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# PRODUCT OF DOMINATED VECTOR MEASURES

MILOSLAV DUCHOŇ

This paper is concerned with vector measures whose values lie in a Banach algebra and we deal with some properties of such measures in product spaces. In [5] and [11] the problem of a product of vector measures is investigated using no vector integral; it is shown there that a sufficient condition for the existence of the bilinear product of vector measures is that one of the factor measures be dominated [6, 11] with respect to the range space of the other measure. A natural question arises: If both vector measures are dominated, is their product also dominated? Using the bilinear vector integral of Bartle [1] we shall show that if both vector measures take their values in the same Banach algebra and if they are both dominated, then their product — as a vector measure with values in the same Banach algebra — is also dominated. Theorems of the Fubini-type are also established.

**1.** Let $X$ be a Banach algebra. Let $(S, \mathscr{S})$ and $(T, \mathscr{T})$ be measurable spaces ($\mathscr{S}$ and $\mathscr{T}$ being $\sigma$-algebras) and let $m$: $\mathscr{S}\to X$ and $n$: $\mathscr{T}\to X$ be ($\sigma$-additive) vector measures. If $p$: $\mathscr{S}\to X$ is a finitely additive set function we define the semivariation of $p$ with respect to $X$ to be the set function

$$\|p\|_X(A)=\sup\left\|\sum_i x_i p(A_i)\right\|,\quad A\in\mathscr{S},$$

where the sup is taken over all finite disjoint families $A_i$ with $A_i$ in $\mathscr{S}$ and $\bigcup_{i=1}^{r}A_i=A$ and all $x_i\in X$ with $\|x_i\|\leqq 1$, $i=1, \ldots, r$. $\|p\|_X$ is a non-negative, monotone and subadditive set function [1, 3].

Let $p$: $\mathscr{S}\to X$ be additive. We say that $p$ is dominated with respect to $X$ if there exists a non-negative finite measure $\lambda$ such that

$$\lim_{\lambda(A)\to 0}\|p\|_X(A)=0,\quad A\in\mathscr{S}.$$

The following lemma is important [6, Th. 2 or 11, Lemma 5].

**Lemma 1.** *Let $m$: $\mathscr{S}\to X$ be dominated. Then* $\|m\|_X(S)<\infty$.

Also the following lemma will be useful [6, Th. 5].

**Lemma 2.** *If $m$: $\mathcal{S}\to X$ is dominated, then there exists a non-negative finite measure $\mu$ on $\mathcal{S}$ such that*

$$\|m\|_X(A)\to 0 \quad \textit{if and only if} \quad \mu(A)\to 0.$$

**2.** We shall now consider the product of vector measures. Let $\mathcal{S}\ \hat{o}\ \mathcal{T}$ denote the algebra generated by the measurable rectangles of $S\times T$, i.e. the algebra generated by sets of the form $A\times B$, $A\in\mathcal{S}$, $B\in\mathcal{T}$. Let $\mathcal{S}\times\mathcal{T}$ be the $\sigma$-algebra generated by $\mathcal{S}\ \hat{o}\ \mathcal{T}$. If $m$: $\mathcal{S}\to X$ and $n$: $\mathcal{T}\to X$ are finitely additive set functions, the product $m\times n$ with respect to $X$ is the set function defined on measurable rectangles $A\times B$ by the formula

$$m\times n(A\times B)=m(A)n(B).$$

If $D\in\mathcal{S}\ \hat{o}\ \mathcal{T}$, then $D$ can be represented in the form $D=\bigcup_{i=1}^{r} A_i\times B_i$, where $A_i\in\mathcal{S}$, $B_i\in\mathcal{T}$ and $A_i\times B_i$ are pairwise disjoint, and $m\times n$ can be extended to $\mathcal{S}\ \hat{o}\ \mathcal{T}$ by setting $m\times n(D)=\sum_{i=1}^{r} m(A_i)n(B_i)$. This definition does not depend upon the representation $D$ and the extension (still denoted by $m\times n$) is finitely additive on $\mathcal{S}\ \hat{o}\ \mathcal{T}$. However, $m\times n$ may fail to be countably additive on $\mathcal{S}\ \hat{o}\ \mathcal{T}$ even though both $m$ and $n$ are countably additive ([10], [11]). A sufficient condition for the countably additivity of $m\times n$ is as follows.

**Theorem 3.** *Let $m$: $\mathcal{S}\to X$ and $n$: $\mathcal{T}\to X$ be vector measures with $n$ dominated with respect to $X$. Then the product $m\times n$ (with respect to $X$) is countably additive on $\mathcal{S}\ \hat{o}\ \mathcal{T}$ and has a unique countably additive extension defined on $\mathcal{S}\times\mathcal{T}$* (still denoted by $m\times n$).

The proof of Theorem 3 is given in [11, Th. 6, cf. also 5, Th. 2].

**3.** In the following we shall use the general bilinear vector integral in the sense of Bartle [1]. We shall obtain an explicit expression for the product analogous to the case of scalar measures.

For every $G\in\mathcal{S}\times\mathcal{T}$ define the function $g^G$: $T\to X$ by the formula

$$g^G(t)=m(G^t). \quad (G^t=\{s\colon (s,t)\in G\}).$$

Denote by $c_F$ the characteristic function of the set $F$. If $G=E\times F\in\mathcal{S}\times\mathcal{T}$, then

$$g^{E\times F}=m(E)c_F.$$

If $n$ is dominated with respect to $X$, then we have $\|n\|_X(T)<\infty$ by Lemma 1. In such a case the function $G^{E\times F}$ is $n$-simple [1, p. 339] and there holds

$$\int_T g^{E\times F}\mathrm{d}n=m(E)n(F)=m\times n(E\times F),$$

where we take the bilinear integral in the sense of Bartle [1]. Further we have

$$g^{E\times F}(t)=m(G^t)=\int_S c_{E\times F}(s,t)\,\mathrm{d}m(s),$$

and $$m\times n(E\times F)=\int_T\left\{\int_S c_{E\times F}(s,t)\,\mathrm{d}m(s)\right\}\mathrm{d}n(t)=\int_T g^{E\times F}\mathrm{d}n.$$

If $G$ and $H$ are disjoint sets in $\mathscr{S}\times\mathscr{T}$, then $g^{G\cup H}=g^G+g^H$ and if $G_n$ is a monotone sequence of the sets in $\mathscr{S}\times\mathscr{T}$ and $G=\lim G_n$, then from the properties of the vector measure $m$ it follows that $g^{G_n}$ converges to $g^G$. Observe that since the measure $m$ is bounded on $S$, for every $G\in\mathscr{S}\times\mathscr{T}$ the function $g^G$ is bounded on $T$.

If $G=\bigcup_{i=1}^{r}E_i\times F_i$ is a disjoint representation of $G$ with $E_i\times F_i\in\mathscr{S}\times\mathscr{T}$, then

$$g^G(t)=m(G^t)=\int_S c_G(s,t)\,\mathrm{d}m(s),$$

and the function $g^G$ is $n$-simple [1, p. 339] and we have

$$m\times n(G)=\sum_{i=1}^{r}m(E_i)n(F_i)=\int_T\left\{\int_S c_G(s,t)\,\mathrm{d}m(s)\right\}\mathrm{d}n(t)=$$

$$=\int_T g^G(t)\,\mathrm{d}n(t).$$

Let $\mathscr{R}$ denote the class of all sets $C\in\mathscr{S}\times\mathscr{T}$ such that $g^C$ is defined on $T$, is $n$-measurable, $n$-integrable [1, p. 347] and there holds

$$m\times n(C)=\int_T g^C\mathrm{d}n.$$

The class $\mathscr{R}$ contains the algebra $\mathscr{S}\,\hat{\circ}\,\mathscr{T}$, $\mathscr{S}\,\hat{\circ}\,\mathscr{T}\subset\mathscr{R}$. If $C_i$ is a monotone sequence of sets from $\mathscr{R}$, then $\bigcup_{i=1}^{\infty}C_i\in\mathscr{R}$. For $g^{C_i}$ is a sequence of $n$-measurable and $n$-integrable functions converging to $g^C$, where $C=\bigcup_{i=1}^{\infty}C_i$, hence $g^C$ is $n$-measurable [1, p. 347]. For all $t\in T$ we have $\|g^{C_r}(t)\|\leqq M<\infty$, $r=1,2,\ldots$, thus $g^C$ is $n$-integrable [1, Th. 3]; according to bounded convergence theorem [1, p. 345] we have

$$\int_T g^C(t)\,\mathrm{d}n(t)=\lim_{r\to\infty}\int_T g^{C_r}(t)\,\mathrm{d}n(t).$$

If $C\in\mathscr{R}$, then $S\times T-C\in\mathscr{R}$, since $g^{(S\times T)-C}=g^{S\times T}-g^C$. By lemma on monotone classes we have $\mathscr{R}=\mathscr{S}\times\mathscr{T}$. We have thus proved the following.

**Theorem 4.** *Let $m: \mathcal{S} \to X$ and $n: \mathcal{T} \to X$ be vector measures with $n$ dominated with respect to $X$. For every $G \in \mathcal{S} \times \mathcal{T}$ the function $g^G$*

$$g^G(t) = m(G^t) = \int_S c_G(s, t)\, \mathrm{d}m(s)$$

*is defined on $T$, $n$-measurable, $n$-integrable and we have*

$$m \times n(G) = \int_T m(G^t)\, \mathrm{d}n(t),$$

i.e.

$$m \times n(G) = \int_T \left\{ \int_S c_G(s, t)\, \mathrm{d}m(s) \right\} \mathrm{d}n(t).$$

For simplicity we shall take $X$ to be commutative. In such a case we have the following result.

**Theorem 5.** *Let $m: \mathcal{S} \to X$ and $n: \mathcal{T} \to X$ be vector measures both dominated with respect to $X$. Then for every $G \in \mathcal{S} \times \mathcal{T}$ the functions*

$$h_G(s) = n(G_s) = \int_T c_G(s, t)\, \mathrm{d}n(t),$$

$$g^G(t) = m(G^t) = \int_S c_G(s, t)\, \mathrm{d}m(s)$$

*are defined on the spaces $S$ and $T$, respectively, $m$-measurable and $m$-integrable, $n$-measurable and $n$-integrable, respectively, and we have*

$$m \times n(G) = \int_S n(G_s)\, \mathrm{d}m(s) = \int_T m(G^t)\, \mathrm{d}n(t),$$

i.e.

$$m \times n(G) = \int_S \left\{ \int_T c_G(s, t)\, \mathrm{d}n(t) \right\} \mathrm{d}m(s) =$$

$$= \int_T \left\{ \int_S c_G(s, t)\, \mathrm{d}m(s) \right\} \mathrm{d}n(t).$$

The main result of this paper is contained in the next theorem. It asserts that if both vector measures $m$ and $n$ are dominated with respect to $X$, then their product $m \times n$ is also dominated (and only in this case) with respect to $X$.

**Theorem 6.** *Let $m: \mathcal{S} \to X$ and $n: \mathcal{T} \to X$ be vector measures both dominated with respect to $X$ by $\mu$ and $\nu$, respectively. Then the product $m \times n: \mathcal{S} \times \mathcal{T} \to X$ is dominated with respect to $X$ by $\mu \times \nu$.*

Proof. Let $e$ and $d$ be such two positive numbers that $\mu(E) < d$, $E \in \mathcal{S}$ implies $\|m\|_X(E) < e$ and $\nu(F) < d$, $F \in \mathcal{T}$ implies $\|n\|_X(F) < e$. We shall show that

$$\mu \times \nu(G) < d^2, \quad G \in \mathcal{S} \times \mathcal{T}$$

implies

$$\|m \times n\|_X(G) \leqq e(\|m\|_X(S) + \|n\|_X(T)).$$

Put

$$A = \{s\colon \nu(G_s) < d,\ s \in S\}.$$

Then

$$d^2 > \mu \times \nu(G) = \int_S \nu(G_s)\,\mathrm{d}\mu(s) \geqq \int_{S-A} \nu(G_s)\,\mathrm{d}\mu(s) \geqq$$

$$\geqq d\mu(S-A),$$

hence $\mu(S-A) < d$ and thus $\|m\|_X(S-A) < e$.

Let $x_i \in X$, $i = 1, \ldots, r$, $\|x_i\| \leqq 1$, be arbitrary. Take arbitrary partition $G = \bigcup_{i=1}^{r} G_i$, $G_i \in \mathcal{S} \times \mathcal{T}$ with $G_i$ disjoint. For every $s \in S$ we have by Lemma 1

$$\left\|\sum_{i=1}^{r} x_i n(G_i)_s\right\| \leqq \|n\|_X(G_s) \leqq \|n\|_X(T) < \infty.$$

The function

$$s \to \sum_{i=1}^{r} x_i n((G_i)_s)$$

is $m$-measurable by Theorem 4 and since it is bounded on $S$, it is $m$-integrable [1, Th. 3] and we have

$$\left\|\int_A \left[\sum_{i=1}^{r} x_i n((G_i)_s)\right] \mathrm{d}m(s)\right\| \leqq \sup_{s \in A} \left\|\sum_{i=1}^{r} x_i n((G_i)_s)\right\| \cdot \|m\|_X(A) \leqq$$

$$\leqq \|m\|_X(A) \sup_{s \in A} \|n\|_X(G_s) \leqq \|m\|_X(S)e,$$

since for $s \in A$ we have $\nu(G_s) < d$ and thus $\|n\|_X(G_s) < e$. Further

$$\left\|\int_{S-A} \sum_{i=1}^{r} x_i n((G_i)_s)\,\mathrm{d}m(s)\right\| \leqq \sup_{s \in S-A} \left\|\sum_{i=1}^{r} x_i n((G_i)_s)\right\| \cdot \|m\|_X(S-A) \leqq$$

$$\leqq \sup_{s \in S-A} \|n\|_X(G_s)\|m\|_X(S-A) \leqq \|n\|_X(T)e.$$

Now using Theorem 4 we have

$$\left\|\sum_{i=1}^{r} x_i m \times n(G_i)\right\| = \left\|\sum_{i=1}^{r} x_i \int_S n((G_i)_s)\,\mathrm{d}m(s)\right\| \leqq$$

$$\leqq \left\| \int_A \left[ \sum_{i=1}^{r} x_i n((G_i)_s) \right] \mathrm{d}m(s) \right\| + \left\| \int_{S-A} \sum_{i=1}^{r} x_i n((G_i)_s \, \mathrm{d}m(s) \right\| \leqq$$

$$\leqq e\|m\|_X(S) + e\|n\|_X(T).$$

Since $G_i$ are arbitrary, it follows that

$$\|m \times n\|_X(G) \leqq e(\|m\|_X(S) + \|n\|_X(T)).$$

**4.** In the following Lemma 7 and Theorem 8 we shall suppose that $\|m \times n\|_X$-zero sets and $\mu \times \nu$-zero sets coincide. We shall use the following fact.

**Lemma 7.** *Let $Z$ be an $\|m \times n\|_X$-zero set in $\mathscr{S} \times \mathscr{T}$ (or equivalently a $\mu \times \nu$-zero set). Then there exists an $\|m\|_X$-zero set (a $\mu$-zero set) $M$ such that for all $s \notin M$ we have $\|n\|_X(Z_s) = 0$ (or equivalently $\nu(Z_s) = 0$).*

This result is well known for scalar measures and since for the dominated measure $m$ ($n$) we may suppose (Lemma 2) that $\|m\|_X$-zero sets and $\mu$-zero sets ($\|n\|_X$-zero sets and $\nu$-zero sets) coincide, our lemma holds for dominated measures.

Let $m$ and $n$ be dominated and suppose that $f$ is an $m \times n$-integrable function on $S \times T$ and let $g$ differ from $f$ only on an $\|m \times n\|_X$-zero set, say $Z$. By Lemma 7 there exists an $\|m\|_X$-zero set $M$ in $\mathscr{S}$ such that for all $s \notin M$ the maps $f_s$ and $g_s$ differ only on an $\|n\|_X$-zero set. Thus $f_s$ is $n$-integrable if and only if $g_s$ is $n$-integrable and if this is the case, the integrals with respect to $n$ will be equal. We shall use this fact.

We shall prove now the Fubini-type theorem for bounded functions integrable in the sense of Bartle [1].

**Theorem 8.** *Let $m\colon \mathscr{S} \to X$ and $n\colon \mathscr{T} \to X$ be dominated vector measures. Let $f$ be a bounded $m \times n$-measurable (hence $m \times n$-integrable) function on $S \times T$ to $X$. Then for $\|m\|_X$-almost all $s \in S$, the map $f_s$ is $n$-integrable, the map given by*

$$s \to \int_T f_s \, \mathrm{d}m$$

*for $\|m\|_X$-almost all $s$ (and defined arbitrarily for other $s$) is $m$-integrable and we* have

$$\int_{S \times T} f \, \mathrm{d}m \times n = \int_S \int_T f_s \, \mathrm{d}n \, \mathrm{d}m(s).$$

Proof. We can find a bounded sequence $f_r$ of $m \times n$-simple functions which converges to $f$ $\|m \times n\|_X$-almost everywhere on $S \times T$ and for every $G \in \mathscr{S} \times \mathscr{T}$ it is true

$$\int_G f \, \mathrm{d}m \times n - \int_G f_r \, \mathrm{d}m \times n \to 0, \quad r \to \infty,$$

in the norm of $X$ [1].

We let $Z$ be an $\|m \times n\|_X$-zero set in $\mathscr{S} \times \mathscr{T}$ such that $f_r$ converges pointwise of $f$ outside $Z$. We let $M$ be an $\|m\|_X$-zero set in $\mathscr{S}$ such that for all $s \notin M$ we have

$$\|n\|_X(Z_s) = 0.$$

If $s \notin M$, it follows that $f_{r,s}$ converges pointwise to $f_s$ on the complement of $Z_s$. For every $s \in S$, $f_{r,s}$ is an $n$-simple function on $T$, and for $x \in X$, the formula

$$(xc_{A \times B})_s = xc_A(s)c_B$$

shows that for each $r$, the map

$$g_r \colon s \to f_{r,s}$$

is an $m$-simple function on $S$ with values in the space of $n$-simple functions on $T$.

If $s \notin M$, then $f_{r,s}(t)$ converges to $f_s(t)$ for $\|n\|_X$-almost all $t \in T$. We conclude that $f_s$ is $n$-measurable and $n$-integrable and

$$\int_F f_{r,s}\,\mathrm{d}n \quad \text{converges to} \quad \int_F f_s\,\mathrm{d}n$$

for all $s \notin M$ and all $F \in \mathscr{T}$ [1, p. 345].

Finally we note that the map

$$h_r \colon s \to \int_T f_{r,s}\,\mathrm{d}n$$

is an $m$-simple function on $S$ with values in $X$.

For all $s \notin M$, the sequence $h_r$ converges to the map $h$,

$$h(s) = \int_T f_s\,\mathrm{d}n,$$

hence $h$ is $m$-measurable [1, p. 346] and $m$-integrable [1, Th. 3]. Further, [1, Th. 7],

$$\int_S \int_T f_{r,s}\,\mathrm{d}n\,\mathrm{d}m(s) \quad \text{converges to} \quad \int_S \int_T f_s\,\mathrm{d}n\,\mathrm{d}m(s).$$

Since $f_r$ is an $m \times n$-simple function and by Theorem 4 we have

$$\int_S \int_T f_{r,s}\,\mathrm{d}n\,\mathrm{d}m(s) = \int_{S \times T} f_r\,\mathrm{d}m \times n,$$

our result follows.

We can prove another Fubini-type theorem. Let $m \colon \mathscr{S} \to X$ and $n \colon \mathscr{T} \to X$ be vector measures dominated by $\mu$ and $\nu$, respectively. Let $f \colon S \times T \to X$ be a $\mu \times \nu$-measurable function, i.e. $f$ is a limit $\mu \times \nu$-almost everywhere (hence $\|m \times n\|_X$-almost everywhere) of a sequence $f_r$ of $\mu \times \nu$-simple (hence

$m \times n$-simple) functions. If, moreover, $f$ is bounded, then $f$ is $m \times n$-integrable [1, Th. 3]. We have then the following.

**Theorem 9.** *Let $f$: $S \times T \to X$ be a bounded $\mu \times \nu$-measurable function. Then $f$ is an $m \times n$-integrable function for $m$ and $n$ dominated. Further, for $\|m\|_X$-almost all $s \in S$, the map $f_s$ is $n$-integrable, the map given by*

$$s \to \int_T f_s \, \mathrm{d}n$$

*for $\|m\|_X$-almost all $s$ (and defined arbitrarily for other $s$) is $m$-integrable and we* have

$$\int_{S \times T} f \, \mathrm{d}m \times n = \int_S \int_T f_s \, \mathrm{d}n \, \mathrm{d}m(s).$$

The proof of this theorem is the same as that of Theorem 8. Note that the assumption made at the beginning of the section is now unnecessary.

**5.** Concluding remarks. In the paper [9] there are considered the product, Fubini-type theorem and convolution for vector measures with finite variation defined on Borel sets of locally compact Hausdorff spaces. For a compact semigroup the similar questions are treated in [7, 8]. The paper [2] generalizes the theorem of Fubini in the context of integration theory presented in the book of N. Dinculeanu [3]. The Fubini-type theorem for vector measures which are indefinite Pettis integrals is established in the paper [10].

Another approach to the problem of the product of vector measures and to the Fubini theorem for operator valued measures is possible in the context of integration theory developed in the paper [4].

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## ПРОИЗВЕДЕНИЕ ДОМИНИРОВАНЫХ ВЕКТОРНЫХ МЕР

Милослав Духонь

Резюме

Пусть $X$ – алгебра Банаха. Векторная мера $m$ определенная на $\sigma$-алгебре $\mathcal{S}$ со значениями в $X$ называется доминированой относительно $X$, если существует неотрицательная конечная мера $\mu$ на $\mathcal{S}$ такая, что

$$\lim_{\mu(A)\to 0} \|m\|_X(A) = \lim_{\mu(A)\to 0} \sup \|\Sigma\, x_i m(A_i)\| = 0, \quad A \in \mathcal{S},$$

где sup берется для всех конечных непересекающихся семейств

$$A_i \in \mathcal{S} \quad \text{и} \quad \cup A_i = A, \quad x_i \in X, \quad \|x_i\| \leqq 1, \quad i = 1, \ldots, r.$$

Главным результатом статьи является следующая теорема.

**Теорема.** *Пусть $m: \mathcal{S} \to X$ и $n: \mathcal{T} \to X$ векторные меры доминированные относительно $X$ с $\mu$ и $\nu$, соответственно. Тогда произведение*

$$m \times n: \mathcal{S} \times \mathcal{T} \to X$$

*существует и доминировано относительно $X$ с $\mu \times \nu$.*

Доказаны также некоторые обобщения теоремы Фубини для векторных функций и векторных мер.