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Charles J. Colbourn; Jeffrey H. Dinitz

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GENERATING SETS IN STEINER TRIPLE SYSTEMS

Charles J. Colbourn* — Jeffrey H. Dinitz**

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ABSTRACT. A k-generating set generalizes the notion of a complete arc. Consider the Steiner quasigroup $(V, \oplus)$ of a Steiner triple system. Let $W \subset V$. Then (some) elements of $V \setminus W$ can be written in the form $w \oplus \widehat{w}$ for $w, \widehat{w} \in W$. In turn, further elements can be written in the form $\tilde{w} \oplus (w \oplus \widehat{w})$, and so on. If every element of V can be written using elements of W and at most $k-1$ applications of $\oplus$, then W is called a *k-generating set*.

When $k = 2$, such a generating set is a spanning or dominating set, and these have been constructed in the course of constructing STSs with complete arcs.

Here we report on the case $k = 3$, focussing on the situation that every element of $V \setminus W$ can be written in *exactly one* way as a word with at most two occurrences of $\oplus$, and using the elements of W. When $x = |W|$, we find that $|V| \le x + \binom{x}{2} + x \cdot \binom{x-1}{2}$ for a 3-generating set, and we examine when equality can hold.

1. Definitions

A *Steiner triple system* of order v (briefly STS(v)) is a pair $(X, \mathcal{B})$ where X is a v-element set and $\mathcal{B}$ is a collection of 3-subsets of X (*triples*), such that every pair of X is contained in exactly one triple of $\mathcal{B}$. It is well known that a necessary and sufficient condition for an STS(v) to exist is that $v \equiv 1$ or $3 \pmod 6$.

Let K and G be sets of nonnegative integers and let λ be a positive integer. A *group divisible design of index λ and order v* ((K, λ)-GDD) is a triple $(V, \mathcal{G}, \mathcal{B})$, where V is a finite set of cardinality v, $\mathcal{G}$ is a partition of V into parts (*groups*) whose sizes lie in G, and $\mathcal{B}$ is a family of subsets (*blocks*) of V which satisfy the properties:

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(1) If $B \in \mathcal{B}$, then $|B| \in K$.
(2) Every pair of distinct elements of V occurs in exactly λ blocks or one group, but not both.
(3) $|\mathcal{G}| > 1$.

The *group-type* (*type*) of the GDD is the multiset $\{|G| : G \in \mathcal{G}\}$. An "exponential" notation describes group-type: $g_1^{u_1} \dots g_s^{u_s}$ denotes u_i groups of size g_i for $1 \le i \le s$. If $K = \{k\}$, then the (K,λ)-GDD is a (k,λ)-GDD. If $\lambda = 1$, the GDD is a K-GDD. Furthermore, a $(\{k\},1)$-GDD is a k-GDD.

LEMMA 1.1. ([11]) *There exists a $\{3\}$-GDD of type g^t if and only if $g(t-1) \equiv 0 \pmod 2$, $g^2t(t-1) \equiv 0 \pmod 3$ and $t \ne 2$.*

Colbourn, Hoffman and Rees [6] prove:

LEMMA 1.2. *Let g, t, and u be nonnegative integers. There exists a 3-GDD of type $g^t u^1$ if and only if the following conditions are all satisfied:*

(i) *if $g > 0$, then $t \ge 3$, or $t = 2$ and $u = g$, or $t = 1$ and $u = 0$, or $t = 0$;*
(ii) *$u \le g(t-1)$ or $gt = 0$;*
(iii) *$g(t-1) + u \equiv 0 \pmod 2$ or $gt = 0$;*
(iv) *$gt \equiv 0 \pmod 2$ or $u = 0$;*
(v) *$\frac{1}{2}g^2t(t-1) + gtu \equiv 0 \pmod 3$.*

A *modified group-divisible design* with block sizes in K (K-MGDD) of *type* (u,v) and *index* λ, is a set $U \times V$ with $|U| = u$ and $|V| = v$, partitioned into *first groups* $\{\{u\} \times V : u \in U\}$, and into *second groups* $\{U \times \{v\} : v \in V\}$, and equipped with a collection $\mathcal{B}$ of blocks with sizes from K, so that every pair of elements appears either in a first or second group together, or in exactly λ blocks in $\mathcal{B}$, but not both.

LEMMA 1.3. ([1]) *Let $u, v \ge 3$. A 3-MGDD of type (u,v) and index λ exists if and only if $\lambda \equiv 0 \pmod{\gcd(u-2, v-2, 6)}$.*

An *incomplete* 3-*GDD*, or 3-IGDD, of type $(g:h)^u$ and index λ is a set X of gu elements, a partition $\mathcal{G}$ of X into u *groups* of size g, a *hole* $H \subset X$ satisfying $|H \cap G| = h$ for each $G \in \mathcal{G}$, and a collection $\mathcal{B}$ of *blocks*, so that every pair of elements not in a group together, and not both in the hole, lies in exactly λ blocks of $\mathcal{B}$. Pairs of elements in a group together or in the hole together appear in no blocks.

THEOREM 1.4. ([12]) *A 3-IGDD of type $(g : h)^u$ and index one exists if and only if*

$$u \equiv 1,3 \pmod 6, \quad g > 0, \quad h > 0$$

or

$$u \equiv 0,4 \pmod 6, \quad g \equiv 0 \pmod 2, \quad h \equiv 0 \pmod 2$$

or

$$u \equiv 5 \pmod 6, \quad \begin{matrix} g \equiv 0 \pmod 3, & h \equiv 0 \pmod 3 \\ g \equiv 1,2 \pmod 3, & h \equiv 1,2 \pmod 3 \end{matrix}$$

or

$$u \equiv 2 \pmod 6, \quad \begin{matrix} g \equiv 0 \pmod 6, & h \equiv 0 \pmod 6 \\ g \equiv 2,4 \pmod 6, & h \equiv 2,4 \pmod 6 \end{matrix}$$

and

$$g \geq 2h, \quad u \geq 3.$$

A *holey group-divisible design* $(V, \mathcal{G}, \mathcal{H}, \mathcal{B})$ is a set V of elements, together with two partitions $\mathcal{G}$ and $\mathcal{H}$ of V into first and second groups, with the property that for $G, G' \in \mathcal{G}$ and $H \in \mathcal{H}$, one has $|G \cap H| = |G' \cap H|$. The blocks $\mathcal{B}$ are 3-subsets of V containing every pair exactly λ times if the pair does not lie in the same first or second group; blocks contain no pairs from a first or second group. A holey group-divisible design is a 3-HGDD of type $(u : g_1^{r_1} \dots g_s^{r_s})$ when the first groups, $\mathcal{G}$, consist of a partition of V into u groups of size $\sum_{i=1}^{s} g_i r_i$; and the second groups, $\mathcal{H}$, consist of r_i groups of size ug_i for $1 \leq i \leq s$, each meeting each first group in g_i elements. A 3-HGDD of type $(u : t^v)$ is *uniform*.

THEOREM 1.5. ([15]) *A 3-HGDD of index λ and type $(u : t^v)$ exists if and only if $u \geq 3$, $v \geq 3$, $\lambda(v-1)(u-1)t \equiv 0 \pmod 2$, and $\lambda t^2 u(u-1)v(v-1) \equiv 0 \pmod 6$.*

The *leave* of a partial triple system of index λ is the multigraph whose vertices are the elements of the partial triple system, in which edge $\{x, y\}$ appears with multiplicity $\lambda - s$ when the pair occurs in exactly s triples of the partial triple system.

THEOREM 1.6. ([7]) *Let G be a graph on $v \equiv 1 \pmod 2$ vertices with every vertex of degree 0 or 2, and having $0 \pmod 3$ edges if $v \equiv 1,3 \pmod 6$, and $1 \pmod 3$ edges if $v \equiv 5 \pmod 6$. Then G is a leave of a partial triple system of index one unless $v = 7$ and $G = 2\{C_3\} \cup K_1$, or $v = 9$ and $G = C_4 \cup C_5$.*

Every Steiner triple system has an associated Steiner quasigroup, obtained by defining $x \oplus x = x$ for all elements x, and $x \oplus y = z$ whenever $\{x, y, z\}$ is a triple; see [8] for a discussion of the connections between triple systems and quasigroups.

When an $\mathrm{STS}(v)$ defined by the quasigroup $(V, \oplus)$ has a 3-generating set of size x, there are

1. x *primary* elements in the generating set;
2. at most $\binom{x}{2}$ *secondary* elements defined as $w \oplus w'$ for w, w' primary;
3. at most $x \cdot \binom{x-1}{2}$ *tertiary* elements defined as $w \oplus z$ where w is primary and z is secondary.

Thus, $|V| \leq x + \binom{x}{2} + x \cdot \binom{x-1}{2}$. We are interested here in cases when equality holds. In principle, this can occur when $x \equiv 1, 2, 3 \pmod 4$, since in these cases we find that $x + \binom{x}{2} + x \cdot \binom{x-1}{2} \equiv 1, 3 \pmod 6$. When $x \equiv 0 \pmod 4$, the threshold quantity is even and hence not the order of a Steiner triple system.

Generating sets generalize the notion of complete arcs and independent sets in designs; see [2], [4], [5], [9], [10], [13], [14] for relevant literature.

We settle the existence of $\mathrm{STS}\big(x + \binom{x}{2} + x \cdot \binom{x-1}{2}\big)$s with a 3-generating set of size x for all $x \equiv 1, 2, 3 \pmod 4$ in the sections that follow, treating the cases when x is odd and those when x is even separately.

2. Minimum generating sets: The odd case

THEOREM 2.1. *Let $x = 2t+1$ be an integer, $x \geq 15$, and $v = x+xt+xt(x-2)$. There is an $\mathrm{STS}(v)$ having a 3-generating set of size x.*

Proof. Let $X = \{\infty\} \cup \{i : 1 \leq i \leq t\} \cup \{(i,j) : 1 \leq i \leq t,\ j \in \mathbb{Z}_x,\ j \not\equiv \pm i \pmod x\}$. We form a Steiner triple system on $X \times \mathbb{Z}_x$ as follows. First we specify the triples containing ∞_0 (this subscript notation is used for $(\infty, 0) \in X \times \mathbb{Z}_x$. They are:

1. $\{\infty_0, \infty_{2k}, k_k\}$ for $1 \leq k \leq t$;
2. $\{\infty_0, k_j, (k,j)_0\}$ for $1 \leq k \leq t$, $j \in \mathbb{Z}_x \setminus \{\pm k \bmod x\}$;
3. $\{\infty_0, (k,j)_y, (k,j)_{-y}\}$ for $1 \leq k \leq t$, $j \in \mathbb{Z}_x \setminus \{\pm k \bmod x\}$, and $1 \leq y \leq t$.

Develop these triples in $X \times \mathbb{Z}_x$ mod $(-, x)$ to obtain a set $\mathcal{D}$, having all triples containing an element of $\{\infty\} \times \mathbb{Z}_x$. If we produce a Steiner triple system of order v containing these triples, then we can readily verify that $\{\infty\} \times \mathbb{Z}_x$ is a 3-generating set.

Consider the structure of the pairs not occurring in a triple of $\mathcal{D}$. No pair on $\{i : 1 \leq i \leq t\} \times \mathbb{Z}_x$ appears in one of these triples. For fixed $1 \leq k \leq t$ and $j \in \mathbb{Z}_x \setminus \{\pm k \bmod x\}$, all pairs of the form $\{(k,j)_y, (k,j)_z\}$ for $y, z \in \mathbb{Z}_x$, $y \neq z$, are in triples of $\mathcal{D}$. The only other pairs in triples of $\mathcal{D}$ form, for fixed $1 \leq k \leq t$ and $j \in \mathbb{Z}_x \setminus \{\pm k \bmod x\}$, a one-factor between the elements $\{k\} \times \mathbb{Z}_x$ and the elements $\{(k,j)\} \times \mathbb{Z}_x$. Each element of $\{(k,j)\} \times \mathbb{Z}_x$ occurs in exactly one such pair, while each element of $\{k\} \times \mathbb{Z}_x$ occurs in $x-2$, one for each selection of j. Our task reduces to showing that the pairs in triples of $\mathcal{D}$ form a leave of a partial STS.

Case 1: $x \equiv 3 \pmod 6$, and hence $t \equiv 1 \pmod 3$. For $1 \le k \le t$, place an STS(x) on $\{k\} \times \mathbb{Z}_x$. Next, for $1 \le k \le t$, place a 3-MGDD of type $(x-1, x)$ on $\big(\{k\} \cup \{(k,j) : \ j \in \mathbb{Z}_x,\ j \not\equiv \pm k \pmod x\}\big) \times \mathbb{Z}_x$, with first groups defined by constant last coordinates, and second groups defined by constant first coordinates. Now place the triples of an STS($x-2$) on $\big(\{(k,j) : \ j \in \mathbb{Z}_x,\ j \not\equiv \pm k \pmod x\}\big) \times \{\ell\}$ for each $\ell \in \mathbb{Z}_x$ and each $1 \le k \le t$.

Pairs used now account for all those on $\big(\{k\} \cup \{(k,j) : \ j \in \mathbb{Z}_x,\ j \not\equiv \pm k \pmod x\}\big) \times \mathbb{Z}_x$ for each $1 \le k \le t$, and for no others. Then form a 3-GDD of type $\big(x(x-1)\big)^t$, with groups aligned on $\big(\{k\} \cup \{(k,j) : \ j \in \mathbb{Z}_x,\ j \not\equiv \pm k \pmod x\}\big) \times \mathbb{Z}_x$ for $1 \le k \le t$. This completes the case $x \equiv 3 \pmod 6$.

Case 2: $x \equiv 1 \pmod 6$, and hence $t \equiv 0 \pmod 3$. Write $s = \frac{t}{3}$. For $1 \le k \le t$, place an STS(x) on $\{k\} \times \mathbb{Z}_x$. Now partition $\{1, \dots, t\}$ into s disjoint classes $C_1, \dots, C_s$, each of size 3. Form a 3-GDD of type $\big(3x(x-1)\big)^s$ whose groups are $\big(\{i : \ i \in C_k\} \cup \{(i,j) : \ i \in C_k,\ j \in \mathbb{Z}_x,\ j \not\equiv \pm i \pmod x\}\big) \times \mathbb{Z}_x$, for $1 \le k \le s$. Next, on each of these s groups, we place a 3-HGDD of type $(x-1 : 3^x)$ with the groups of size $3(x-1)$ on $\big(\{i : \ i \in C_k\} \cup \{(i,j) : \ i \in C_k,\ j \in \mathbb{Z}_x,\ j \not\equiv \pm i \pmod x\}\big) \times \{\ell\}$ for each $\ell \in \mathbb{Z}_x$. The groups of size $3x$ are on $\{i : \ i \in C_k\} \times \mathbb{Z}_x$ and $\{(i,j) : \ i \in C_k\} \times \mathbb{Z}_x$ for $j \in \mathbb{Z}_x$, $j \not\equiv \pm i \pmod x$. These two sets of groups intersect in sets of size three, namely $\big(\{i\} \times \{\ell\}$ and $\{(i,j)\}\big) \times \{\ell\}$ for each $\ell \in \mathbb{Z}_x$, $i \in C_k$, and $j \in \mathbb{Z}_x$, $j \not\equiv \pm i \pmod x$.

On each of the groups of size $3x$, when one examines pairs already used, one finds that within three disjoint sets of size x, all pairs are used, and that no other pairs are used. Place a 3-MGDD of type $(3, x)$ on each such group, so that the groups of size x align on these three previously used sets, and the groups of size 3 align within the groups of size $3(x-1)$ of the HGDD selected earlier.

It remains only to fill the groups of size $3(x-1)$. Again, considering pairs already used within such a group, we find that in a group $\big(\{i : \ i \in C_k\} \cup \{(i,j) : \ i \in C_k,\ j \in \mathbb{Z}_x,\ j \not\equiv \pm i \pmod x\}\big) \times \{\ell\}$, the pairs used thus far are $\big\{\{i_\ell, (i,j)_\ell\} : \ i \in C_k,\ j \in \mathbb{Z}_x,\ j \not\equiv \pm i \pmod x\big\}$, and no others. To cover the remaining pairs, we select a partial triple system of order $3(x-2)$ on $\{(i,j) : i \in C_k,\ j \in \mathbb{Z}_x,\ j \not\equiv \pm i \pmod x\} \times \{\ell\}$, whose leave is a $3(x-2)$-cycle Y. Such a partial triple system exists by Theorem 1.6. When $C_k = \{a, b, c\}$, we place the partial triple system so that, reading the first coordinate of the points sequentially along the cycle forming the leave, one always encounters b after a, c after b, and a after c (which can be done since the cycle is hamiltonian and has a number of edges which is a multiple of three. To complete filling the hole of size $3(x-1)$, whenever $\{(b,j)_\ell,(c,j')_\ell\}$ is an unused pair, the triple $\{a_\ell,(b,j)_\ell,(c,j')_\ell\}$ is added; this process is repeated for (b,c,a) and (c,a,b) in place of (a,b,c). This completes the proof for $x \equiv 1 \pmod 6$.

Case 3: $x \equiv 11 \pmod{12}$, and hence $t \equiv 5 \pmod 6$. Place an $\mathrm{STS}(x-2)$ on $\{(k,j)_i : j \in \mathbb{Z}_x,\ j \neq \pm k\}$ for $1 \le k \le t$ and $i \in \mathbb{Z}_x$. Next place a 3-MGDD of type $(x, x-1)$ on $(\{k\} \cup \{(k,j) : j \in \mathbb{Z}_x,\ j \not\equiv \pm k \pmod x\}) \times \mathbb{Z}_x$, with first groups defined by constant last coordinates, and second groups defined by constant first coordinates. Then form the triples of a 3-IGDD of type $(x(x-1):x)^t$ with the hole on $\{1,\dots,t\} \times \mathbb{Z}_x$ and the groups on $(\{k\} \cup \{(k,j) : j \in \mathbb{Z}_x,\ j \not\equiv \pm k \pmod x\}) \times \mathbb{Z}_x$ for $1 \le k \le t$. Finally place the triples of an $\mathrm{STS}(xt)$ on $\{1,\dots,t\} \times \mathbb{Z}_x$.

Case 4: $x \equiv 5 \pmod{12}$, and hence $t \equiv 2 \pmod 6$. Form a 3-GDD of type $(2x(x-1))^{t/2}$ with groups $\{\{k_i\} \cup \{(k,j)_i\} : i \in \mathbb{Z}_x,\ j \in \mathbb{Z}_x,\ j \neq \pm k,\ k \in \{2\ell-1, 2\ell\}\}$ for $1 \le \ell \le t/2$. Evidently all as yet uncovered pairs now lie within a group of this 3-GDD. We describe how to handle the pairs when $\ell = 1$, observing that the same method treats all other values $2 \le \ell \le t/2$.

Next we place a 3-GDD of type 2^{x-2} on $\{\{(k,j)_i\} : i \in \mathbb{Z}_x,\ j \in \mathbb{Z}_x,\ j \neq \pm k,\ k \in \{1,2\}\}$, aligning the groups on $\{(k,j)_i,(k,-j)_i\}$ for $i \in \mathbb{Z}_x$, $k \in \{1,2\}$, and $j \in \mathbb{Z}_x \setminus \{0,\pm k\}$. Now add triples $\{(3-k)_i,(k,j)_i,(k,-j)_i\}$ for $i \in \mathbb{Z}_x$, $k \in \{1,2\}$, and $j \in \mathbb{Z}_x \setminus \{0,\pm k\}$. It is easily verified that, among elements with the same subscript i, all pairs are covered except $\{1_i,2_i\}$, $\{1_i,(2,0)_i\}$, $\{2_i,(1,0)_i\}$, and $\{(1,0)_i,(2,0)_i\}$ — i.e., a 4-cycle remains uncovered.

Now since $2x-7 \equiv 3 \pmod{24}$, there is a Kirkman triple system (resolvable Steiner triple system) of order $2x-7$ ([11]). To each of five resolution classes, add an infinite point to each block in the class to form a $\{3,4\}$-GDD of type $1^{2x-7}5^1$. Using 3-MGDDs of type $(3,x)$ and $(4,x)$ to give weight x to the $\{3,4\}$-GDD, produce a 3-HGDD of type $(x : 1^{2x-7}5^1)$. Place this on the points $\{\{k_i\} \cup \{(k,j)_i\} : i \in \mathbb{Z}_x,\ j \in \mathbb{Z}_x,\ j \neq \pm k,\ k \in \{1,2\}\}$ so that the first groups align on points with the same subscript and the second groups align on $\{1,2,(1,0),(2,0),(1,1)\} \times \mathbb{Z}_x$ (for the group of size $5x$) and on groups of the form $\{(k,j)\} \times \mathbb{Z}_x$ (for the groups of size x).

It remains only to handle the pairs that remain on $\{1,2,(1,0),(2,0),(1,1)\} \times \mathbb{Z}_x$, noting that all pairs with the same entry from $\mathbb{Z}_x$ are already covered except those in the 4-cycle $(1_i,(2,0)_i,(1,0)_i,2_i)$; and that all pairs of elements with the same entry from $\{1,2,(1,0),(2,0),(1,1)\}$ are already covered.

Now form a 3-GDD of type $3^{(x-5)/3}5^1$ on $\mathbb{Z}_x$. For each block B of this 3-GDD, place a 3-MGDD of type $(3,5)$ on $\{1,2,(1,0),(2,0),(1,1)\} \times B$, with groups aligned on elements with constant first, or constant second, coordinate. It remains only to handle pairs on $\{1,2,(1,0),(2,0),(1,1)\} \times G$ where G is a group of size three or five. The two ingredients needed have 15 and 25 points, respectively, and were found via a hillclimbing procedure. We present the solutions on lowercase letters to enhance readability.

The 15 point ingredient has five groups of size three on afk, bgl, chm, din, and ejo, and on the five points ABCDE (for ABCDE equal to each of abcde, fghij, klmno), the only pairs covered are BC, CD, DE, and BE. Its triples are:

agj aho ail amn bck bef bhn bim bjo cdg cfn cio cjl dek dfo dhl din djm egm ehi ejn elo flm ghk gno ijk.

The 25 point ingredient has five groups of size five on afkpu, bglqv, chmrw, dinsx, ejoty, and on the five points ABCDE (for ABCDE equal to each of abcde, fghij, klmno, pqrst, uvwxy), the only pairs covered are BC, CD, DE, and BE. Its triples are:

agr ahi ajv aly amq anw aox ast bck beh bfn bip bjo bmy bru bsx btw cdv cfq cgp cin cjt clx cou csy dek dfy dgw dhx diq djm dlt dnu dop drs efv egt eim ejw elr enp eos equ exy flm for fsw ftx ghu gjk gms gnx goy hkt hls hno hpv hqy ijr iky ilo isu itv iwx jlu jns jpy jqx kqw krx ksv lpw mnv mpx mtu nqr nty oqt ovw rvy.

□

3. Minimum generating sets: The even case

In this section, we address the problem when the size of the 3-generating set is even. In order to suggest the similarities with the odd case, we suppose that the generating set has size $x+1$, so that x is odd. Indeed, since $x+1 \equiv 2 \pmod 4$, we have that $x \equiv 1 \pmod 4$. We also write $t = \frac{x-1}{2}$ as before.

THEOREM 3.1. *Let $x \equiv 1 \pmod 4$, $x \geq 13$, and let $v = 1 + x + \binom{x+1}{2} + (x+1)\binom{x}{2}$. Then there exists an STS(v) having a 3-generating set of size $x+1$.*

Proof. Let $X = \{\infty\} \cup \big(\{i : 1 \leq i \leq t\} \cup \{\alpha\}\big) \cup \big(\{(i,j) : 1 \leq i \leq t, j \in \mathbb{Z}_x \text{ or } j = \infty \text{ and } j \neq \pm i\}\big) \cup \big(\{(\alpha, j) : j \in \mathbb{Z}_x, j \neq 0\}\big)$. The element set is $\{\infty\} \cup (X \times \mathbb{Z}_x)$, and we write x_i for $(x,i) \in (X \times \mathbb{Z}_x)$. There are two types of "fixed" elements, those involving α and those involving ∞. The $x+1$ primary elements are $\{\infty\} \cup \big(\{\infty\} \times \mathbb{Z}_x\big)$. The $\binom{x+1}{2} = x + \binom{x}{2}$ secondary elements are $\big(\{i : 1 \leq i \leq t\} \cup \{\alpha\}\big) \times \mathbb{Z}_x$. They are defined by the primary elements by including the following triples:

1. $\{\infty, \infty_i, \alpha_i\}$ for $i \in \mathbb{Z}_x$;
2. $\{\infty_i, \infty_{2k+i}, k_{k+i}\}$ for $1 \leq k \leq t$, $i \in \mathbb{Z}_x$.

The tertiary elements are $\Big(\big(\{(i,j): 1 \le i \le t,\ j \in \mathbb{Z}_x \text{ or } j = \infty \text{ and } j \ne \pm i\}\big) \cup \big(\{(\alpha,j): j \in \mathbb{Z}_x,\ j \ne 0\}\big)\Big) \times \mathbb{Z}_x$. Each is defined by a primary and a secondary element by including the triples:

3. $\{\infty, k_j, (k,\infty)_j\}$ for $1 \le k \le t$, $j \in \mathbb{Z}_x$;
4. $\{\infty_i, \alpha_{i+j}, (\alpha,j)_i\}$ for $i \in \mathbb{Z}_x$, $j \in \mathbb{Z}_x \setminus \{0\}$;
5. $\{\infty_i, k_{i+j}, (k,j)_i\}$ for $i \in \mathbb{Z}_x$, $1 \le k \le t$, $j \in \mathbb{Z}_x \setminus \{\pm k\}$.

Our goal is to establish that there is a Steiner triple system of the desired order containing the triples 1.$-$5. specified. We specify the remaining triples containing the primary elements next. They are:

6. $\{\infty_i, (k,j)_{i+y}, (k,j)_{i-y}\}$ for $1 \le k \le t$, $j \in \mathbb{Z}_x \setminus \{\pm k\}$ or $j = \infty$, $1 \le y \le t$, $i \in \mathbb{Z}_x$;
7. $\{\infty_i, (2k-1,\infty)_i, (2k,\infty)_i\}$ for $1 \le k \le t/2$, $i \in \mathbb{Z}_x$;
8. $\{\infty, (2k-1,j)_i, (2k,\ell)_i\}$ for $1 \le k \le t/2$, $i \in \mathbb{Z}_x$, $j \in \mathbb{Z}_x$, $j \ne \pm(2k-1)$,
$$\ell = \begin{cases} j & \text{if } j \ne \pm 2k, \\ 2k-1 & \text{if } j = 2k, \\ -(2k-1) & \text{if } j = -2k; \end{cases}$$
9. $\{\infty, (\alpha,j)_i, (\alpha,-j)_i\}$ for $1 \le j \le t$, $i \in \mathbb{Z}_x$;
10. $\{\infty_i, (\alpha,j)_{i+y}, (\alpha,j)_{i-y}\}$ for $1 \le y \le t$, $j \in \mathbb{Z}_x \setminus \{0\}$, and $i \in \mathbb{Z}_x$.

Now we treat cases for x individually.

Case 1: $x+1 \equiv 2, 10 \pmod{12}$. Place an STS($x$) on $\{\alpha\} \times \mathbb{Z}_x$. Form an STS($x$) on $\{\alpha\} \cup \{(\alpha,j): j \in \mathbb{Z}_x \setminus \{0\}\}$, so that $\{\alpha, (\alpha,j), (\alpha,-j)\}$ is a triple for $j \in \mathbb{Z}_x \setminus \{0\}$. Use this to form a set of triples on x^2 points $\big(\{\alpha\} \cup \{(\alpha,j): j \in \mathbb{Z}_x \setminus \{0\}\}\big) \times \mathbb{Z}_x$ as follows. When B is a triple of the STS(x) and $\alpha \notin B$, place a 3-GDD of type x^3 on $B \times \mathbb{Z}_x$ with groups aligned on $\{b\} \times \mathbb{Z}_x$ for $b \in B$. When $B = \{\alpha, (\alpha,j), (\alpha,-j)\}$, place a 3-MGDD of type $(3,x)$ on $B \times \mathbb{Z}_x$ with groups aligned on $\{x\} \times \mathbb{Z}_x$ for $b \in B$ and on $B \times \{j\}$ for $j \in \mathbb{Z}_x$. Include all of these triples in the system being constructed.

Next form a 3-GDD of type $(2x^2)^{t/2}\big(x(x-1)\big)^1$ with groups $\big(\{2k, 2k-1, (2k,\infty), (2k-1,\infty)\} \cup \{(\ell,j): \ell \in \{2k-1,2k\},\ j \in \mathbb{Z}_x \setminus \{\pm\ell\}\}\big) \times \mathbb{Z}_x$ for $k = 1,\dots,t/2$, and short group $\{(\alpha,j): j \in \mathbb{Z}_x \setminus \{0\}\} \times \mathbb{Z}_x$. Now all pairs involving any element of the form $(\alpha,j)_i$ is covered.

For $k = 1,\dots,t/2$, add the triples $\{\alpha_i, (2k-1)_j, (2k)_{i+j}\}$ for $i \in \mathbb{Z}_x$ and $j \in \mathbb{Z}_x$. Then add the triples $\{\alpha_i, (2k-1,j)_y, (2k,\ell)_{y+i}\}$ for $i \in \mathbb{Z}_x \setminus \{0\}$, $j \in \mathbb{Z}_x \setminus \{\pm(2k-1)\}$, $y \in \mathbb{Z}_x$, and
$$\ell = \begin{cases} j & \text{if } j \ne \pm 2k, \\ 2k-1 & \text{if } j = 2k, \\ -(2k-1) & \text{if } j = -2k. \end{cases}$$

Next place an STS(x) on $\{k\} \times \mathbb{Z}_x$ for $1 \leq k \leq t$. Next place an STS(x) on $\{\alpha_0\} \cup \{(k,j)_i : j \in \mathbb{Z}_x \setminus \{\pm k\} \text{ or } j = \infty\}$ when $i \in \mathbb{Z}_x$ and $1 \leq k \leq t$. Finally for $k = 1, \ldots, t/2$, place on $(\{2k, 2k-1, (2k, \infty), (2k-1, \infty)\} \cup \{(\ell, j) : \ell \in \{2k-1, 2k\},\ j \in \mathbb{Z}_x \setminus \{\pm \ell\}\}) \times \mathbb{Z}_x$ the triples of a 3-MGDD of type $(2x, x)$ in which one set of groups is $\{2k-1, 2k\} \times \mathbb{Z}_x$, $\{(2k, \infty), (2k-1, \infty)\} \times \mathbb{Z}_x$, and $\{(\ell, j) : \ell \in \{2k-1, 2k\}\} \times \mathbb{Z}_x$ for $j \in \mathbb{Z}_x \setminus \{\pm \ell\}$. The second set of groups is defined, for $\ell \in \{2k-1, 2k\}$ and $m \in \mathbb{Z}_x$, by $(\{\ell, (\ell, \infty)\} \cup \{(\ell, j) : j \in \mathbb{Z}_x \setminus \{\pm \ell\}\}) \times \{m\}$. It is now possible to verify that the resulting collection of triples forms an STS, and hence the case when $x \equiv 1, 3 \pmod 6$ is completed.

Case 2: $x + 1 \equiv 6 \pmod{12}$. On $(\{1, \ldots, t\} \cup \{\alpha\}) \times \mathbb{Z}_x$, place an STS$(x(t+1))$. For $1 \leq j \leq t$, place an 3-MGDD of type $(x, 3)$ on $\{\alpha, (\alpha, j), (\alpha, -j)\} \times \mathbb{Z}_x$, aligning the larger groups on $\{y\} \times \mathbb{Z}_x$ for $y \in \{\alpha, (\alpha, j), (\alpha, -j)\}$ and the groups of size three on $\{\alpha, (\alpha, j), (\alpha, -j)\} \times \{i\}$ for $i \in \mathbb{Z}_x$. For $1 \leq i < x$, $1 \leq k \leq t/2$, $j \in \mathbb{Z}_x$, $j \neq \pm(2k-1)$,

$$\ell = \begin{cases} j & \text{if } j \neq \pm 2k, \\ 2k-1 & \text{if } j = 2k, \\ -(2k-1) & \text{if } j = -2k \end{cases}$$

and $y \in \mathbb{Z}_x$ place the triple $\{\alpha_i, (2k-1, j)_y, (2k, \ell)_{y+i}\}$.

For $i \in \mathbb{Z}_x$, $1 \leq k \leq t$, $1 \leq j \leq t$, and $j \neq \pm k$, include the triple $\{\alpha_0, (k,j)_i, (k,-j)_i\}$. For $i \in \mathbb{Z}_x$, $1 \leq k \leq t$, include the triple $\{\alpha_0, (k,0)_i, (k,\infty)_i\}$. At this point, all pairs involving elements of the form α_i have been handled.

Next, on $(\{k : 1 \leq k \leq t\} \cup \{(\alpha, j) : j \in \mathbb{Z}_x \setminus \{0\}\}) \times \mathbb{Z}_x$, form a 3-GDD of type $(2x)^t (x(x-1)/2)^1$ with $\{k : 1 \leq k \leq t\} \times \mathbb{Z}_x$ as the group of size $x(x-1)/2$, and for $1 \leq y \leq t$ a group of size $2x$ is $\{(\alpha, y), (\alpha, -y)\} \times \mathbb{Z}_x$. On $(\{(k,j) : 1 \leq k \leq t,\ j \in \mathbb{Z}_x \setminus \{\pm k\} \text{ or } j = \infty\} \cup \{(\alpha, j) : j \in \mathbb{Z}_x \setminus \{0\}\}) \times \mathbb{Z}_x$, form a 3-GDD of type $(x(x-1))^{t+1}$, where $\{(\alpha, j) : j \in \mathbb{Z}_x \setminus \{0\}\} \times \mathbb{Z}_x$ forms one group. The other t groups are formed by partitioning, for each $1 \leq k \leq t$, the set $\mathbb{Z}_x \setminus \{\pm k\} \cup \{\infty\}$ into t sets $(P_{k1}, \ldots, P_{kt})$, where each class in the partition contains either two elements y and $-y$, or two elements 0 and ∞. Then the t groups of the GDD constructed are formed by taking $\left(\bigcup_{k=1}^{t} \{(k,a), (k,b) : P_{ky} = \{a, b\}\} \right) \times \mathbb{Z}_x$ for $1 \leq y \leq t$.

At this point, all pairs involving elements of the form $(\alpha, j)_i$ have also been handled. To complete the construction, we consider the groups of size $2xt = x(x-1)$ in the last GDD constructed. We place two 3-GDDs of type $(2x)^{t/2}$ on each group of the form $\left(\bigcup_{k=1}^{t} \{(k,a), (k,b) : P_{ky} = \{a, b\}\} \right) \times \mathbb{Z}_x$, one on the points of the form (k,a) and the other on points of the form (k,b). The groups are for $1 \le z \le t/2$ in each, containing points of the form $(2k-1,c)$ and $(2k,c)$ where c is one of a or b.

Each group of size $2xt = x(x-1)$ is also used together with the xt elements of the form $\{k : 1 \le k \le t\} \times \mathbb{Z}_x$ to form $\big(\{k,(k,a),(k,b) : 1 \le k \le t,$ $P_{ky} = \{a,b\}\}\big) \times \mathbb{Z}_x$ for some y with $1 \le y \le t$. We place a 3-MGDD of type $(xt,3)$ with the three groups of size xt on elements of the form $\{y\} \times \mathbb{Z}_x$ for $y = k,(k,a),(k,b)$, respectively. The groups of size 3 are aligned on $\{k,(k,a),(k,b)\} \times \mathbb{Z}_x$ for the choices of k, a, b given. This completes the (most involved) case. $\square$

4. Conclusion

The treatment of the odd and even cases yield our main theorem:

THEOREM 4.1. *An* $\mathrm{STS}\big(x + \binom{x}{2} + x\binom{x-1}{2}\big)$ *having a* 3*-generating set exists if and only if* $x \equiv 1,2,3 \pmod 4$.

Proof. If $x \le 13$, a suitable STS can be easily found using hillclimbing. Solutions are available from the authors, and are not displayed here due to their length. For larger values of x, apply Lemma 2.1 when x is odd, and apply Lemma 3.1 when $x \equiv 2 \pmod 4$. $\square$

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* *Department of Computer Science*
Votey Building
University of Vermont
Burlington, VT 05405
U. S. A.
E-mail: Charles.Colbourn@uvm.edu

** *Department of Mathematics and Statistics*
University of Vermont
Burlington, VT 05405
U. S. A.