Ľubica Holá
On determining sets for the class of somewhat continuous functions

Mathematica Slovaca, Vol. 39 (1989), No. 3, 257--259

Persistent URL: http://dml.cz/dmlcz/129533

Math. Slovaca 39, 1989, No. 3, 257—259

ON DETERMINING SETS FOR THE CLASS OF SOMEWHAT CONTINUOUS FUNCTIONS

ĽUBICA HOLÁ

This paper positively answers a question asked in [1].

Let X, Y be sets. Let $F_{X,Y}$ be a class of functions from X to Y. A set $D \subset X$ is called a determining set for $F_{X,Y}$ if each two members of $F_{X,Y}$ which agree on D must agree on all of X. Denote by $\mathscr{D}(F_{X,Y})$ the family of all determining sets for $F_{X,Y}$.

Let X, Y be topological spaces. A function $f\colon X \to Y$ is said to be somewhat continuous if for each set $V \subset Y$ open in Y such that $f^{-1}(V) \neq \emptyset$ there exists a nonempty open set $U \subset X$ so that $U \subset f^{-1}(V)$. (See [2].) In the sequel $S_{X,Y}$ denotes the set of all somewhat continuous functions from X to Y.

Let e, $t \in Y$, $e \neq t$. If A is a subset of X, then the characteristic function of A is the function $\chi_A^{e,t}\colon X \to Y$

$$\chi_A^{e,t}(x) = \begin{cases} e & \text{for } x \in X - A \\ t & \text{for } x \in A\,. \end{cases}$$

Denote by $\chi_{X,Y}^{e,t}$ the class of all characteristic functions of the form $\chi_A^{e,t}\colon X \to Y$. We assume throughout this paper that the set Y has at least two elements.

Let X, Y be topological spaces and A be a nonempty subset of X. Let e, $t \in Y$, $e \neq t$. Consider the following statements:

(i) $A \in \mathscr{D}(S_{X,Y})$

(ii) $A \in \mathscr{D}(S_{X,Y} \cap \chi_{X,Y}^{e,t})$

(iii) for each $L \subset K \subset X$, $\emptyset \neq K - L \subset X - A$, some of the following assertions holds:

(a) $\operatorname{Int} K = \emptyset$

(b) L is dense in X

(c) $\operatorname{Int} L = \emptyset$ and $L \neq \emptyset$

(d) K is dense in X and $K \neq X$.

If Y is a Urysohn space, then the statements (i), (ii) and (iii) are equivalent. (See [1].)

We show that if Y is a Hausdorff space, then the statements (i), (ii) and (iii) are also equivalent. This positively answers a question asked in [1].

Theorem 1. *Let X, Y be topological spaces, Y be a Hausdorff space. Let A be a nonempty subset of X. Let $e, t \in Y$, $e \neq t$. Then the statements* (i), (ii) *and* (iii) *are equivalent.*

Proof. The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are obvious from [1].

(iii) $\Rightarrow$ (i): By contradiction. Suppose that (iii) holds and there exist two different functions $f, g \in S_{X,Y}$ which agree on the set A. Choose $a, b \in X$ such that

(1) $f(a) = g(a)$ and $f(b) \neq g(b)$.

First we shall prove that for each nonempty open set $G \subset X$ such that Cl $G \neq X$ we have

(2) $G \cap A \neq \emptyset$.

Suppose that there exists a nonempty open subset G of X such that Cl $G \neq X$ and $G \cap A = \emptyset$. Then for the sets $\emptyset$ and G no assertion from (iii) holds. Thus $G \cap A \neq \emptyset$.

By (1) either $f(a) \neq f(b)$ or $g(a) \neq g(b)$. Suppose that $f(a) \neq f(b)$. The other case is similar. Choose U, V and T open neighbourhoods of the points $f(b)$, $g(b)$ and $f(a)$ respectively, such that

(3) $U \cap V = \emptyset$ and $U \cap T = \emptyset$.

The somewhat continuity of f implies that $\operatorname{Int} f^{-1}(U) \neq \emptyset \neq \operatorname{Int} f^{-1}(T)$. By (3) $\operatorname{Int} f^{-1}(U) \cap \operatorname{Int} f^{-1}(T) = \emptyset$, thus $\operatorname{Int} f^{-1}(U)$ is not dense in X. Since f, $g \in S_{X,Y}$ we have

(4) $\operatorname{Int} f^{-1}(U) \neq \emptyset \neq \operatorname{Int} g^{-1}(V)$. Put $W = \operatorname{Int} f^{-1}(U) \cap \operatorname{Int} g^{-1}(V)$. We shall prove that

(5) $W = \emptyset$.

By contradiction. Suppose that $W \neq \emptyset$. Since $W \subset \operatorname{Int} f^{-1}(U)$ and $\operatorname{Int} f^{-1}(U)$ is not dense in X, W is not dense in X. Thus by (2) we have $W \cap A \neq \emptyset$. Choose $z \in W \cap A$. Then $f(z) \in U$, $g(z) \in V$ and $f(z) = g(z)$. This is contrary to (3). In the following we distinguish three cases

e) Suppose that $\operatorname{Int} f^{-1}(U) - \operatorname{Cl}\{b\} = \emptyset = \operatorname{Int} g^{-1}(V) - \operatorname{Cl}\{b\}$. By (4) we obtain $b \in \operatorname{Int} f^{-1}(U)$ and $b \in \operatorname{Int} g^{-1}(V)$. This is contrary to (5).

f) Suppose that $\operatorname{Int} f^{-1}(U) - \operatorname{Cl}\{b\} \neq \emptyset$. Putting in (iii) $K = \operatorname{Int} f^{-1}(U) \cup \{b\}$, $L = \operatorname{Int} f^{-1}(U) - \{b\}$, we obtain that

(6) $\operatorname{Int} f^{-1}(U) \cup \{b\}$ is dense in X.

Then by (5) we have $\emptyset \neq (\operatorname{Int} f^{-1}(U) \cup \{b\}) \cap \operatorname{Int} g^{-1}(V) = \{b\} \cap \operatorname{Int} g^{-1}(V)$. Thus

(7) $b \in \operatorname{Int} g^{-1}(V)$.

We distinguish two cases. First, suppose that $\operatorname{Int} g^{-1}(V) - \operatorname{Cl}\{b\} = \emptyset$. By (4) and (5) $\operatorname{Int} g^{-1}(V)$ is a nonempty open set which is not dense in X. Thus by (2) $\operatorname{Int} g^{-1}(V) \cap A \neq \emptyset$. Choose a point z in this intersection. Then $f(z) = g(z) \in V$. Hence $f^{-1}(V) \neq \emptyset$. Since $f \in S_{X,Y}$, we have $\operatorname{Int} f^{-1}(V) \neq \emptyset$. Then by (6) $\emptyset \neq (\operatorname{Int} f^{-1}(U) \cup \{b\}) \cap \operatorname{Int} f^{-1}(V) \subset f^{-1}(U) \cap f^{-1}(V)$, which contradicts (3).

Now, suppose that $\operatorname{Int} g^{-1}(V) - \operatorname{Cl}\{b\} \neq \emptyset$. Analogously as for (7) we obtain $b \in \operatorname{Int} f^{-1}(U)$. Thus $b \in W$. This is contrary to (5). This shows that the case f) is not true.

g) Suppose that $\operatorname{Int} g^{-1}(V) - \operatorname{Cl}\{b\} \neq \emptyset$. Analogously as for f) we obtain that the case g) is not true.

The proof is complete.

The following theorem is obvious.

Theorem 2. *Let X, Y be Hausdorff topological spaces.* Then $\mathscr{D}(S_{X,Y}) = \{X\}$.

REFERENCES

[1] J. DOBOŠ: On determining sets for certain generalizations of continuity. Math. Slovaca 37, 1987, 137—145.

[2] K. R. GENTRY—H. B. HOYLE: Somewhat continuous functions. Czech. Math. J. 21, 96, 1971, 5—12.

Received October 12, 1987

Katedra teórie pravdepodobnosti
a matematickej štatistiky MFF UK
Mlynská dolina
842 15 Bratislava

ОБ ОПРЕДЕЛЯЮЩИХ МНОЖЕСТВАХ ДЛЯ НЕМНОЖКО-НЕПРЕРЫВНЫХ ФУНКЦИЙ

Ľubica Holá

Резюме

Мы даем характеризацию определяющих множеств для немножко-непрерывных функций из топологического пространства X в топологическое пространство Хаусдорфа Y.