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Ján Andres

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SOLUTION WITH PERIODIC SECOND DERIVATIVE OF A CERTAIN THIRD ORDER DIFFERENTIAL EQUATION

JAN ANDRES

1. There are not so many results concerning the higher kind oscillations problems. As far as we know, although the second kind periodic solutions (in Minorsky's terminology) have been partly investigated especially by M. Farkas [1—3] (see also the references included), the solutions with the periodic second derivative (and not necessarily the first) have not yet been studied.

With respect to this the present note should have been originally devoted to the problem inserted in the title for the equation

$$x''' + ax'' + g(t, x') + h(x) = p(t), \tag{0}$$

where a is a constant, the functions $h(x), p(t) \in \mathscr{C}^1(\mathrm{R}^1)$ and the function $g(t, y) \in \mathscr{C}^1(\mathrm{R}^2)$ is bounded. But as we will show, the θ-periodicity of the function $g(t, y)$, i.e. $g(t + \theta, x'(t + \theta)) \equiv g(t, x'(t))$, implies the special degenerate type of the restoring term $h(x)$, namely $h(x) = cx$ (c-const.), and consequently our problem is simplified into the one for

$$x''' + ax'' + g(t, x') + cx = p(t). \tag{1}$$

We call *a periodic solution of the third kind* (PS3.K) such a solution $x(t)$ of the respective equation that

$$x(t) = x_0(t) + \frac{\omega_1}{2\theta}t^2 + \frac{\omega_2}{2\theta}t, \quad x_0(t + \theta) \equiv x_0(t) \tag{2}$$

is satisfied with suitable constants θ, ω_1, ω_2. Similarly for $\omega_1 = 0$ or $\omega_1 = 0 = \omega_2$ $x(t)$ will be explicitly called *a periodic solution of the second or the first kind* (PS2.K or PS1.K), respectively.

Note 1. *It is obvious that requiring the existence of* PS3.K *of* (0) *with* (2) *and a bounded* $g(t, y)$, *we will assume*

$$g(t + \theta, y) \equiv g(t, y) \equiv g(t, y + \omega_1), \tag{3}$$

$$h(x + \omega(t)) - h(x) \equiv p(t + \theta) - p(t), \tag{4}$$

where $\omega(t) = \omega_1(t + \theta/2) + \omega_2$.

Consequence 1. *Taking into account* (3), (4), *the necessary condition for the existence of* PS3.K *of* (0) *with* $\omega_1 \neq 0$ *is for the function* $h(x)$ *to be linear and hence*

$$p(t + \theta) \equiv p(t) + \alpha t + \beta, \tag{5}$$

where $\alpha = c\omega_1, \quad \beta = \alpha\theta/2 + c\omega_2$.

Indeed, from (4) it follows for any fixed $t_0 \in \mathrm{R}^1$ that

$$h(x + \omega(t_0)) - h(x) \equiv p(t_0 + \theta) - p(t_0) \ldots \text{const}.$$

or

$$\frac{\partial h(x + \omega(t_0))}{\partial x} \equiv \frac{\mathrm{d}h(x)}{\mathrm{d}x},$$

which implies that

$$\frac{\mathrm{d}h(x)}{\mathrm{d}x} \equiv c \quad \text{or} \quad h(x) \equiv cx + d$$

with suitable constants c, d.

Note 2. *Just in the same way there can be verified the necessary condition for the existence of* PS2.K *of* (0) *under* $\omega := \omega_2$ *and*

$$g(t + \theta, y) \equiv g(t, y), \tag{3'}$$

namely $h(x) \equiv cx + h_0(x)$, *where* $h_0(x + \omega) \equiv h_0(x)$.

Consequence 2. *Taking into account* (3′) *and the monotone function* $h(x)$, *the necessary condition for the existence of* PS2.K *of* (0) *is for the function* $h(x)$ *to be linear and hence*

$$p(t + \theta) \equiv p(t) + c\omega. \tag{5'}$$

Consequence 3. *The problem of the existence of* PS2.K *of* (1) *with* (5′) *is equivalent to the one of* PS1.K *of the equation*

$$x''' + ax'' + g_0(t, x') + cx = p_0(t), \tag{1_0}$$

where

$$g_0(t, y) = g(t, y + \omega/\theta) \text{ and } p_0(t + \theta) \equiv p_0(t).$$

Althoug we have got already some earlier results (cf. [4], [5]) dealing with the existence of PS1.K of (1) or (1_0), we will improve them in the line with our investigation in section 3 as well.

2. Consider (1) satisfying (3), (5) with respect to the problem of the existence of PS3.K, i.e. $x(t)$ with (2), resp.

$$(2') \qquad x(t+\theta) \equiv x(t) + \omega_1(t+\theta/2) + \omega_2$$

or for $\mu = 1$ and $k > 1$ ... a fixed real:

$$(2_\mu) \qquad \left| \begin{array}{l} x(\mu\theta) - x(0) - \mu^k(\mu\omega_1\theta/2 + \omega_2), \\ x'(\mu\theta) - x'(0) = \mu^k\omega_1, \\ x''(\mu\theta) - x''(0) = 0. \end{array} \right.$$

Solving the problem $(1) \cap (2_1)$, we employ modified *Poincaré's* (*or Levinson's*, resp. T-) *operator*:

$$T_\mu(X_0) := \left| \begin{array}{l} (x(\mu\theta;\ X_0) - x(0) = \mu^k(\mu\omega_1\theta/2 + \omega_2), x'(\mu\theta; X_0) - x'(0) - \mu^k\omega_1, \\ \qquad\qquad x''(\mu\theta; X_0) - x''(0)(\mu\theta)^{-1} \text{ for } \mu \in (0, 1\rangle, \\ (x'(0), x''(0), -ax''(0) - g(0, x'(0)) - cx(0) + p(0)) \text{ for } \mu = 0, \end{array} \right.$$

where $x(t; X_0) = x(t; x_0, x_0', x_0'') = x(t; x(0), x'(0), x''(0))$ is the solution $x(t)$ of (1) satisfying Cauchy's initial values:

$$x^{(j)}(0) = x_0^{(j)} \qquad j = 0, 1, 2.$$

It is clear that the problem $(1) \cap (2_1)$ is sollvable if and only if $T_1(X_0) = (0, 0, 0) := \boldsymbol{0}$.

Lemma 1. *The problem* $(1) \cap (2_1)$ *is solvable provided all solutions of* $(1) \cap (2_\mu)$ *are a priori bounded, uniformly with respect to* $\mu \in (0,1)$, *when* $c \neq 0$.

Proof. We will proceed here by a technique similar to that developed in [6]. The employed degree arguments can be found, e.g., in [7].

If the following relation is satisfied for $X_0 \in \operatorname{cl} I \setminus I$, where $I \subset \mathrm{R}^3$ is an open set symmetrical with respect to the origin $\boldsymbol{0}$:

$$(6) \qquad T_\mu(X_0) \neq \boldsymbol{0},$$

uniformly with respect to $\mu \in (0,1)$, then evidently instead of $T_1(X_0) \neq \boldsymbol{0}$ we can require

$$(7) \qquad T_0(X_0) \neq \boldsymbol{0},$$

or, resp., since the topological degree $\mathrm{d}[T_0(X_0) - T_0(-X_0), \operatorname{cl} I, \boldsymbol{0}]$ is for $X_0 \in \operatorname{cl} I \setminus I$ always different from zero, it is enough to require besides (6) instead of (7) only that

$$T_0(X_0) - (1-v)T_0(-X_0) \neq \boldsymbol{0} \qquad v \in (0, 1),$$

resp. for $|(p(0) - g(0,0))/c| \neq |x_0| > R$ ... a great enough number even only

$$\frac{p(0)-g(0,0)-cx_0}{|p(0)-g(0,0)-cx_0|} \neq \frac{p(0)-g(0,0)+cx_0}{|p(0)-g(0,0)+cx_0|},$$

which is certainly trivially fulfilled.

Since (6) may, however, be substituted for a suitable I under the condition of a priori boundedness of all solutions of $(1) \cap (2_\mu)$, uniformly with respect to $\mu \in (0,\ 1)$, the proof is complete.

Denoting

$$(8) \qquad x^* := \begin{cases} x & \text{for } |x| \leq R \\ R \operatorname{sgn} x & \text{for } |x| \geq R, \end{cases}$$

$$(9) \qquad G := \max_{\substack{t \in \langle 0, \theta \rangle \\ y \in \langle 0, \omega_1 \rangle}} |g(t,y)|, \quad P := \max_{t \in \langle 0, \theta \rangle} |p(t)|,$$

we can give

Lemma 2. *All solutions of* $(1) \cap (2_\mu)$ *are a priori bounded, uniformly with respect to* $\mu \in (0, 1)$, *provided*

$$(10) \qquad 0 \neq |c| < |\theta|^{-3}.$$

Proof. Let $x(t)$ be a fixed solution of

$$(1^*) \qquad x''' + ax'' + g(t,x') + cx^* = p(t)$$

satisfying (2_μ) for some $\mu \in (0, 1)$.

Substituting $x(t)$ into (1^*) and multiplying (1^*) by $x'''(t)$ we obtain after integration the identity

$$\int_0^{\mu\theta} x'''^2(t)\,\mathrm{d}t = \int_0^{\mu\theta} (p(t) - g(t,x'(t)) + cx^*(t))x'''(t)\,\mathrm{d}t$$

and from it by means of the Schwarz inequality and (8), (9) the relation

$$(11) \qquad \left| \int_0^{\mu\theta} x'''^2(t)\,\mathrm{d}t \right| \leq |\theta|(P + G + |cR|)^2.$$

Since such a point $t_1 \in \langle 0, \theta \rangle$ surely exists that

$$|x''(t)| \leq \left|\frac{\omega_1}{\theta}\right| + \left|\int_{t_1}^{t} |x'''(s)|\,\mathrm{d}s\right| \leq \left|\frac{\omega_1}{\theta}\right| + \sqrt{|\theta|}\left(\int_0^{|\theta|} x'''^2(t)\,\mathrm{d}t\right)^{\frac{1}{2}}$$

holds, from (11) it follows that

$$(12) \qquad |x''(t)| \leq |\theta|(P + G + |cR|) + |\omega_1/\theta| := D_2(R).$$

Similarly there certainly holds the inequality

$$\max_{t\in\langle 0,\theta\rangle} |x'(t)| \le |\omega_1| + |\omega_2/\theta| + |\theta D_2(R)| := D_1(R). \tag{13}$$

At last, after integration (1*), we obtain for $|x(t)| > R_1 \ge R$ with respect to (2_μ), (8) — (10) the inequality

$$|\mu c\theta R_1| < |c\int_0^{\mu\theta} |x^*(t)|\mathrm{d}t| \le |\mu a\omega_1| + |\mu\theta(P+G)|,$$

leading for $R_1 \ge (|a\omega_1| + |\theta|(P+G))/|c\theta| := R_2$
to a contradiction.

This time

$$\min_{t\in\langle 0,\theta\rangle} |x(t)| \le R$$

and with respect to (11) — (13) also

$$\max_{t\in\langle 0,\theta\rangle} |x(t)| \le R_2 + |\theta D_1(R)| \le R + 2|\theta\omega_1| + |\omega_2| + |\theta|^3(P+G+|cR|) :=$$
$$:= R_3 + R_4(R_5 + |cR|)$$

is valid, which implies for $|c| < R_4^{-1} := |\theta|^{-3}$, i.e. (10), (for more detail see, e.g., [8]) the existence of such a positive constant ε that

$$\max_{t\in\langle 0,\theta\rangle} |x(t)| \le D_0 := R > (R_3 + R_4 R_5)/\varepsilon \tag{14}$$

is satisfied. Hence from (12) — (14) there follows the existence of such a constant $D := 3\max(D_0, D_1, D_2)$, uniformly with respect to $\mu \in (0,1)$, that

$$\max_{t\in\langle 0,\theta\rangle} (|x(t)| + |x'(t)| + |x''(t)|) \le D.$$

Q. e. d.

As a direct consequence of the preceding two lemmas we can give the following principal result.

Theorem 1. *The equation* (1) *admits under* (3), (5) *and* (10) *a periodic solution of the third kind.*

3. Since PS2.K or PS1.K is evidently a special case of PS3.K, Theorem 1 may certainly be extended to the above types of solutions; nevertheless, condition (10) should not be too restrictive, as it can be practically omitted like in the following

Theorem 2. *The equation* (1) *possesses under* (3′), (5′), *resp.* (3′), *and* $p(t + \theta) \equiv p(t)$ *a periodic solution of the second kind, resp. the first kind, provided* $c \neq 0$ *only.*

Proof. Since (11) can be obviously rewritten here for $R = \infty$ and a fixed PS1.K $x(t)$ *as*

$$\left|\int_0^{\mu\theta} x'''^2(t)\,\mathrm{d}t\right| \leq |\theta|(P + G),$$

the relations (12) — (14) are satisfied without any modification, but independently of R in (12), (13). Hence all the foregoing arguments hold for any $c \neq 0$ without any loss of generality and therefore the assertion of our theorem follows directly from Lemma 1, Lemma 2 and Consequence 3. This completes the proof.

4. Example. The equation

$$x''' + ax'' - \lambda \operatorname{arc\,tg} x' + cx = p(t) + (1 - \lambda)\sin(2\pi x'/\omega_1)$$

has for $c \neq = \lambda$ according to Theorem 1 and Theorem 2 PS3.K and PS2.K or PS1.K assuming only (5) together with $|c| < |\theta|^{-3}$ and (5′) or $p(t + \theta) \equiv p(t)$, respectively; while for $c = 0$, $\lambda = 1$ and $p(t + \theta) \equiv p(t)$ it admits *simple continua* of PS1.K or PS2.K with respect to Consequence 3 and the assertion from [9].

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Katedra matematické analýzy a numer. matematiky
přírodoveděcká fakulta UP, Gottwaldova 15,
771 46 Olomouc

РЕШЕНИЕ С ПЕРИОДИЧЕСКОЙ ВТОРОЙ ПРОИЗВОДНОЙ ОДНОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ТРЕТЬЕГО ПОРЯДКА

Jan Andres

Резюме

В работе даются достаточные условия существования периодических решений третьего рода, т.е. решений, вторая призводная которых периодична, уравнения (1) на основании использования теории топологической степени отображения.

Эта задача решаема, если коэфициент $c \neq 0$ из (1) в третьей степени достаточно мал в сравнении с θ-периодом второй призводной решения.

Показывается тоже, что для любого $c \neq 0$ существуют периодические траектории второго и первого родов при экстремально слабых условиях.