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# EQUIVALENCES IN PRESHEAVES, FUNCTIONAL SEPARATION OF THEIR INDUCTIVE LIMITS AND REPRESENTATION BY SECTIONS

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## Introduction

In [1], [2], [3] we have dealt with the question of when the topology $t$ of the inductive limit $(I, t)$ of a presheaf $\mathcal{S} = \{(X_\alpha, t_\alpha)\,|\varrho_{\alpha\beta}|\langle A \leqslant \rangle\}$ from the category of topological spaces is Hausdorff. We have also studied there a slightly more general question of when there is a topology $\tilde{t}$ in $I$, such that $(I, \tilde{t})$ is functionally separated, meaning that for any different $x, y$ from $I$ there is a $t$-continuous function separating them. Several theorems have been found answering the question in an even more general setting. But these suffer from two drawbacks. The one is that we need $\leqslant$ to be a well order, the other is that we need the $\varrho_{\alpha\beta}$'s to be 1—1. These drawbacks have been taken over by [4], where we try to solve the question of the representation of a presheaf by sections in its covering space by means of the theorems of [1]—[3], and they thus set some unwanted boundaries to the usefulness of the theory.

In this paper we try to get rid of these shortcomings. We seek some ways of extending the validity of the theorems of [1]—[3] to the case when $\leqslant$ is not a well order on the one hand, and when the $\varrho_{\alpha\beta}$'s are not 1—1 on the other. A separation theorem (Th. 4) is found here, in which neither the $\varrho_{\alpha\beta}$'s need be 1—1 nor $\leqslant$ needs be a well order. The method used in [1] fails to bring such an outcome, therefore the method of equivalences which leads to Th. 4 brings us to the solution of the representation questions in some cases when the former ones fail to do so. From Th. 4 we get a theorem on the representation of sheaves by sections (Th. 5) which could not be obtained by means of [1]—[3] and could not thus occur in [4], and thus enlargen the set of the representation theorems from [4] and help so our knowledge of the problems to become more complete.

Remark 1. Let us have a presheaf $\mathcal{S} = \{(X_\alpha, t_\alpha)|\varrho_{\alpha\beta}|\langle A \leqslant \rangle\}$ from the category of topological spaces meaning that the $\varrho_{\alpha\beta}$'s are continuous. In every $X_\alpha$ we shall define the relation "$xR_\alpha y$ iff there is $\beta \geqslant \alpha$ with $\varrho_{\alpha\beta}(x) = \varrho_{\alpha\beta}(y)$" – as $\langle A \leqslant \rangle$ is right directed, it is an equivalence. Let $(Y_\alpha, v_\alpha)$ be the quotient space of

$(X_\alpha, t_\alpha)$ by $R_\alpha$, let $p_\alpha:(X_\alpha, t_\alpha)\to(Y_\alpha, v_\alpha)$ be the natural maps. If $\alpha, \beta \in A$, $\alpha \leqslant \beta$, then clearly $\varrho_{\alpha\beta}: X_\alpha \to X_\beta$ agrees with $R_\alpha$, $R_\beta$; there thus is $\varrho_{\alpha\beta}: (Y_\alpha, v_\alpha) \to (Y_\beta, v_\beta)$ — the quotient of $\varrho_{\alpha\beta}$. Clearly it is continuous and $\varrho_{\alpha\gamma} = \varrho_{\beta\gamma}\varrho_{\alpha\beta}$ for $a \leqslant \beta \leqslant \gamma$, hence $\mathcal{S} = \{(Y_\alpha, v_\alpha)\ |\varrho_{\alpha\beta}|\ \langle A \leqslant\rangle\}$ is a presheaf from the category of topological spaces. It shall be called quotient presheaf of $\mathcal{S}$. Clearly the following diagram commutes for any $\alpha, \beta \in A$, $a \leqslant \beta$:

$$\begin{array}{ccc} (X_\alpha, t_\alpha) & \longrightarrow & (X_\beta, t_\beta) \\ \downarrow{\scriptstyle p_\alpha} & & \downarrow{\scriptstyle p_\beta} \\ (Y_\alpha, v_\alpha) & \longrightarrow & (Y_\beta, v_\beta) \end{array}$$

**Lemma 2.** *Let $\mathcal{S} = \{(X_\alpha, t_\alpha)\ |\varrho_{\alpha\beta}|\ \langle A \leqslant\rangle\}$ be a presheaf from the category of topological spaces, let $\tilde{\mathcal{S}} = \{(Y_\alpha, v_\alpha)\ |\varrho_{\alpha\beta}|\ \langle A \leqslant\rangle\}$ be its quotient presheaf. Then*

(a): *$\varrho_{\alpha\beta}$ is 1—1 for any $\alpha, \beta \in A$, $a \leqslant \beta$,*

(b): *If $\varinjlim \mathcal{S} = \langle (I, t)\ |\{\xi_\alpha | a \in A\}\rangle$, $\varinjlim \tilde{\mathcal{S}} = \langle (J, v) | \{\eta_\alpha | \alpha \in A\}\rangle$ then $(I, t)$ is homeomorphic to $(J, v)$.*

Proof. It is streightforward and, therefore, left to the reader.

Remark 3. Let $\mathcal{S}$ be as in Lemma 2.

(a): For the relation $R'_\alpha$ on $X_\alpha$, defined as "$xR'_\alpha y$ if $\xi_\alpha(x) = \xi_\alpha(y)$", we have $R'_\alpha = R_\alpha$.

(b): If $\alpha \in A$, then for every $\beta \geqslant \alpha$ an equivalence $R_{\alpha\beta}$ can be defined as "$xR_{\alpha\beta}y$ if $\varrho_{\alpha\beta}(x) = \varrho_{\alpha\beta}(y)$". If $M \subset X_\alpha$, then the $R_\alpha$ — saturation $\mathcal{S}_\alpha(M)$ of $M$ is related to the $R_{\alpha\beta}$ — saturations $\mathcal{S}_{\alpha\beta}(M)$ of $M$ as follows: $\mathcal{S}_\alpha(M) = \cup\{\mathcal{S}_{\alpha\beta}M | \beta \geqslant \alpha\}$. Thus $M$ is $R_\alpha$ — saturated iff it is such for every $R_{\alpha\beta}$ with $\beta \geqslant \alpha$.

(c): If $\alpha \leqslant \beta$, $M \subset X_\alpha$, then $\mathcal{S}_\alpha(M) = \xi_\alpha^{-1}\xi_\alpha(M)$, $\mathcal{S}_{\alpha\beta}(M) = \varrho_{\alpha\beta}^{-1}\varrho_{\alpha\beta}(M)$. $\mathcal{S}_{\alpha\beta}(M) \subset \mathcal{S}_{\alpha\gamma}(M) \subset \mathcal{S}_\alpha(M)$ if $\beta \leqslant \gamma$.

(d) If $M, N \subset X_\alpha$, then $\mathcal{S}_\alpha(M) \cap \mathcal{S}_\alpha(N) \neq \emptyset$ iff there is $\beta \geqslant \alpha$ with $\mathcal{S}_{\alpha\beta}(M) \cap \mathcal{S}_{\alpha\beta}(N) \neq \emptyset$.

(e): (i) If all the $\varrho_{\alpha\beta}$'s are open maps, then $\mathcal{S}_\alpha(M)$ is open for any $\alpha \in A$ and any open $M \subset X_\alpha$.

(ii) If moreover $\varrho_{\alpha\beta}$ maps any $t_\alpha$ — open set onto a $t_\beta$ — open one for each $\alpha, \beta \in A$, $\alpha \leqslant \beta$, then $\xi_\alpha(M)$ is $t$ — open for any $\alpha \in A$ and any $t_\alpha$ — open set $M$.

Proof. (a) follows directly from the construction of $\varinjlim \to \mathcal{S}$, (b) from the way of definition of $R_\alpha$ and $R_{\alpha\beta}$, (c) is easy to check. (d): If $x \in \mathcal{S}_\alpha(M) \cap \mathcal{S}_\alpha(N)$, then, by (b) there are $\beta, \gamma \geqslant \alpha$ such that $x \in \mathcal{S}_{\alpha\beta}(M) \cap \mathcal{S}_{\alpha\gamma}(N)$. Take $\delta \geqslant \beta, \gamma$; by (c) we have $x \in \mathcal{S}_{\alpha\beta}(M) \cap \mathcal{S}_{\alpha\delta}(N)$. The converse is clear. (e): If $M \subset X_\alpha$ is open then for any $\beta \geqslant \alpha \varrho_{\alpha\beta}(M)$ is open in $\varrho_{\alpha\beta}(X_\alpha)$ (openness of $\varrho_{\alpha\beta}$) and $\varrho_{\alpha\beta}^{-1}\varrho_{\alpha\beta}(M)$ is $t_\alpha$

— open (continuity of $\varrho_{\alpha\beta}$) hence (i) follows from (b). (ii): It is known that $S \subset I$ is $t$ — open iff $\xi_\alpha^{-1}(S)$ is $t_\alpha$ — open for every $\alpha \in A$. If $M$ is $t_\alpha$ — open, $\beta \in A$, then there is $\gamma \geqslant \beta$, $\alpha$ and we have $\xi_\gamma^{-1}\xi_\alpha(M) = \xi_\gamma^{-1}\xi_\gamma\varrho_{\alpha\gamma}(M) = \mathscr{S}_\gamma(\varrho_{\alpha\beta}(M))$. As $N = \varrho_{\alpha\gamma}(M)$ is $t_\gamma$ — open, so is $\mathscr{S}_\gamma(N) = \xi_\gamma^{-1}\xi_\alpha(M)$ by (i). Now, $\xi_\beta^{-1}\xi_\alpha(M) = \varrho_{\beta\gamma}^{-1}\xi_\gamma^{-1}\xi_\alpha(M)$ and it is open as $\varrho_{\beta\gamma}$ is continuous.

**Theorem 4.** *Let $\mathscr{S} = \{(X_\alpha, t_\alpha) | \varrho_{\alpha\beta} | \langle A \leqslant \rangle\}$ be a presheaf from the category of topological spaces, let $\tilde{\mathscr{S}} = \{(Y_\alpha, v_\alpha) | \varrho_{\alpha\beta} | \langle A \leqslant \rangle\}$ be its quotient (see 1), let $\varinjlim \mathscr{S} = \langle (I, t) | \{\xi_\alpha | \alpha \in A\}$, $\varinjlim \tilde{\mathscr{S}} = \langle (J, v) | \{\eta_\alpha | \alpha \in A\} \rangle$. Suppose that for any $\alpha \in A$ and any $x, y \in X_\alpha$ with $\varrho_{\alpha\beta}(x) \neq \varrho_{\alpha\beta}(y)$ for all $\beta \geqslant \alpha$ there is an open nbd $U$ of $x$ and $V$ of $y$ with $\varrho_{\alpha\beta}^{-1}\varrho_{\alpha\beta}(U) \cap \varrho_{\alpha\beta}^{-1}\varrho_{\alpha\beta}(V) = \emptyset$. Then*

(a): *If any $\varrho_{\alpha\beta}$ maps any $t_\alpha$ — open set onto a $t_\beta$ — open one, then $(I, t)$ and $(J, v)$ are Hausdorff.*

(b): *If all the $\varrho_{\alpha\beta}$'s are open maps, $\leqslant$ a well order, if there is a countable cofinal subset of A, and if every $(X_\alpha, t_\alpha)$ is compact and Hausdorff, then $(J, v)$ and $(I, t)$ are functionally separated.*

Proof. (a): If $a, b \in I$, $a \neq b$, then there is $\alpha \in A$ and $x, y \in X_\alpha$ with $\xi_\alpha(x) = a$, $\xi_\alpha(y) = b$. Then $\varrho_{\alpha\beta}(x) \neq \varrho_{\alpha\beta}(y)$ for all $\beta \geqslant \alpha$, and so there are open $U$, $V \subset X_\alpha$ with $\varrho_{\alpha\beta}^{-1}\varrho_{\alpha\beta}(U) \cap \varrho_{\alpha\beta}^{-1}\varrho_{\alpha\beta}(V) = \emptyset$. By 3b, we have $\mathscr{S}_{\alpha\beta}(U) \cap \mathscr{S}_{\alpha\beta}(V) = \emptyset$, for all $\beta \geqslant \alpha$, and 3d yields $\mathscr{S}_\alpha(U) \cap \mathscr{S}_\alpha(V) = \emptyset$. By 3c $\xi_\alpha^{-1}\xi_\alpha(U) \cap \xi_\alpha^{-1}\xi_\alpha(V) = \mathscr{S}_\alpha(U) \cap \mathscr{S}_\alpha(V) = \emptyset$ and so $\xi_\alpha(U) \cap \xi_\alpha(V) = \emptyset$. By 3e ii $\xi_\alpha(U)$, $\xi_\alpha(V)$ is an open neighborhood of $a, b$ respectively hence $(I, t)$ is Hausdorff.

(b): We may assume that the ordinal type of $A$ is $\omega_0$ as there is a cofinal subset of $A$ of the ordinal type $\omega_0$. Any $x, y \in X_\alpha$ with non $xR_\alpha y$ have open *nbds* $U$, $V$ with $\mathscr{S}_\alpha(U) \cap \mathscr{S}_\alpha(V) = \emptyset$; by 3e (i) $\mathscr{S}_\alpha(U)$, $\mathscr{S}_\alpha(V)$ are open; plainly they are saturated; thus $p_\alpha\mathscr{S}_\alpha(U) = p_\alpha(U)$ and $p_\alpha\mathscr{S}_\alpha(V) = p_\alpha(V)$ are $v_\alpha$ — open disjoint *nbds* of $p_\alpha(x)$, $p_\alpha(y)$ — here $p_\alpha : X_\alpha \to Y_\alpha$ are the natural maps (see 1). Thus we have the $(Y_\alpha, v_\alpha)$'s Hausdorff, hence compact because such are the $(X_\alpha, t_\alpha)$'s. By 2a the $\varrho_{\alpha\beta}$'s are 1—1, hence they are homeomorphisms. Thus $\mathscr{S}$ fulfils the conditions of 1.5.5 or of 1.5.6b of [1] and hence $(J, v)$ is functionally separated. Now we use 2b, which fiishes the proof.

Th. 4 yields the following representation theorem:

**Theorem 5.** *Let $\mathscr{S} = \{(X_U, t_U) | \varrho_{UV} | X\}$ be a sheaf from the category of topological spaces such tah every $(X_U, t_U)$ is compact. Suppose each $x \in X$ has a filter base $Ax$ of open nbds such that*

(a): *either each $\varrho_{UV}$ maps any $t_U$ — open set onto a $t_V$ — open one, or all the $\varrho_{UV}$'s are open and $x$ is of a countable local character.*

(b): *For any $U \in Ax$ and any $x, y \in X_U$ with $\varrho_{UV}(x) = \varrho_{UV}(y)$ for all $V \in Ax$,*

*$V \subset U$ there is an open nbd $U$ of $x$ and $V$ of $y$ such that $\varrho_{UV}^{-1}\varrho_{UV}(U) \cap \varrho_{UV}^{-1}\varrho_{UV}(V) = \emptyset$ for all $V \in Ax$, $V \subset U$. Then there is a separated closure $\hat{t}$ in the covering space P of S such that for the set $\Gamma(U, \hat{t})$ of all continuous sections over U, for the natural map $p_U: X_U \to \Gamma(U, \hat{t})$ mapping $a \in X_U$ onto $\hat{a} \in \Gamma(U, \hat{t})$, where $\hat{a}(x)$ is the germ of a in the stalk over x, and for the topology $b_U(\hat{t})$ projectively defined in $A_U$ by the maps $\{r_{Ux}(x) | x \in U\}$ where $r_{Ux}(\hat{a}) = \hat{a}(x)$ for $x \in U$, we have for any open $U \subset X$:*

(a): $p_U: (X_U, t_U) \to (A_U, b_U(\hat{t}))$ *is a homeomorphism,*

(b): $\Gamma(U, \hat{t}) = A_U$;

(for a more complete statement see [4, Th. 4.2.2]).

Proof. Clearly $\mathcal{S}_{Ax}$ fulfils the conditions of Th. 4 whence if $\underrightarrow{\lim}\, \mathcal{S}_{Ax} = \langle (I_x, t_x) | \{\xi_{Ux} | U \in Ax\} \rangle$, then $(I_x, t_x)$ is a Hausdorff topological space; thus $(A_U, \tilde{b}_U)$ is also one for the topology $\tilde{b}_U$ projectively defined in $A_U$ by the maps $\{r_{Ux}: A_U \to (I_x, t_x) | x \in U\}$ (see [4, Th. 4.2.2], where $\tilde{b}_U$ is denoted by $b_U(st_x)$). As $p_U: (X_U, t_U) \to (A_U, \tilde{b}_U)$ are 1—1 because $\mathcal{S}$ is a sheaf, and continuous, they are homeomorphisms, $(X_U, t_U)$ being compact. The rest can be proven in the same way as [4, Th. 4.2.2].

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# ЭКВИВАЛЕНЦИИ В ПРЕДПУЧКАХ, ФУНЦИОНАЛЬНАЯ ОТДЕЛИМОСТЬ ИХ ИНДУКТИВНЫХ ПРЕДЕЛОВ И ПРЕДСТАВЛЕНИЕ СЕЧЕНИЯМИ

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Резюме

Доказана теорема, при условиях которой индуктивный предел $\mathcal{I}$ предпучка $\mathcal{S} = \{X_\alpha | \varrho_{\alpha\beta} | \langle A \leq \rangle\}$ функционально отделим (для всяких $p, q \in \mathcal{I}$, $p \neq q$ существует непрерывная функция $f$ такая, что $f(p) = f(q)$) без предположения, что $\leq$ есть хорошее упорядочение и что $\varrho_{\alpha\beta}$-простые. Это является обобщением подобной теоремы 1.1.7 из [1], в которой упомянутые предположения должны выполнятся, а также обобщает некоторые теоремы о представлении пучков сечениями из [4], позволяя отбросить предположение, что $\varrho_{UV}$-простые и что точки $x \in X$ имеют хорошо упорядоченные фильтер — базици открытых окрестностей.