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Mathematica Slovaca, Vol. 42 (1992), No. 1, 65--84

Persistent URL: `http://dml.cz/dmlcz/129519`

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Mathematica Slovaca

PERIODIC BOUNDARY VALUE PROBLEM IN HILBERT SPACE FOR DIFFERENTIAL EQUATION OF SECOND ORDER WITH REFLECTION OF THE ARGUMENT

BORIS RUDOLF

ABSTRACT. The differential equation $-x''+\alpha^2x+f\big(t,x(t),x(-t)\big)=h(t)$ with periodic boundary conditions is studied. The existence of a solution in case when f is a completely continuous operator and in case when f is only continuous and bounded is proved. The connectedness of the set of solutions is studied.

The aim of this paper is to extend the results of Chaitan P. Gupta [1] for the boundary value problems in a Hilbert space involving the reflection of the argument to the case of the periodic boundary conditions.

1. Some preliminary results

We deal with the differential equation

$$-x''+\alpha^2x+f\big(t,x(t),x(-t)\big)=h(t) \tag{1}$$

with periodic boundary conditions

$$x(-\pi)=x(\pi), \qquad x'(-\pi)=x'(\pi), \tag{2}$$

where $h(t)\colon \langle-\pi,\pi\rangle \to H$, $f(t,x,y)\colon \langle-\pi,\pi\rangle\times H\times H\to H$ and H is a *real Hilbert space with norm* $\|\cdot\|$.
We assume $\alpha\in\mathbb{R}$, $\alpha>0$.

We use the following function spaces:

$$L_1\big((-\pi,\pi),H\big) \quad \text{with norm} \quad \|u\|_1=\int_{-\pi}^{\pi}\|u(t)\|\,\mathrm{d}t,$$

AMS Subject Classification (1991): Primary 34G20.
Key words: Periodic BVP, Leray-Schauder theorem, Hilbert space.

$$L_2\big((-\pi,\pi),H\big) \quad \text{with norm} \quad \|u\|_2 = \Big(\int_{-\pi}^{\pi} \|u(t)\|^2\,\mathrm{d}t\Big)^{\frac{1}{2}},$$

$$C\,\big(\langle-\pi,\pi\rangle,H\big) \quad \text{with norm} \quad \|u\|_0 = \sup_{t\in\langle-\pi,\pi\rangle} \|u(t)\|,$$

and assume $h(t) \in L_1$ and f is a *completely continuous function.*

In the case $H = \mathbb{R}$ we obtain the scalar problem (1), (2), for which the homogeneous problem

$$-x'' + \alpha^2 x = 0 \tag{2}$$

has only trivial solution.

That means we can find the Green function

$$G(t,s) = \frac{1}{2\alpha}\frac{1}{\mathrm{e}^{2\alpha\pi}-1}\begin{cases} \mathrm{e}^{2\alpha\pi}\,\mathrm{e}^{\alpha(t-s)} + \mathrm{e}^{\alpha(s-t)} & -\pi \leqq t \leqq s \leqq \pi \\ \mathrm{e}^{\alpha(t-s)} + \mathrm{e}^{2\alpha\pi}\,\mathrm{e}^{\alpha(s-t)} & -\pi \leqq s \leqq t \leqq \pi \end{cases} \tag{3}$$

such that the scalar problem (1), (2) is equivalent to the equation

$$x(t) = \int_{-\pi}^{\pi} G(t,s)\big[h(s) - f\big(s,x(s),x(-s)\big)\big]\,\mathrm{d}s. \tag{4}$$

For reference to our first lemma see [1, p. 377]. (Though this lemma is not explicitly formulated there.)

LEMMA 1. *If the scalar problem* (1), (2) *is equivalent to the equation* (4), *then also the problem* (1), (2) *in the Hilbert space* H *is equivalent to the equation* (4), *and the Green function* $G(t,s)\colon \langle-\pi,\pi\rangle \times \langle-\pi,\pi\rangle \to \mathbb{R}$ *is given by* (3).

Using the Lemma 1 we obtain that the existence of a solution to the problem (1), (2) is equivalent to the existence of a fixed point for a completely continuous operator T.

LEMMA 2. *Let* $f\colon \langle-\pi,\pi\rangle \times H \times H \to H$ *be a completely continuous operator and* $h(t) \in L_1\big((-\pi,\pi),H\big)$.

Then the problem (1), (2) *is equivalent to the operator equation*

$$x = Tx \tag{5}$$

where T *is a completely continuous operator,* $T\colon C\big(\langle-\pi,\pi\rangle,H\big) \to C\big(\langle-\pi,\pi\rangle,H\big)$.

Proof. We define

$$Tx(t) = \int_{-\pi}^{\pi} G(t,s)\big[h(s) - f\big(s,x(s),x(-s)\big)\big]\,\mathrm{d}s. \tag{6}$$

Continuity of $G(t,s)$ on $\langle -\pi,\pi\rangle \times \langle -\pi,\pi\rangle$ implies the continuity of the function Tx, i.e. $Tx \in C(\langle -\pi,\pi\rangle, H)$.

Continuity of the operator T.
Let $x_n \to x$ in $C(\langle -\pi,\pi\rangle, H)$. Then

$$f(t,x_n(t),x_n(-t)) \to f(t,x(t),x(-t)) \qquad \text{for every} \quad t \in \langle -\pi,\pi\rangle.$$

Moreover, for every $n \in \mathbb{N}$, and every $t \in \langle -\pi,\pi\rangle$ there is

$$\|f(t,x_n(t),x_n(-t))\| \leqq M.$$

Then the Lebesgue convergence theorem implies

$$\int_{-\pi}^{\pi} G(t,s)\big[h(s) - f(s,x_n(s),x_n(-s))\big]\,\mathrm{d}s \to$$

$$\to \int_{-\pi}^{\pi} G(t,s)\big[h(s) - f(s,x(s),x(-s))\big]\,\mathrm{d}s.$$

From the inequality

$$\begin{aligned}\|Tx_n(t_1)-Tx_n(t_2)\| &\leqq \int_{-\pi}^{\pi} |G(t_1,s)-G(t_2,s)|\big(\|h(s)\|+\|f(s,x_n(s),x_n(-s))\|\big)\,\mathrm{d}s\\ &\leqq 2\pi\varepsilon\big(\|h(s)\| + M\big)\end{aligned}$$

we obtain that Tx_n converges uniformly to Tx.

Compactness of T.
Let $\{x_n\}$ be bounded in $C(\langle -\pi,\pi\rangle, H)$. Then $\{Tx_n\}$ is equicontinuous. The set $\{Tx_n(t), n \in \mathbb{N}\} \subset H$ is a relatively compact set for every $t \in \langle -\pi,\pi\rangle$. The *Theorem of Ascoli* [5, p. 18] implies now the complete continuity of the operator T.

The relative compactness of the set $\{Tx_n(t), n \in \mathbb{N}\}$ is proved in the following way. Denote the integral sum associated with the partition $[s_0 = -\pi, \dots, s_i, \dots, s_k = \pi]$, $s_{i+1} - s_i = \dfrac{2\pi}{k}$ as

$$I_k = \sum_{i=0}^{k} G(t,s_i) f(s_i, x_n(s_i), x_n(-s_i))(s_{i+1} - s_i).$$

The complete continuity of f and the continuity of G implies that for every $t \in \langle -\pi, \pi \rangle$

$$G(t, s_i) f\big(s_i, x_n(s_i), x_n(-s_i)\big) \in K,$$

where K is a compact subset of H. Then

$$I_k \in \operatorname{conv}(2\pi k)$$

and

$$T x_n(t) \in \overline{\operatorname{conv}(2\pi k)}$$

where the set $\overline{\operatorname{conv}(2\pi k)}$ is a compact subset.

The assumption of the complete continuity of the function f is essential. For further references to the preceding lemma see [6, pp. 281–282].

2. The estimations

In this section we derive the inequalities which we use to estimate the norm of a solution to the equation (5).

LEM[illegible] *Let* $y(t) \in AC(\langle -\pi, \pi \rangle, H)$, $y'(t) \in L_2((-\pi, \pi), H)$, $\int_{-\pi}^{\pi} y(t)\,\mathrm{d}t = 0$ *and* $y(t)$ *satisfies the periodic boundary conditions* (2). *Then*

$$\|y(t)\|_0 \leqq \sqrt{\frac{\pi}{2}} \|y'(t)\|_2. \tag{7}$$

P r o o f. We consider the real function $z(t) \in AC(\langle -\pi, \pi \rangle, \mathbb{R})$ such that $z'(t) \in L_2((-\pi, \pi), \mathbb{R})$, $\int_{-\pi}^{\pi} z(t)\,\mathrm{d}t = 0$, $z(-\pi) = z(\pi)$, $z'(-\pi) = z'(\pi)$.

The mean value theorem implies the existence of $t_0 \in (-\pi, \pi)$ such that $z(t_0) = 0$.

We consider now the function $z(t)$ on $\langle t_0, t_0 + 2\pi \rangle$, defined by $z(t) = z(t - 2\pi)$ for $t > \pi$. The inequality

$$|z(t)| \leqq \sqrt{\frac{\pi}{2}} \|z'(t)\|_{L_2}$$

is for such $z(t)$ derived in [4]. For $y(t)$ satisfying the assumptions of the lemma, there is $t_0 \in \langle -\pi, \pi \rangle$ such that

$$\|y\|_0 = \sup_{t \in \langle -\pi, \pi \rangle} \|y(t)\| = \|y(t_0)\|.$$

The Hahn-Banach theorem implies the existence of $w \in H$ with $\|w\| = 1$ such that

$$\|y(t_0)\| = \big(y(t_0), w\big).$$

We denote

$$z(t) = \big(y(t), w\big)$$

and we obtain

$$\big|\big(y(t), w\big)\big|^2 \leqq \frac{\pi}{2} \int_{-\pi}^{\pi} \big(y'(t), w\big)^2 \,\mathrm{d}t \leqq \frac{\pi}{2} \int_{-\pi}^{\pi} \|y'(t)\|^2 \,\mathrm{d}t,$$

$$\|y(t)\|_0 = \big(y(t_0), w\big) \leqq \sqrt{\frac{\pi}{2}} \|y'(t)\|_2$$

The following estimation is in a real case well known as the *Wirtinger inequality* [2, p. 185].

In the rest of this part we assume that H is a *separable Hilbert space* and $\{e_i\}$ is an *orthogonal basis in* H.

LEMMA 4. *Let* $y(t) \in C\big(\langle -\pi, \pi\rangle, H\big)$. *Then*

$$y(t) = \sum_{i=1}^{\infty} a_i(t) e_i,$$

where $a_i(t)$ *are uniformly continuous functions and*

$$\|y(t)\|_2^2 = \sum_{i=1}^{\infty} \int_{-\pi}^{\pi} |a_i(t)|^2 \,\mathrm{d}t.$$

Proof. For every $t_0 \in \langle -\pi, \pi\rangle$ is $y(t_0) \in H$, $y(t_0) = \sum\limits_{i=1}^{\infty} a_i(t_0) e_i$ and $a_i(t) = \big(a_i(t)e_i, e_i\big) = \big(y(t), e_i\big)$. The uniform continuity of $y(t)$ implies the uniform continuity of $a_i(t)$.

For $y(t) \in H$ we use the Parseval equality

$$\|y(t)\|^2 = \sum_{i=1}^{\infty} |a_i(t)|^2.$$

The sequence $\sum_{i=1}^{n} |a_i(t)|^2 \to \|y(t)\|^2$ for $n \to \infty$, for every t and

$$\sum_{i=1}^{n} |a_i(t)|^2 \leqq \|y(t)\|^2.$$

The Lebesgue dominated convergence theorem implies that

$$\|y(t)\|_2^2 = \sum_{i=1}^{\infty} \int_{-\pi}^{\pi} |a_i(t)|^2 \, dt.$$

LEMMA 5. *Let $y(t) \in C^1(\langle -\pi, \pi \rangle, H)$. Then*

$$y'(t) = \sum_{i=1}^{\infty} a_i'(t) e_i,$$

where $a_i'(t)$ are uniformly continuous functions and

$$\|y'(t)\|_2^2 = \sum_{i=1}^{\infty} \int_{-\pi}^{\pi} |a_i'(t)|^2 \, dt.$$

Proof. From the Lemma 4 we obtain that

$$y(t) = \sum_{i=1}^{\infty} a_i(t) e_i \qquad \text{and}$$

$$y'(t) = \sum_{i=1}^{\infty} b_i(t) e_i, \qquad \text{where } b_i(t) \text{ are uniformly continuous functions.}$$

Moreover $y \in C^1$ implies that

$$a_i(t) = \big(y(t), e_i\big) \in C^1\big(\langle -\pi, \pi \rangle, \mathbb{R}\big) \quad \text{and}$$

$$b_i(t) = \Big(\sum_{i=1}^{\infty} b_j(t) e_y, e_i\Big) = \big(y'(t), e_i\big) = a_i'(t).$$

The rest of the proof is similar to the proof of the Lemma 4.

LEMMA 6. *Let* $y(t) \in C^1(\langle -\pi, \pi\rangle, H)$ *,* $y(t)$ *satisfies* (2) *and* $\int_{-\pi}^{\pi} y(t)\,\mathrm{d}t = 0$ *.*
Then

$$\|y(t)\|_2 \leqq \|y'(t)\|_2 \tag{8}$$

Proof. Obviously

$$\int_{-\pi}^{\pi} a_i(t)\,\mathrm{d}t = 0 \qquad \text{for every} \quad i \in \mathbb{N}$$

and $a_i(t)$ satisfies (2).
From the Wirtinger inequality we obtain

$$\int_{-\pi}^{\pi} |a_i(t)|^2\,\mathrm{d}t \leqq \int_{-\pi}^{\pi} |a_i'(t)|^2\,\mathrm{d}t$$

and the inequality (8) follows now from the Lemmas 4 and 5.

LEMMA 7. *Let* H *be a separable Hilbert space and* $\{e_i\}$ *the orthonormal basis in* H *. Then* $\{e_i, \cos kt \cdot e_i, \sin kt \cdot e_i\}_{i,k=1}^{\infty}$ *is the orthogonal basis in* $L_2((-\pi, \pi), H)$ *.*

Proof. The orthogonality is obvious. We prove the completeness. Let $y(t) \in C(\langle -\pi, \pi\rangle, H)$. Then $y(t) = \sum_{i=1}^{\infty} a_i(t)e_i$, $a_i(t)$ are uniformly continuous functions and

$$a_i(t) = \frac{a_0^i}{2} + \frac{1}{\pi}\sum_{k=1}^{\infty} a_k^i \cos kt + b_k^i \sin kt.$$

Supposing
$\int_{-\pi}^{\pi} (y(t), e_i)\,\mathrm{d}t = 0$, $\int_{-\pi}^{\pi} (y(t), \cos kt \cdot e_i)\,\mathrm{d}t = 0$, $\int_{-\pi}^{\pi} (y(t), \sin kt \cdot e_i)\,\mathrm{d}t = 0$
we obtain $\int_{-\pi}^{\pi} a_i(t)\,\mathrm{d}t = 0$, $\int_{-\pi}^{\pi} \cos kt \cdot a_i(t)\,\mathrm{d}t = 0$, $\int_{-\pi}^{\pi} \sin kt \cdot b_i(t)\,\mathrm{d}t = 0$ and then $a_0^i = 0$, $a_k^i = 0$, $b_k^i = 0$.
This implies that $a_i(t) = 0$ for every $i \in \mathbb{N}$ and then also $y(t) = 0$.

Since the space $C(\langle -\pi, \pi\rangle, H)$ is dense in $L_2((-\pi, \pi), H)$, then the system $\{e_i, \cos kt \cdot e_i, \sin kt \cdot e_i\}$ is complete.

Lemma 8. *Let* $y(t) \in C^1(\langle -\pi, \pi\rangle, H)$. *Then*

$$\|y(t)\|_0 \leqq a\|y'(t)\|_2 + b\|y(t)\|_2 \qquad a, b \in \mathbb{R} \tag{9}$$

P r o o f . The continuity of $y(t)$ implies that there is t_0 such that

$$\|y(t_0)\| = \sup_{t\in\langle -\pi,\pi\rangle} \|y(t)\| = \|y(t)\|_0 .$$

We choose again $w \in H$, $\|w\| = 1$ such that

$$\big(y(t_0), w\big) = \|y(t_0)\|.$$

Then

$$\big(y(t), w\big) = \big(y(t_1), w\big) + \int_{t_1}^{t} \big(y(s), w\big)' \,\mathrm{d}s = \big(y(t_1), w\big) + \sqrt{2\pi}\|y'(t)\,|_2 .$$

Using the mean-value theorem we take $t_1 \in \langle -\pi, \pi\rangle$ such that

$$\int_{-\pi}^{\pi} \big(y(t), w\big)^2 \,\mathrm{d}t = \big(y(t_1), w\big)^2 2\pi$$

and

$$\big(y(t), w\big) \leqq \sqrt{\frac{1}{2\pi}}\|y(t)\|_2 + \sqrt{2\pi}\|y'(t)\|_2 \qquad \text{for every} \quad t \in \langle -\pi, \pi\rangle .$$

3. Existence theorems

Theorem 1. *Let* $f\colon \langle -\pi, \pi\rangle \times H \times H \to H$ *be completely continuous operator and for every* $(t, x, y) \in \langle -\pi, \pi\rangle \times H \times H$ *is*

$$\big(f(t, x, y), x\big) \geqq -a\|x\|^2 - b\|x\|\|y\|, \qquad \textit{where} \quad a + |b| < \alpha^2 .$$

Then there is a solution to the problem (1), (2) *for every* $h(t) \in L_1\big((-\pi, \pi), H\big)$.

P r o o f . The problem (1), (2) is equivalent to the equation

$$x = Tx \tag{5}$$

where T is a completely continuous operator.

Let x be a solution to the equation

$$x = \lambda T x \qquad \text{for} \quad \lambda \in (0,1). \tag{10}$$

Then

$$-\int_{-\pi}^{\pi} \big(x''(t), x(t)\big)\,\mathrm{d}t + \int_{-\pi}^{\pi} \alpha^2 \big(x(t), x(t)\big)\,\mathrm{d}t + \int_{-\pi}^{\pi} \lambda\big(f\big(t, x(t), x(-t)\big), x(t)\big)\,\mathrm{d}t$$

$$= \int_{-\pi}^{\pi} \big(h(t), x(t)\big)\,\mathrm{d}t,$$

and

$$\|x'\|_2^2 + \alpha^2\|x\|_2^2 - \big(a + |b|\big)\|x\|_2^2 \leqq \|h(t)\|_1 \left(\sqrt{\frac{1}{2\pi}}\|x\|_2 + \sqrt{2\pi}\|x'\|_2\right).$$

The last inequality can be rewritten in the form

$$\|x'\|_2^2 - A\|x'\|_2 + \big(\alpha^2 - (a + |b|)\big)\|x\|_2^2 - B\|x\|_2 \leqq 0 \tag{11}$$

where A, B are constants.

Obviously if (11) is valid then $\|x\|_2 \leqq C_1$ and $\|x'\|_2 \leqq C_2$, C_1, C_2 are suitable constants.

Then if x is a solution to (10), there holds

$$\|x\|_0 \leqq \sqrt{\frac{1}{2\pi}} C_1 + \sqrt{2\pi} C_2 = C.$$

The existence of the solution to the equation (5) follows from the *Leray-Schauder theorem.*

THEOREM 2. *Let H be a separable Hilbert space, $f\colon \langle -\pi, \pi\rangle \times H \times H \to H$ be a completely continuous operator. Suppose that there are $a, b, c, d, e \in \mathbb{R}$ such that $a + |b| < 1 + \alpha^2$ and*

$$\big(f(t,x,y), x\big) \geqq -a\|x\|^2 - b\|x\|\|y\| - c\|x\| - d\|y\| - e$$

for every $(t,x,y) \in \langle -\pi, \pi\rangle \times H \times H$.

Suppose that either

(i) *if* $\int\limits_{-\pi}^{\pi} x(t)\,\mathrm{d}t = 0$ *and* $\int\limits_{-\pi}^{\pi} y(t)\,\mathrm{d}t = 0$, *then* $\int\limits_{-\pi}^{\pi} f\big(t, x(t), y(t)\big)\,\mathrm{d}t = 0$,

or

(ii) $\int\limits_{-\pi}^{\pi}\int\limits_{-\pi}^{\pi} \big(f\big(t, x(t), x(-t)\big), x(s)\big)\,\mathrm{d}t\,\mathrm{d}s \geqq 0$,

or

(iii) $$\liminf_{\substack{x\in S\\ \|x(t)\|_2\to\infty}} \frac{\int\limits_{-\pi}^{\pi}\int\limits_{-\pi}^{\pi} \big(f\big(t, x(t), x(-t)\big), x(s)\big)\,\mathrm{d}t\,\mathrm{d}s}{\Big\|\int\limits_{-\pi}^{\pi} x(t)\,\mathrm{d}t\Big\|^2} > -\alpha^2,$$

where $S = \Big\{x(t),\ \int\limits_{-\pi}^{\pi} x(t) \neq 0\Big\}$,

holds.

Then there is a solution to the problem (1), (2) *for every* $h(t) \in L_1\big((-\pi,\pi), H\big)$, $\int\limits_{-\pi}^{\pi} h(t)\,\mathrm{d}t = 0$.

P r o o f. The problem (1), (2) is equivalent to the equation (5). At first we prove that under the condition (i) there is

$$T(K) \subset K \tag{12}$$

where

$$K = \Big\{x(t) \in C,\ \int_{-\pi}^{\pi} x(t)\,\mathrm{d}t = 0\Big\}.$$

Operator T is given b (6) and it is ea y to prove that

$$\int_{-\pi}^{\pi} h(t)\,\mathrm{d}t - 0 \qquad \text{impli} \qquad \int_{-\pi}^{\pi}\left[\int_{-\pi}^{\pi} G(t,s)h(s)\,\mathrm{d}s\right] \text{lt} \quad 0.$$

It ıs obvious no to s e that the conditıon (i) impl'e (12)

Let $x(t) \in K$ be a solution to th equatıon

$$\lambda T\ , \qquad \lambda \in (0\ 1) \tag{10}$$

Then

$$-x''(t) + \alpha^2 x(t) + \lambda f(t\quad (t)\quad (-t))\quad \lambda\quad t) \tag{13}$$

and

$$\|x'\|_2^2 + \alpha^2\|x\|_2^2 - a\|x\|_2^2 - |b|\|x\|_2^2 - 2\pi(c+d)\|x\|_0 - \|h(t)\|_1\|x\|_0 - \mathrm{e} \leqq 0. \quad (14)$$

We use the inequalities (7) and (8) from Lemmas 3 and 6. Supposing $\alpha^2 - (a+|b|) < 0$, we obtain

$$\bigl(1+\alpha^2 - (a+|b|)\bigr)\|x'\|_2^2 - B\|x'\|_2 - \mathrm{e} \leqq 0$$

where $B = \bigl(2\pi(c+d) + \|h(t)\|_1\sqrt{\frac{\pi}{2}}$ is a constant.

The last inequality implies that $\|x'(t)\|_2 \leqq C_1$, where C_1 is a suitable constant.

In case $\alpha^2 - (a+|b|) \geqq 0$ we argue similarly as in the proof of the preceding theorem.

In both cases we obtain the estimation

$$\|x(t)\|_0 \leqq C,$$

and we can use the Leray-Schauder theorem in subspace K. This theorem implies the existence of a solution $x(t) \in K$ to the equation (5).

In case that (ii) or (iii) holds, we prove that if $x(t) \in C(\langle -\pi,\pi\rangle, H)$ is a solution to (10) then

$$\int_{-\pi}^{\pi} x(t)\,\mathrm{d}t = 0.$$

Equation (13) implies that

$$\begin{aligned} &\alpha^2 \int_{-\pi}^{\pi} x(t)\,\mathrm{d}t + \lambda \int_{-\pi}^{\pi} f\bigl(t, x(t), x(-t)\bigr)\,\mathrm{d}t = 0 \qquad \text{and} \\ &\alpha^2 \Bigl\| \int_{-\pi}^{\pi} x(t)\,\mathrm{d}t \Bigr\|^2 + \lambda \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \bigl(f(t,x(t),x(-t)), x(s)\bigr)\,\mathrm{d}t\,\mathrm{d}s = 0. \end{aligned} \quad (15)$$

Condition (ii) implies that $\int_{-\pi}^{\pi} x(t)\,\mathrm{d}t = 0$.

Now using the same argumentation as in the preceding part we obtain that for a solution $x(t)$ to the equation (10) holds

$$\|x(t)\|_0 \leqq C.$$

The existence of a solution to the equation (5) follows again from the Leray-Schauder theorem.

Case (iii).

It follows from (15) that

$$-\alpha^2\left\|\int\limits_{-\pi}^{\pi} x(t)\,\mathrm{d}t\right\|^2 > \int\limits_{-\pi}^{\pi}\int\limits_{-\pi}^{\pi}\big(f\big(t,x(t),x(-t)\big),x(s)\big)\,\mathrm{d}t\,\mathrm{d}s.$$

We use (iii) and choose C_1 such that for every $x(t)$, $\|x(t)\|_2 > C_1$ is

$$\frac{\int\limits_{-\pi}^{\pi}\int\limits_{-\pi}^{\pi}\big(f\big(t,x(t),x(-t)\big),x(s)\big)\,\mathrm{d}t\,\mathrm{d}s}{\left\|\int\limits_{-\pi}^{\pi} x(t)\,\mathrm{d}t\right\|^2} \geqq -\alpha^2.$$

Last two inequalities are in a contradiction, which implies that $\|x(t)\|_2 \leqq C_1$ for every solution to (10).

This estimation and the inequality (14) give the inequality

$$\|x'\|_2^2 - A\|x'\|_2 - B \leqq 0,$$

where A, B are constants. From the last inequality we obtain the estimation

$$\|x'\|_2 \leq C_2.$$

Finally, from the inequality (9) follows that

$$\|x\|_0 \leqq \sqrt{\frac{1}{2\pi}}C_1 + \sqrt{2\pi}C_2 = C,$$

and we can again use the Leray-S hauder theorem.

4. Existence when f is continuous

The continuity instead of the complete continuity of the operator f is assumed in this part. The operator T, defined by (6), is not nece arily completely continuous. Adding other assumptions for the operator f we can prove the existence and uniquene s of the solution to the problem (1) (2) al o in this ca e.

In following we a ume that

(A) H is s parable Hilbert space, $\{e_i\}$ is the orthonormal ba is in H, the operator $f\colon \langle -\pi,\pi\rangle \times H \quad H \to H$ is contin us and bounded and $h(t) \in L_2\big((-\pi,\pi),H\big)$.

THEOREM 3. *Assume that* (A) *holds and that for every* $x, y, u, v \in H$ *and every* $t \in \langle -\pi, \pi \rangle$

$$\big(f(t,x,y) - f(t,u,v), x - u\big) \geqq -a\|x-u\|^2 - b\|x-u\|\|y-v\|, \qquad (16)$$

where $a + |b| < \alpha^2$.

Then there is a unique solution to the problem (1), (2).

Proof.

Uniqueness.

Let x_1, x_2 be two solutions to the problem (1), (2), $x_1, x_2 \in C_1\big(\langle -\pi, \pi\rangle, H\big)$. Then

$$-(x_1 - x_2)'' + \alpha^2(x_1 - x_2) + f\big(t, x_1(t), x_1(-t)\big) - f\big(t, x_2(t), x_2(-t)\big) = 0$$

and

$$\|x_1' - x_2'\|_2^2 + \alpha^2\|x_1 - x_2\|_2^2 + \int_{-\pi}^{\pi} \Big(f\big(t, x_1(t), x_1(-t)\big) - f\big(t, x_2(t), x_2(-t)\big), x_1(t) - x_2(t)\Big)\,\mathrm{d}t = 0.$$

Using (16) we obtain

$$\|x_1' - x_2'\|_2^2 + \big(\alpha^2 - a - |b|\big)\|x_1 - x_2\|_2^2 \leqq 0. \qquad (17)$$

The last inequality implies that

$$\|x_1' - x_2'\|_2^2 = 0 \qquad \text{and} \qquad \|x_1 - x_2\|_2^2 = 0.$$

Then $x_1(t) = x_2(t)$ for every $t \in \langle -\pi, \pi\rangle$.

Existence.

Denote by $E_n \subset H$, $E_n = [e_1, \dots, e_n]$ the finite-dimensional subspace of H, by P_n the orthogonal projection onto E_n, $F_n = \{x \in L_2,\ x(t) \colon \langle -\pi, \pi\rangle \to E_n\}$, $\mathcal{P}_n$ the orthogonal projection of L_2 onto F_n, and denote $x_n = \mathcal{P}_n x$. (We use simply L_2, C instead of $L_2\big((-\pi,\pi), H\big)$, $C\big(\langle -\pi, \pi\rangle, H\big)$.)

Denote also $L \colon D(L) \to L_2$ the operator $Lx = -x'' + \alpha^2 x$ and $N \colon C \to C$ the operator $Nx = f\big(t, x(t), x(-t)\big)$, where $D(L) = \{x \in C,\ x' \in AC$ and $x'' \in L_2\}$.

Let us consider the problem

$$-x_n''(t) + \alpha^2 x_n(t) + P_n f\big(t, x_n(t), x_n(-t)\big) = P_n h(t) \qquad (18)$$

$$x_n(-\pi) = x_n(\pi), \qquad x_n'(-\pi) = x_n'(\pi). \qquad (2)$$

Obviously the operator $P_n f\colon \langle -\pi, \pi\rangle \times E_n \times E_n \to E_n$ is continuous and bounded. Since E_n is the finite-dimensional subspace, $P_n f$ is completely continuous.

From the inequality (16) for $u = v = 0$ we obtain

$$\big(P_n f(t,x,y), x\big) \geqq -a\|x\|^2 - b\|x\|\|y\| - c\|x\|, \qquad (19)$$

where $c = \max\limits_{t\in\langle -\pi,\pi\rangle} \|P_n f(t,0,0)\|$.
Theorem 1 implies the existence of a solution to the problem (18), (2) and a priori estimations

$$\|x\|_2 \leqq C_1, \qquad \|x'\|_2 \leqq C_2, \qquad \|x\|_0 \leqq C$$

for the solution, where C_1, C_2, C are suitable constants independent of F_n.

The complete continuity of the operator $T_n\colon C \to C$ and the a priori estimations mean that the set of the solution to the problem (18), (2) is compact in $(C, \|\cdot\|_0)$ for every $n \in \mathbb{N}$. Moreover the set of solutions is compact in $(L_2, \|\cdot\|_2)$. (These statements are trivial in case when $T_n x = x$ has a unique solution. The proof is to be used also in a more general case.)

Denote by U_n the set of solutions to (18), (2) and $V_n = \bigcup\limits_{k=n}^{\infty} U_k$. Obviously $V_n \supset V_{n+1}$ and V_n is a bounded set for every $n \in \mathbb{N}$.

Let $W_n = \overline{V}_n$ be the weak closure of V_n in L_2. Then W_n is weakly compact and $W_n \supset W_{n+1}$. Then means there is

$$x_0 \in \bigcap_{n=1}^{\infty} W_n$$

and the sequence $x_n \in V_n$ such that $x_n \rightharpoonup x_0$.

Obviously $\|x_n''\|_2 \leqq c$, where c is a suitable constant. That means we can choose from $\{x_n\}$ such subsequence that

$$Lx_n = -x_n'' + \alpha^2 x_n \rightharpoonup v \qquad \text{in} \quad L_2.$$

Since the graph of L is a closed convex set it is weakly closed and $v = Lx_0$, $x_0 \in D(L)$.

We prove the inequality

$$\langle (L+N)u - h, u - x_0 \rangle \geqq 0. \tag{20}$$

Let $u \in D(L) \cap F_m$, $x_n \in F_n$ and $n \geqq m$. Inequality (16) implies

$$\begin{aligned}
&\langle (L+N)x - (L+N)y, x - y \rangle \\
&= \|x' - y'\|_2^2 + \alpha^2 \|x - y\|_2^2 + \int_{-\pi}^{\pi} \big(f(t, x(t), x(-t)) - f(t, y(t), y(-t)), x(t) - y(t) \big) \, \mathrm{d}t \\
&\qquad \geqq \|x' - y'\|_2^2 + \big(\alpha^2 - a - |b|\big) \|x - y\|_2^2 \geqq 0.
\end{aligned}$$

Then

$$0 \leqq \langle (L+N)u - (L+N)x_n, u - x_n \rangle = \langle (L+N)u - h, u - x_n \rangle - \langle (L+N)x_n - h, u - x_n \rangle.$$

Since $H = F_n \oplus F_n^{\perp}$, $u - x_n \in F_n$, $\mathcal{P}_n\big((L+N)x_n - h\big) \in F_n$, then

$$\langle (L+N)x_n - h, u - x_n \rangle = \langle \mathcal{P}_n\big((L+N)x_n - h\big), u - x_n \rangle = 0.$$

The last equality follows from the fact that $x_n(t)$ is a solution to (18). Then

$$0 \leqq \langle (L+N)u - h, u - x_n \rangle,$$

and for $n \to \infty$ we obtain

$$0 \leqq \langle (L+N)u - h, u - x_0 \rangle.$$

Now we prove the inequality (20) for every $u \in D(L)$.
Using the Fourier series from Lemma 4 and 5 we obtain

$$u(t) = \sum_{i=1}^{\infty} a_i(t) e_i, \qquad u'(t) = \sum_{i=1}^{\infty} a_i'(t) e_i, \qquad u''(t) = \sum_{i=1}^{\infty} a_i''(t) e_i$$

where $a_i(t) = \big(u(t), e_i\big) \in C^1\big((-\pi, \pi), R\big)$ and $a_i''(t) \in L_2\big((-\pi, \pi), R\big)$.
We denote

$$u_n(t) = \sum_{i-1}^{n} a_i(t) e_i.$$

The sequence $u_n(t) \to u(t)$ in H for every $t \in \langle -\pi, \pi \rangle$. Since

$$\|u_n(s) - u_n(t)\| = \|P_n u(s) - P_n u(t)\| \leqq \|u(s) - u(t)\|,$$

the sequence $\{u_n\}$ is equicontinuous. The same is true for $\{u'_n\}$.

Then

$$u_n \rightrightarrows u, \qquad u'_n \rightrightarrows u', \qquad \text{and} \quad u''_n \to u'' \qquad \text{in} \quad L_2.$$

As $u_n \in F_n$ the inequality

$$0 \leqq \langle (L+N)u_n - h, u_n - x_0 \rangle \qquad \text{is valid.}$$

The fact that $Lu_n \to Lu$ and $Nu_n \to Nu$ in L_2 implies that

$$0 \leqq \langle (L+N)u - h, u - x_0 \rangle \qquad \text{for every} \quad u \in D(L).$$

Let now $v \in D(L)$, $\tau \geqq 0$ and $u = x_0 + \tau v$. Then

$$0 \leqq \langle (L+N)(x_0 + \tau v) - h, v \rangle$$

and for $\tau \to 0$

$$0 \leqq \langle (L+N)x_0 - h, v \rangle.$$

The density of $D(L)$ in L_2 implies that

$$(L+N)x_0 = h.$$

THEOREM 4. *Assume that* (A) *holds and that* (16) *holds for* $a + |b| < 1 + \alpha^2$. *Further assume that* (i), (ii), *or* (iii) *holds. Then there is a solution to the problem* (1), (2) *for every* $h(t)$ *such that* $\int_{-\pi}^{\pi} h(t)\,\mathrm{d}t = 0$. *In the case* (i) *or* (ii) *the solution is unique.*

Proof. Let x, y be two solutions to (1), (2). By the same method as in proof of Theorem 2 we obtain in case (i) or (ii) that

$$\int_{-\pi}^{\pi} x(t)\,\mathrm{d}t = \int_{-\pi}^{\pi} y(t)\,\mathrm{d}t = 0.$$

Using the inequality (8) in (17) we obtain

$$(1 + \alpha^2 - a - |b|)\|x - y\|_2^2 \leqq 0.$$

Then $\|x - y\|_2^2 = 0$ and since $x, y \in C^1$, $x(t) = y(t)$ for every t.

The proof of the existence of a solution is similar to that of Theorem 3, only the existence of a solution to the finite-dimensional problem (18), (2) follows now from Theorem 2.

5. Critical case

Let $\alpha = 0$ i.e. we consider the problem

$$-x''(t) + f\big(t, x(t), x(-t)\big) = h(t) \tag{21}$$

$$x(-\pi) = x(\pi), \quad x'(-\pi) = x'(\pi). \tag{2}$$

The homogeneous problem has a nontrivial solution in this case and there is no Green's function associated to the problem (21), (2).

Instead of (21) we consider the equation

$$-x''(t) + \alpha^2 x(t) + f_1\big(t, x(t), x(-t)\big) = h(t), \tag{22}$$

where

$$f_1(t, x, y) = f(t, x, y) - \alpha^2 x. \tag{23}$$

Because the function $\alpha^2 x$, as function $H \to H$, is only continuous and bounded (and is not completely continuous), we have the same assumptions for f, and use the same method as in Theorem 3 and 4.

THEOREM 5. *Assume that* (A) *and the inequality* (16) *hold for* $a + |b| < 0$. *Then the problem* (21), (2) *has a unique solution.*

P r o o f. We use the equation (22), where f_1 is given by (23). Inequality (16) implies that

$$\big(f_1(t, x, y) - f_1(t, u, v), x - u\big) \geqq -\alpha^2\|x - u\|^2 - a\|x - u\|^2 - b\|x - u\|\|y - v\|,$$

and obviously $\alpha^2 + a + |b| < \alpha^2$.
Theorem 3 implies the existence and uniqueness of the solution to the problem (22), (2) and then also to (21), (2).

THEOREM 6. *Assume that* (A) *and* (16) *hold for* $a + |b| < 1$. *Assume that* (i) *or*

(ii') *there is* $\beta > 0$ *such that*

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Big(f\big(t, x(t), x(-t)\big), x(s)\Big)\,\mathrm{d}t\,\mathrm{d}s \geq \beta \Big\|\int_{-\pi}^{\pi} x(t)\,\mathrm{d}t\Big\|^2$$

or

(iii') $$\liminf_{\substack{x \in S \\ \|x(t)\|_2 \to \infty}} \frac{\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Big(f\big(t, x(t), x(-t)\big), x(s)\Big)\,\mathrm{d}t\,\mathrm{d}s}{\Big\|\int_{-\pi}^{\pi} x(t)\,\mathrm{d}t\Big\|^2} > 0,$$

where $S = \{x(t), \int\limits_{-\pi}^{\pi} x(t) \neq 0\}$ *holds.*

Then there is a solution to the problem (21), (2) *for every* $h(t)$ *such that* $\int\limits_{-\pi}^{\pi} h(t)\,\mathrm{d}t = 0$ *and if* (i) *or* (ii') *holds, the solution is exactly one.*

P r o o f . We use again the equation (22). Assumptions (i), (ii') resp. (iii') for the function f imply that (i), (ii) resp. (iii) is true for the function f_1. In case (ii) we choose α such that $0 < \alpha < \beta$. Using Theorem 4 we obtain Theorem 6.

6. Connectedness of the set of solutions

LEMMA 9. *Let the assumptions of Theorem 1 hold. Assume that* (16) *holds for* $a + |b| = \alpha^2$.
Then the set of solutions to the problem (1), (2) *is nonempty, compact and connected. If* x, y *are solutions to* (1), (2), *then* $x - y = \mathrm{const}$.

We omit the proof of the lemma since it is similar to the one of the following

THEOREM 7. *Let the assumptions of Theorem 2 hold. Assume that* (16) *holds for* $a + |b| = \alpha^2 + 1$.

Then the set of solutions to the problem (1), (2) *is nonempty. Moreover it is compact and connected in case* (i) *or* (ii).

P r o o f . The existence of a solution follows from Theorem 2. Proving that theorem we have obtained the estimation $\|x(t)\|_0 < c$ for a solution to the equation

$$x = \lambda T x \qquad \lambda \in \langle 0, 1\rangle, \tag{10}$$

where T is given by

$$Tx(t) = -L^{-1}Nx(t) + L^{-1}h(t).$$

Moreover for every solution $x(t)$ $\int\limits_{-\pi}^{\pi} x(t)\,\mathrm{d}t = 0$ is valid when (i) or (ii) holds.

For x, y solutions to (1), (2) we obtain

$$0 = \int\limits_{-\pi}^{\pi} \big(x'' - y'' + \alpha^2(x - y) + f(t, x(t), x(-t)) - f(t, y(t), y(-t)), x(t) - y(t)\big)\,\mathrm{d}t$$

$$= \|x' - y'\|_2^2 + \alpha^2\|x - y\|_2^2 + \int\limits_{-\pi}^{-\pi} f(t, x(t), x(-t)) - f(t, y(t), y(-t)), x(t) - y(t)\big)\,\mathrm{d}t$$

and using the estimation (8) we get

$$(1+\alpha^2)\|x-y\|_2^2+\int_{-\pi}^{\pi}\big(f\big(t,x(t),x(-t)\big)-f\big(t,y(t),y(-t)\big),x(t)-y(t)\big)\,\mathrm{d}t \leqq 0. \quad (24)$$

We use Krasnosel'skij's theorem [7, p. 155]. We choose $f_n(t,x,y) = \lambda_n f(t,x,y)$, where $0 < \lambda_n < 1$ and $\lambda_n \to 1$. Obviously f_n satisfies the same assumptions as f, i.e. f_n is completely continuous and satisfies (i) resp. (ii).

We define the operator T_n by

$$T_n x(t) = -L^{-1}N_n x(t) + L^{-1}h(t),$$

where

$$N_n x(t) = f_n\big(t,x,(t),x(-t)\big).$$

Then the sequence $\{T_n\}$ and the operator T satisfies the assumptions of Krasnosel'skij's theorem.

Really, if we choose $\Omega = \{x(t) \in C,\ \|x(t)\|_0 < c\}$, then

$$\sup_{x\in\Omega} \|T_n(x) - T(x)\|_0 \to 0,$$

the estimation $\|x\|_0 < c$ implies that the Leray-Schauder degree

$$d(I-T,\,\Omega,\,0) \neq 0 \qquad \text{and} \qquad Tx \neq x \quad \text{on} \quad \partial\Omega.$$

Using the estimation (24), we obtain that there is at most one solution to the equation

$$x = T_n x + z \qquad \text{for every} \quad z \in C.$$

The Krasnosel'skij theorem implies that the set of solutions is compact and connected.

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Received May 18, 1989

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