of references can be found).

In the present paper some classes of m.m. algebras are studied. It is shown (Theorem 3.1) that the m.m. algebra $\mathscr{A}$ satisfies the identity

$$\text{(U)} \qquad ((xyz)xt)=(xy(zxt))$$

if and only if neither a special algebra nor its specific homomorphic image is a subalgebra of $\mathscr{A}$. An important variety of m.m. algebras is the *variety* **T** given by the identity

$$\text{(T)} \qquad ((xyz)uv)=((xuv)(yuv)(zuv)).$$

This variety is closely related to the dual discriminator variety studied by E. Fried and A. F. Pixley [3]. Namely, if $\mathscr{A}$ is an algebra of such a variety and q is the corresponding ternary polynomial, then $(A\,;\,\bar{q})$ is an m.m. algebra of the variety **T** where $\bar{q}(x, y, z) = q(z, y, x)$. We shall show that an algebra of a dual discriminator variety has all congruence relations permutable if and only if each of its congruence relations is regular. This is equivalent to the condition that the algebra $(A\,;\,\bar{q})$ is relatively complemented (see section 2). The author wishes to thank J. Hedlíková for her remarks which contributed to a better formulation of some parts of the section 3 of the present paper.

1.

For terminology in this paper we shall generally follow G. Grätzer [4].

Definition 1.1. *We shall call a set A with one ternary operation (xyz) a modular median algebra (an m.m. algebra and notation $\mathscr{A}=(A\,;(\,))$) if the following identities are satisfied in A:*

(1) $(abb)\quad b$,

(2) $((adc)bc)-(ac(bcd))$.

An m.m. algebra $\mathscr{A}$ is called distributive if $(abc)=(bac)$ is fulfilled in A.

Lemma 1.1 [9]. *The following identities and implications are valid in each m.m. algebra*

(3) $(aba)\quad a$.

(4) $(abc)\quad (acb)$.

(5) $(aab)=a$.

(6) $((abc)bc)=(acb)$.

(7) $((abc)ac)\quad (ac(abc))\quad (abc)$.

(8) $(ab(cab))-(abc)$.

(9) $(abc)-c$ implies $(bac)=c-(cab)$.

(10) $(bac)=(cab)$ *implies* $(abc)=(bac)$.

(11) $(a(abc)(dbc))=(abc)$.

Remark 1.1. In virtue of (1), (3) and (5) the algebra in Definition 1.1 is a median algebra in the sense of [4, Ex. 4, p. 356] and by [8] *m.m. algebras have distributive congruence lattices.*

Remark 1.2. The following results are given in [9]: Given a modular lattice $\mathscr{L}=(L\,;\wedge,\vee)$, the operation $(xyz)=(x\wedge(y\vee z))\vee(y\wedge z)$ satisfies (1) and (2). The algebra $(L\,;(\,))$ will be reffered to as the *m.m. algebra corresponding to (or derived from) the lattice $\mathscr{L}$.* Conversely, any m.m. algebra $\mathscr{A}$ with the elements o, u satisfying $(oxu)=x$ for each $x\in A$ gives a modular lattice in which $x\wedge y=(xoy)$ and $x\vee y-(xuy)$.

In an m.m. algebra $(A\,;(\,))$ we say that *x is between a and b* and write axb if $x=(axb)$. From (4) and (9) it follows that axb implies bxa. *The interval* (a,b) is defined as the set of all elements between a and b, i.e. $(a,b)=\{x\in A\colon axb\}$. We shall write $abcd$ instead of the next four betweenness relations: abc, acd, abd, bcd. Evidently $abcd$ implies $dcba$. We say that the elements a, b, c, d form a *cyclic quadruple* in an m.m. algebra when they satisfy abc, bcd, cda, dab.

Lemma 1.2. *The following identities and implications are valid in each m.m. algebra.*

(12) $((abc)(bac)(cab))=(abc)$.

(13) $((acd)cb)=(ac(dcb))\quad (ac(bcd))-((acb)cd)$.

(14) $(ab\ cda))=(a(bda)(cda))$.

(15) $(ab(cda))=(ac(bda))$.

(16) $(ab\ acd))=(a(bac)(acd))\quad (a(bad)(acd))$.

(17) *abc and acd imply bcd and abd* (i.e. $abcd$).

(18) $(axb)=x$ *and* $(ayb)=y$ *imply* $(xay)=(yax)=(axy)$.

(19) $((abc)de)=(((ade)(bde)(cde))de)$ *implies*
$((abc)de)=((ade)(bde)(cde))$.

Proof. (12), (13), (14), (15) and (19) follow from the results of J. Hedlíková [7]. For (17) see [6]. For the proof of (16) we shall use (7), (4); (15), (4): $(ab(acd)) = (ab((acd)ca)) = (a(acd)(bca)) = (a(bac)(acd))$. Hence using (4) we get $(ab(acd)) = (ab(adc)) = (a(bad)(adc))$. (18) can be deduced from the proof of Theorem 1 (to be exact from the condition (24)) in [9].

Lemma 1.3. *Let Θ be a congruence relation of an m.m algebra $\mathcal{A}$ and let the elements x, y, z, u of A satisfy xyz and yzu. Then $x\Theta u$ implies $y\Theta z$. In particular, $x=u$ implies $y=z$.*

Proof. $y=(xyz)\Theta(uyz)=(uzy)=z$.

2.

In paper [3] the dual discriminator variety was studied. *The dual discriminator* is the ternary function d from A^3 to A, defined by $d(x, y, z)=x$ if $x=y$ and $d(x, y, z)=z$ if $x\neq y$. A variety $\mathbf{V}$ is a *dual discriminator variety* if $\mathbf{V}$ has a ternary polynomial symbol $q(x, y, z)$ such that the corresponding polynomial $q^{\mathcal{A}}(x, y, z)$ is the dual discriminator on each subdirectly irreducible (SI) algebra $\mathcal{A}$ from $\mathbf{V}$.

Theorem 2.1 [3 ; 3.2 Theorem]. *For a variety $\mathbf{V}$ and a ternary polynomial symbol $q(x, y, z)$ the following are equivalent:*

1) *$\mathbf{V}$ is a dual discriminator variety with $q(x, y, z)$ the dual discriminator on each SI member of $\mathbf{V}$.*
2) *The following are equations of $\mathbf{V}$:*

a) $q(x, z, z)=z,\ q(x, y, x)=x,\ q(x, x, z)=x$,

b) $q(x, y, q(x, y, z))=q(x, y, z)$,

c) $q(z, q(x, y, z), q(x, y, w))=q(x, y, z)$,

d) *for each operation symbol f of $\mathbf{V}$ (where f is k-ary)*

$$q(x, y, f(z_1, \ldots, z_k))=q(x, y, f(q(x, y, z_1), \ldots, q(x, y, z_k))).$$

If $\mathbf{V}$ is an idempotent variety, the equation d) *may be replaced by*

d′) $$q(x, y, f(z_1, \ldots, z_k))=f(q(x, y, z\ ), \ldots, q(x, y, z_k)).$$

Lemma 2.1. *Let $\mathbf{T}$ be the variety of all m.m. algebras satisfying the identity* (T). *Then $\mathbf{T}$ is a dual discriminator variety. An algebra $\mathcal{A}$ from $\mathbf{T}$ is* SI *if and only if for every $x, y, z\in A$*

(t) $$(xyz)=x \text{ if } y\neq z \text{ and } (xyz)=y \text{ if } y=z.$$

The following identities hold in $\mathbf{T}$: $((xyz)uv) = (x(yuv)(zuv)) = ((xuv)y(zuv))$

$= ((xuv)(yuv)(zuv)) = (((xuv)(yuv)(zuv))uv)$. *The variety* **T** *has the congruence extension property* (CEP).

Proof. Taking (xyz) for $q(z, y, x)$ we can show (using (1), (3), (5), (6) and (11)) that a), b), c), d') from Theorem 2.1 are satisfied. According to [3] $q(x, y, z)$ is a dual discriminator on SI members of **T**, hence (t) holds. Clearly all m.m. algebras satisfying (t) are SI. The last statement is true, because each its identity can be verified on each SI algebra from **T**. To show CEP for **T** it suffices to use [3, 3.16 Corollary].

In [3, Theorem 3.1] it is proved that in an algebra $\mathscr{A}$ from a dual discriminator variety $(x, y) \in \Theta(a, b)$ if and only if

i) for each $u \in A$ $(uxy) = ((uab)xy)$).

Lemma 2.2. *In the variety* **T**, $(x, y) \in \Theta(a, b)$ *if and only if*

ii) $(xab)xy) = x$ *and* $((yab)xy) = y$

Proof. Taking $u = x$ and $u = y$ in i) we get ii). Now suppose ii). Then the identities of Lemma 2.1 give

$$(uxy) = (u((xab)xy)((yab)xy)) = ((u(xab)(yab))xy)$$
$$= (((uxy)ab)xy) = ((uxy)(axy)(bxy)) = ((uab)xy)$$

for each element u, hence i) holds

Lemma 2.3. *Let* $\mathscr{A} = (A; (\ ))$ *belong to the variety* **T**. *Let* $a, b \in A$. *Then the interval* (a, b) *is a median subalgebra of* $\mathscr{A}$ *which is distributive, hence* (a, b) *is a distributive lattice with the operations* $x \wedge y = (xay)$, $x \vee y = (xby)$.

Proof.[2]) First we shall show that (a, b) is a subalgebra of $\mathscr{A}$. Let $x, y, z \in (a, b)$, i.e. $(axb) = x$, $(ayb) = y$, $(azb) = z$. By (9) we get $(xab) = x = (bax)$ and analogously for y and z. Using (16), (7) and (4) we get $(ba(bxy)) = (b(abx)(bxy)) = (bx(bxy)) = (bxy)$, hence $(bxy) \in (a, b)$. By Lemma 2.1, $(a(zxy)(bxy)) = ((azb)xy) = (zxy)$, i.e. $a(zxy)(bxy)$ and this together with $a(bxy)b$ gives (by (17)) $a(zxy)b$, hence $(zxy) \in (a, b)$ and (a, b) is a subalgebra of A. Now we shall show that (a, b) is distributive, i.e. $(xyz) = (yxz)$ is valid for each $x, y, z \in (a, b)$. By Remark 1.2, $((a, b); \wedge, \vee)$, where $x \wedge y = (xay)$, $x \vee y = (xby)$ for $x, y \in (a, b)$, is a modular lattice. It is distributive, otherwise it would contain a five-element nondistributive modular sublattice in which the identity (T) is not fulfilled.

Definition 2.1. *An m.m. algebra* $\mathscr{A} = (A; (\ ))$ *is called relatively complemented if for each* $a, b, c \in A$ *satisfying abc* (i.e. $(abc) = b$), *there is an element* $d \in A$ *such*

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[1]) We write (xyz) for $q(z, y, x)$

[2]) Lemma 2.3 is a corollary of Theorem 3.2 because the identity (U) is a consequence of the identity (T). Nevertheless we give an independent proof, because it is much simpler than that of Theorem 3 2.

that the elements a, b, c, d form a cyclic quadruple. Then we call such element d a complement of the element b in the interval (a, c).

Lemma 2.4. *Let $\mathscr{A}=(A; \wedge, \vee)$ be a modular lattice and $\mathscr{A}_1=(A; (\,))$ the corresponding m.m. algebra. Then $(abc)=b$ in $\mathscr{A}_1$ if and only if $(a\wedge b) \vee (b\wedge c) = b = (a\vee b) \wedge (b\vee c)$.*

Proof. Let $(abc)=b$. Then $(a\wedge(b\vee c)) \vee (b\wedge c) = b = (a\vee(b\wedge c)) \wedge (b\vee c)$. It implies $a\vee b = a\vee(b\wedge c)$, $b\vee c = (a\wedge(b\vee c))\vee c = (a\vee c) \wedge (b\vee c)$. Hence $(a\vee b) \wedge (b\vee c) = (a\vee(b\wedge c)) \wedge (a\vee c) \wedge (b\vee c) = (a\vee(b\wedge c)) \wedge (b\vee c)=b$. $b=(a\wedge b) \vee (b\wedge c)$ can be proved dually. Conversely, let $(a\wedge b) \vee (b\wedge c) = b = (a\vee b) \wedge (b\vee c)$. Then $b\vee c = (a\wedge b)\vee c$, $a\wedge(b\vee c) = a\wedge((a\wedge b)\vee c) = (a\wedge b) \vee (a\wedge c)$. Hence $(abc) = (a\wedge(b\vee c)) \vee (b\wedge c) = (a\wedge b) \vee (a\wedge c) \vee (b\wedge c) = (a\wedge c) \vee (a\wedge b) \vee (b\wedge c) = (a\wedge c)\vee b = (a\wedge c) \vee ((a\vee b) \wedge (b\vee c)) = (a\vee b) \wedge (b\vee c)=b$.

The following Lemma 2.5 gives examples of relatively complemented m.m. algebras.

Lemma 2.5. *Let $\mathscr{A}=(A; \wedge, \vee)$ be a modular lattice and $\mathscr{A}_1=(A; (\,))$ the corresponding m.m. algebra. $\mathscr{A}_1$ is relatively complemented if and only if $\mathscr{A}$ is a relatively complemented lattice.*

Proof. Let $\mathscr{A}_1$ be a relatively complemented m.m. algebra and let $a\leqslant b\leqslant c$, a, b, $c\in A$. Then $(abc)=b$ and there exists $d\in A$ such that a, b, c, d form a cyclic quadruple, i.e. bcd, cda, dab. Using Lemma 2.4, from $a\leqslant c$ and adc we get $a\leqslant d\leqslant c$, from bcd, $b\leqslant c$, $d\leqslant c$ we get $b\vee d=c$. $b\wedge d=a$ can be proved similarly. Hence d is a complement of b in the interval (a, c) of the lattice $\mathscr{A}$. Conversely, let the lattice $\mathscr{A}$ be relatively complemented and let abc. Denote by d_1 a complement of the element $a\wedge b$ in the interval $(a\wedge c, a)$ (for Lemma 2.4 gives $a\wedge c\leqslant a\wedge b$) and by d_2 a complement of the element $b\wedge c$ in the interval $(a\wedge c, c)$. We shall prove that $d=d_1\vee d_2$ is a complement of the element b in the interval (a, c) of the m.m. algebra $\mathscr{A}_1$, i.e. adc, bcd, bad. First we shall show adc (using Lemma 2.4): $a\wedge d = a\wedge(d_1\vee d_2) = d_1\vee(a\wedge d_2) = d_1\vee(a\wedge c)=d_1$. Analogously $d\wedge c=d_2$. Then $(a\wedge d) \vee (d\wedge c) = d_1\vee d_2=d$. Similarly the rest can be proved. Now we shall show bcd: $(b\wedge c) \vee (c\wedge d) = (b\wedge c) \vee (c\wedge(d_1\vee d_2)) = (b\wedge c) \vee (d_2\vee(c\wedge d_1)) = (b\wedge c) \vee (d_2\vee(a\wedge c)) = (b\wedge c) \vee d_2=c$. The following relations hold: $(b\vee c)\wedge a = ((a\wedge b) \vee (b\wedge c)\vee c)\wedge a = ((a\wedge b)\vee c)\wedge a = (a\wedge b) \vee (a\wedge c) = a\wedge b$, $(b\vee c)\wedge d_1 = (b\vee c) \wedge (a\wedge d_1) = ((b\vee c)\wedge a)\wedge d_1 = (a\wedge b)\wedge d_1 = a\wedge c$. Hence $(b\vee c) \wedge (c\vee d) = (b\vee c) \wedge (c\vee d_1) = c\vee((b\vee c)\wedge d_1) = c\vee(a\wedge c)=c$. bad can be proved analogously to bcd.

Definition 2.2. *A congruence relation Θ of an algebra $\mathscr{A}$ will be called regular if for each congruence relation Φ of $\mathscr{A}$, $a\in A$ and $[a]\Theta=[a]\Phi$ imply $\Theta=\Phi$.*

Theorem 2.2. *Let $\mathscr{A}$ be an m.m. algebra. Consider the following conditions.*

(i) *$\mathscr{A}$ is relatively complemented.*

(ii) *Each congruence relation of $\mathscr{A}$ is regular.*

(iii) *The congruence relations of $\mathscr{A}$ are pairwise permutable.*

In each m.m. algebra, (i) *implies* (ii) *and* (iii). *In algebras of the variety* **T** *(from Lemma 2.1) the conditions* (i), (ii), (iii) *are equivalent.*

Proof. (i) implies (ii): Let Θ, Φ be congruence relations of $\mathscr{A}$ such that there exists $a \in A$ with $[a]\Theta = [a]\Phi$. With respect to symmetry, it suffices to show $\Theta \leqslant \Phi$. Let $x, y \in A$. Using (4) and (7) we get $(axy) \in (a, x)$, $(axy) \in (a, y)$. Let u, v be relative complements of the element (axy) in the intervals (a, x), (a, y), respectively. $x\Theta y$ implies $x = (axx)\Theta(axy)\Theta(ayy) = y$. With respect to Lemma 1.3, this implies $a\Theta u$, $a\Theta v$. Using our assumption we get $a\Phi u$, $a\Phi v$. By Lemma 1.3, we have $(axy)\Phi x$, $(axy)\Phi y$, hence $x\Phi y$.

(i) implies (iii): Let Θ, Φ be congruence relations of $\mathscr{A}$ and $a\Theta b$, $b\Phi c$. Then $a = (aac)\Theta(abc)$, $(abc)\Phi(acc) - c$. Denote $d - (abc)$. Then we have $a\Theta d$, $d\Phi c$ and $(adc) = d$ (by (7) and (4)). Since $\mathscr{A}$ is relatively complemented, there exists an element $e \in A$ such that a, d, c, e form a cyclic quadruple. With respect to Lemma 1.3 we get $a\Phi e$, $e\Theta c$, hence Θ, Φ are permutable.

(iii) implies (i): Let $\mathscr{A} \in \mathbf{T}$ have permutable congruence relations and let abc. Then $a \equiv b\ (\Theta(a, b))$, $b \equiv c\ (\Theta(b, c))$. Using permutability we get that there exists $d \in A$ such that $a \equiv d\ (\Theta(b, c))$ and $d \equiv c\ (\Theta(a, b))$. With respect to Lemma 2.2, $((abc)ad) = a$, $((dab)dc) = d$, $((cab)dc) = c$. Hence $(bad) = ((abc)ad) = a$ and using (9) we have $(dab) = a$. Similarly $(bcd) = ((cab)cd) = c$, $(adc) = ((dab)dc) = d$. Thus the elements a, b c, d form a cyclic quadruple, q.e.d.

(ii) implies (i): Let $\mathscr{B} \in \mathbf{T}$, $a, b, c \in B$ and abc. According to Lemma 2.3 the interval (a, c) is a distributive median algebra and a distributive lattice in which $x \vee y = (xcy)$. Denote $\Theta = \Theta(b, c)$, $A = \{x \in B\colon x = a(\Theta)\}$. Given $x \in A$ we get, using Lemma 2.2, $((abc)ax) = a$, i.e. $(bax) - a$. Thus

(a) $\qquad\qquad x \in A \quad \text{implies} \quad bax.$

Denote $\Psi = \bigvee\{\Theta(a, t)\colon t \in A\}$. Evidently $\psi \leqslant \Theta$ and A is a class of the congruence relation Ψ. $\Psi = \Theta$ follows from (ii). Thus $b \equiv c(\Psi)$, hence there are sequences $c = e_0, e_1, \dots, e_n = b$ and $t'_1, t'_1, \dots, t'_n \in A$ such that $e_i \equiv e_{i+1}(\Theta(a, t'_i))$, $i = 0, 1, \dots, n-1$. Denote $(bce_i) = f_i$. Then

$$f_i \in (b, c), \quad f_i = (bce_i) = (bce_{i+1}) \quad f_{i+1}(\Theta(a, t'_i)), \quad f_0 = c, \quad f_n = b.$$

According to Lemma 2.2, this implies $((f_i a t'_i) f_i f_{i+1}) = f_i$ for $i = 0, 1, \dots, n-1$. Denote $(f_i a t'_i) = t_i$. Evidently $f_i t_i a$ (by (7)). By (17), abc and $bf_i c$ imply $af_i c$ which together with $at_i f_i$ implies $at_i c$, hence $t_i \in (a, c)$. We get $t_i \in A \cap (a, c)$ because of $t_i \equiv (f_i aa) = a(\Theta)$. It was shown before that .

(b) $$(t_i f_i f_{i+1}) = f_i, \quad i = 0, 1, \dots, n-1.$$

For $i=0$ we have $c = (t_0 c f_1) = t_0 \vee f_1$. Proceeding by induction suppose $c = t_{i-1} \vee t_{i-2} \vee \dots \vee t_0 \vee f_i = (s_i c f)$, where $s_i = t_{i-1} \vee t_{i-2} \vee \dots \vee t_0$. Using the identity (T) we have

$$c = (f_i c s_i) = ((t_i f_i f_{i+1}) c s_i) = ((t_i c s_i)(f_i c s_i)(f_{i+1} c s_i)) =$$
$$= ((t_i c s_i) c (f_{i+1} c s_i)) = t_i \vee s_i \vee f_{i+1} = t_i \vee t_{i-1} \vee \dots \vee t_0 \vee f_{i+1}.$$

Hence for all $i = 0, 1, \dots, n-1$, $c = t_i \vee t_{i-1} \vee \dots \vee t_0 \vee f_{i+1}$. In particular $c = t_{n-1} \vee t_{n-2} \vee \dots \vee t_0 \vee f_n = t \vee b = (tcb)$, where $t = t_{n-1} \vee t_{n-2} \vee \dots \vee t_0 \in (a, c)$. Then $t_1 \vee t_0 = (t_1 c t_0) \equiv (aca) = a(\Theta)$ and using induction we get $t \equiv a(\Theta)$. Hence $t \in A$. According to (a), bat. Because of $t \in (a, c)$, we get atc. Hence a, b, c, t form a cyclic quadruple, q.e.d.

Corollary 2.1 [2, Theorem 6.1]. *The conditions* (i), (ii), (iii) *from Theorem 2.2 are equivalent in algebras of the variety of all distributive median algebras.*

It is easy to see that a modular lattice has the same congruence relations as the corresponding m.m. algebra. Hence we have:

Corollary 2.2. *The conditions* (i), (ii), (iii) *from Theorem 2.2 are equivalent in any distributive lattice.*

Corollary 2.3. *Let $\mathscr{A}$ be an algebra of a dual discriminator variety with the dual discriminator $q(x, y, z)$. The conditions* (ii), (iii) *(from Theorem 2.2) and the following condition are equivalent:*

(iv) *The m.m. algebra $(A; \bar{q})$, where $\bar{q}(x, y, z) = q(z, y, x)$, is relatively complemented.*

Proof. Indeed, the equations a), b), c), d) in Theorem 2.1 imply that d) holds in the variety $\mathbf{V}$ under consideration also if f stands for a polynomial symbol (it suffices to use induction on the number of operation symbols occuring in f). Hence $\bar{q}(a, b, b) = b$ (i.e. (1)) and

$$\bar{q}(\bar{q}(a, b, c), d, e) = \bar{q}(\bar{q}(\bar{q}(a, d, e), \bar{q}(b, d, e), \bar{q}(c, d, e)), d, e).$$

According to (19) (Lemma 1.2) $\bar{q}$ satisfies the identity (T). Since q is a dual discriminator one can easily check that $q(x, y, q(x, z, u)) = q(q(z, x, y), x, u)$ holds in each SI member of $\mathbf{V}$. Hence this identity holds in $\mathbf{V}$. It follows that, given an algebra $\mathscr{A} = (A; (f_i))$ of $\mathbf{V}$, the algebra $(A; \bar{q})$ satisfies the identities (1), (2) and (T), hence it belongs to $\mathbf{T}$. Obviously $(A; \bar{q})$ and $(A; q)$ have the same congruence relations. According to [3, Theorem 3.11] the same holds for the algebras $(A; q)$ and $\mathscr{A} = (A; (f_i))$.

Theorem 2.3. *Each m.m. algebra $\mathscr{A}$ from the variety $\mathbf{T}$ (see Lemma 2.1) can be embedded into an m.m. algebra $\mathscr{L} = (L; (\,))$ derived from a modular lattice $\mathscr{L}_1 = (L; \wedge, \vee)$ (see Remark 1.2).*

Proof. Obviously it suffices to show that SI members of **T** (which are given in Lemma 2.1) can be embedded in modular lattices. The two-element m.m. algebra can be bijectively embedded in the two-element lattice. In the other case a SI m.m. algebra $\mathscr{A}$ can be embedded into a modular lattice consisting of the elements 0, 1 and the set of atoms of cardinality card A. Each element of A will be mapped into an atom. From this the assertion follows.

Corollary 2.4 [10]. *Each distributive median algebra can be embedded into a distributive median algebra derived from a distributive lattice.*

3.

Throughout this section we shall call the m.m. algebra $(\{a, b, c\}; ())$, a, b, c different, in which $a=(abc)$, $b=(bac)$, $c=(cab)$ hold, a *triangle algebra* and $\mathscr{H}$ will denote the following six-element m.m. algebra (observed by J. Hedlíková): $\mathscr{H}=(\{a, b, c, x, y, z\}; ())$ where $\{a, b, c\}$ and $\{x, y, z\}$ form triangle subalgebras and the relations $axyb$, $byzc$, $czxa$ hold. From this it can be easily derived that

$$(ayz)=(ayc)=(azb)=x, \quad (yza)=(yca)=y, \quad (zya)=(zba)=z$$

and the relations symmetrical to these ones. Further by $\mathscr{W}$ we will denote the four-element m.m. algebra $(\{a, b, c, 0\}; ())$, $a, b, c, 0$ different, derived from the five-element modular nondistributive lattice (diamond) $(\{a, b, c, 0, 1\}; \wedge, \vee)$ (see Remark 1.2). It can be easily checked that the m.m. algebra $\mathscr{W}$ and the triangle algebra are SI m.m. algebras, $\mathscr{W}$ is a homomorphic image of $\mathscr{H}$ and $\mathscr{H}$ is subdirectly reducible.

Lemma 3.1. *In any m.m. algebra the following implication holds:* $((abc)ad \neq (ab(cad))$ *implies* $(abc) \neq (bac)$.

Proof. Let $(abc)=(bac)$. Then the identity (13) gives $((abc)ad) = ((bac)ad) = (ba(cad))$. Using (10) and (13) we get $((abc)ad) = ((cab)ad) = (ca(bad)) = ((cad)ab)$. Hence $(ba(cad)) = ((cad)ab) = ((abc)ad)$ and (10) gives $(ab(cad)) = (ba(cad)) = ((abc)ad)$, which proves the assertion.

Lemma 3.2. *Let x, y, z, v be elements of an m.m. algebra satisfying the following conditions.*

a) *x, y, z form a triangle subalgebra (hence x, y, z are different).*

b) *xvy, xvz and $x \neq v$ hold.*

Then the subalgebra generated by $\{x, y, z, v\}$ is either isomorphic to the algebra $\mathscr{W}$ or it contains a subalgebra isomorphic to the algebra $\mathscr{H}$.

Proof. By (13) $v=(xyv)=((xyz)yv)=((xyv)yz)=(vyz)$. Now we have two possibilities:

i) $(yvz)=(vyz)$.

ii) $(yvz) \neq (vyz)$.

In case i), $(yvz)=(vyz)=v$, hence yvz holds. Further, $y\neq v$ and $z\neq v$ hold (for $y=v$ implies $x=(xyz)=(xvz)=v$ and $z=v$ implies $x=(xzy)=(xvy)=v$). Hence the elements x, y, z, v form a subalgebra isomorphic to $\mathcal{W}$. In case ii), according to (10), the elements (yzv), (zvy), $(vyz)=v$ form a triangle subalgebra (see (12)). From (7) we get $(yv(yzv))=(yzv)$, hence $y(yzv)v$ holds. With respect to (17), it imples (together with yvx) that $y(yzv)vx$ hold. Because of the symmetry, $xv(zyv)z$. Using (7) and (11) we get $(y(yzv)(zyv)) = (y(yzv)((zyv)zv)) = (yzv)$, hence $y(yzv)(zyv)$ holds. With respect to (7) and (9), $(zy(zyv)) = (zyv)$ and $y(zyv)z$. Because of (17) $y(yzv)(zyv)z$. The elements x, y, z, v, (yzv), (zvy) are different: According to (13), $y=(yzv)$ implies $x=(xzy) = (xz(yzv)) = ((xzv)zy) = (vzy)=v$, which is a contradiction. Analogously $x=(yzv)$ implies (by (7)) $v=(yvx) = (yv(yzv)) = (yzv)=x$. Hence $y\neq(yzv)\neq x$. $y=v$ implies $x=v$ as in part i). Hence the elements x, y, v, (yzv) are different. Symmetrically x, z, v, (zyv) are different. The proof of $y\neq(zvy)$ and $z\neq(yzv)$ is similar. Hence the above six elements form the algebra $\mathcal{H}$.

Lemma 3.3. *Let a, b, c, d be elements of an m.m. algebra. Denote $i=(abc)$, $j=(bac)$, $k=(cab)$, $m=((abc)b(cad))$, $t=((abc)ad)$ and $r=(ab(cad))$. The following identities hold.*

(20) $r=(rad)$.
(21) $t=(ia(cad))$.
(22) $t=(ai(cad))=((cad)ai)$.
(23) $r=(tb(cad))$.
(24) $t=(itr)$.
(25) $r=(trm)$.
(26) $m=(rbc)$.
(27) $m=(ic(bad))$.
(28) $m=(ij(cad))$.
(29) $m=(ik(bad))$.
(30) $m=(imj)$.
(31) $m=(imk)$.
(32) $i=(tbc)$.

Proof. We shall use (4) freely in proofs.

(20): According to (14), (7), (14), (13) and (7)

$$r=(a(bad)(cad))=((a(bad)(cad))a(bad))=((ab(cad))a(bad))= \\ =(((ab(cad))ab)ad)=((ab(cad))ad)=(rad).$$

(21): By (7) and (13)

$$t=(((abc)ac)ad)=((abc)a(cad))=(ia(cad)).$$

(22): By (7) $(aci)=i$ and $(ca(cad))=(cad)$. According to (9) $(ac(cad))=(cad)$.

Using (18) and (21) $((cad)ai) = (ai(cad)) = t$.

(23): $(tb(cad)) =$ (use (22))
$= ((a(cad)i)(cad)b) =$ (use (13))
$= (a(cad)(b(cad)(abc))) =$ (by (15))
$= (a(cad)(ba((cad)bc))) =$ (use (15))
$= (ab((cad)a((cad)bc))) =$ (by (16))
$= (ab((cad)(a(cad)c)((cad)bc))) =$ (by (7), (9) and (5))
$= (ab((cad)(cad)((cad)bc))) = (ab(cad)) = r$.

(24): By (20) and (11) $(itr) = ((abc)((abc)ad)(rad)) = t$.

(25): According to (23), (11) and (23) $(trm) =$
$= (t(tb(cad))((abc)b(cad))) = (tb(cad)) = r$.

(26): Using (13) $m = (ab(cb(cad))) = ((ab(cad))bc) = (rbc)$.

(27): According to (26), (15) and (13) $m = (rbc) =$
$= ((ac(bad))cb) = ((acb)c(bad)) = (ic(bad))$.

(28): Using (26) we get $m = (rbc) =$ (by (13) and (7))
$= (ab((cad)bc)) = (ab(((cad)ca)cb))$ (by (13) and (14))
$(ab(((cad)cb)ca)) = (a(bca)(((cad)cb)ca)) =$ (use (13) twice)
$= (a(bca)((cad)c(bca))) = ((a(bca)c)(bca)(cad)) =$ (by (8))
$= ((abc)(bca)(cad)) = (ij(cad))$.

(29): According to (28) $((abc)b(cad)) = m =$
$= ((abc)(bca)(cad))$, hence $((acb)c(bad)) = ((acb)(cba)(bad))$.
This together with (27) gives $m = (ic(bad)) = (ik(bad))$.

(30): With respect to (28), (7) and (28) $(imj) =$
$= (ij(ij(cad))) = (ij(cad)) = m$.

(31): The proof is analogous to that of (30) (use (29)).

(32): By (7) $(abi) = i$. It implies $(bai) = i$ (by (9)). It gives together with (13) that $(bat) = (ba(iad)) = ((bai)ad) = (iad) = t$. Hence $(abt) = t$ holds by (9). Using (13) $(tbc) = ((abt)bc) = ((abc)bt) = (ib(iad))$ and by (16) $(ib(iad)) = (i(bia)(iad)) = (ii(iad)) = i$. Hence $(tbc) = i$ holds.

Theorem 3.1. *Let* **U** *be the variety of all m.m. algebras satisfying the identity* (U). *The m.m. algebra* $\mathscr{A}$ *does not belong to the variety* **U** *if and only if* $\mathscr{A}$ *contains a subalgebra isomorphic to one of the m.m. algebras* $\mathscr{W}$ *and* $\mathscr{H}$.

Proof. We shall use the notation from Lemma 3.3. Let $\mathscr{A}$ not belong to **U** and let a, b, c, d be such elements of A that $t = ((abc)ad) \neq (ab(cad)) = r$. With respect to Lemma 3.1, (10) and (12), the elements i, j, k form a triangle subalgebra of $\mathscr{A}$. Applying (25), (24) and Lemma 1.3 to the elements i, t, r, m we get $i \neq m$. Because of (31) and (30), the assumptions of Lemma 3.2 for the elements i, j, k, m are fulfilled. Hence the subalgebra generated by $\{i, j, k, m\}$ has the desired property. As to the converse assertion, it suffices to check that the algebras $\mathscr{W}$ and $\mathscr{H}$ do not satisfy the identity (U).

Theorem 3.2. *Let* U *be the variety from Theorem 3.1,* $\mathscr{A} \in \mathbf{U}$, $a, b \in A$. *Then the interval* (a, b) *is a subalgebra of* $\mathscr{A}$ *which is distributive, hence* (a, b) *is a distributive lattice with the operations* $x \wedge y = (xay)$, $x \vee y = (xby)$.

For the proof of Theorem 3.2 the following Lemmas are useful:

Lemma 3.4. *In any m.m. algebra* $\mathscr{A}$, $a, b \in A$ *the following implications hold.* $x, y, z \in (a, b)$ *implies* $(ax(xyz)) = (xa(xyz)) = ((xyz)xa)$ *and* $(by(xyz)) = (yb(xyz)) = ((xyz)yb)$.

Proof. By (7), (9), (13), (18), (13), (7) and (9) we get $((xyz)ax) = ((yx(xyz))ax) = ((yxa)x(xyz)) = ((axy)x(xyz)) = (ax(yx(xyz))) = (ax(xyz))$, hence by (10) $(ax(xyz)) = (xa(xyz))$ holds. Using (13), (18) and (13) we get $(by(xyz)) = ((byx)yz) = ((xyb)yz) = ((xyz)yb)$, which according to (10) gives $(by(xyz)) = (yb(xyz))$.

Lemma 3.5. *In any m.m. algebra* $\mathscr{A}$, $a, b \in A$, $x, y, z \in (a, b)$ *imply* $((xyz)ba) \in (\,(ax(xyz)), (xyz)\,) \cap (\,(xyz), (by(xyz))\,)$.

Proof. Using Lemma 3.4, (13) and (11) we have

$((xyz)((xyz)ba)(by(xyz))) = ((xyz)((xyz)((xyz)ba)(yb(xyz))) =$
$= ((xyz)((xyz)ba)((yba)b(xyz))) =$
$= ((xyz)((xyz)ba)((xyz)ba)((yb(xyz))ba)) = ((xyz)ba)$

thus $((xyz)ba) \in (\,(xyz), (by(xyz))\,)$. Using (15), (13) twice and (11) we get

$((xyz)((xyz)ba)(ax(xyz))) =$
$= ((xyz)((xyz)ba)(a(xab)(xyz))) = ((xyz)((xyz)ba)(a((xyz)ab)x)) =$
$= ((xyz)((xyz)ba)((xa(xyz))ba)) = ((xyz)ba)$, hence
$((xyz)ba) \in (\,(ax(xyz)), (xyz)\,)$.

Lemma 3.6. *In any m.m. algebra* $\mathscr{A}$, $a, b \in A$, $x, y, z \in (a, b)$ *imply* $(xyz) \in (\,(ax(xyz)), (by(xyz))\,)$ *and* $(xyz) \in (\,(ay(xyz)), (bx(xyz))\,)$.

Proof. Using Lemma 3.4, (13) twice, (7) and (5) we get

$((ax(xyz))(xyz)(by(xyz))) = ((a(xyz)x)(xyz)(y(xyz)b)) =$
$= (a(xyz)(x(xyz)(y(xyz)b))) = (a(xyz)((x(xyz)y)(xyz)b)) =$
$= (a(xyz)((xyz)(xyz)b)) = (a(xyz)(xyz)) = (xyz)$,

hence $(xyz) \in (\,(ax(xyz)), (by(xyz))\,)$. The second assertion follows by symmetry.

Lemma 3.7. *Let* $\mathscr{A}$ *be an m.m. algebra and let* $(uvt) = v$ *for* $u, v, t \in A$. *Then* $l \in (u, v) \cap (v, t)$ *implies* $l = v$.

Proof. According to (9) $(ulv) = l = (vlu) = (luv)$ and $(vlt) = l = (tlv) = (lvt)$. ulv together with uvt gives lvt by (17), hence $l = (lvt) = v$.

Lemma 3.8. *In any m.m. algebra* $\mathscr{A}$, $a, b \in A$, $x, y, z \in (a, b)$ *imply* $(xyz) = ((xyz)ab)$.

Proof. It is a corollary of Lemmas 3.5, 3.6 and 3.7.

Proof of Theorem 3.2. Let $\mathscr{A} \in \mathbf{U}$, $c, d, e, f \in A$. Using (26), (U) and (32) we have

$((cde)d(ecf)) = ((cd(ecf))de) = (((cde)cf)de) = (cde)$, hence

(33) $(cde) = ((cde)d(ecf))$

holds in each $\mathscr{A} \in \mathbf{U}$. We have to prove that (a, b) is a subalgebra of $\mathscr{A}$, i.e.

$((ab(xyz)) = (ba(xyz)) = ((xyz)ab) = (xyz)$

for $x, y, z \in (a, b)$. Using (33), the assumptions, (13), Lemma 3.4 and (13) twice we get

$(a(xyz)b) - ((a(xyz)b)(xyz)(bax)) - ((a(xyz)b)(xyz)x) - ((a(xyz)x)(xyz)b) =$
$= ((x(xyz)a)(xyz)b) - (x(xyz)(a(xyz)b)) = (x(xyz)(b(xyz)a))$.

Symmetrically $(b(xyz)a) = (x(xyz)(b(xyz)a))$. Hence $(ab(xyz)) = (ba(xyz))$, which together with (10) and Lemma 3.8 gives $(ab(xyz)) = ((xyz)ba) - (xyz)$, thus $(xyz) \in (a, b)$, q.e.d. Further assertions of Theorem 3.2 can be proved analogously as the corresponding assertions in Lemma 2.3.

Added in proof. After this paper was elaborated the author learned of the paper by J. R. Isbell [11], where various properties of the variety **T** were established. (In [11] the members of **T** are called isotropic media.)

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МОДУЛЯРНЫЕ МЕДИАННЫЕ АЛГЕБРЫ

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Резюме

Модулярная медианная алгебра (м.м. алгебра) есть множество с одной тернарной операцией (abc), удовлетворяющей тождествам $(abb)=b$ и $((adc)bc) = (ac(bcd))$. Всякая модулярная структура с операцией $(xyz) = (x\wedge(y\vee z)) \vee (y\wedge z)$ является м.м. алгеброй и наоборот, всякая ограниченная модулярная структура может быть охарактеризована как м.м. алгебра с двумя специальными элементами. Рассматриваются многообразия **T** и **U** м.м. алгебр, определенные соответственно тождествами $((xyz)uv) = ((xuv)(yuv)(zuv))$, $((xyz)xt) = (xy(zxt))$. **T** тесно связано с многообразиями с дуальным дискриминатором. Для алгебр, принадлежащих **T**, перестановочность и регулярность конгруэнций эквивалентны. Алгебра $\mathscr{A}$ принадлежит **U** тогда и только тогда, когда не содержит подалгебру, изоморфную одной из двух указанных алгебр.