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OSCILLATIONS OF DIFFERENTIAL EQUATION WITH RETARDED ARGUMENT

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In the present paper we shall investigate the second order nonlinear differential equation of the form

$$(r(t)y'(t))' + p(t)f(y(\varrho_1(t)))h(y'(\varrho_2(t))) = 0. \tag{1}$$

Many authors studied the properties of solutions of the equation (1) with $r(t) \equiv 1$, $p(t) \geq 0$, $f(y) = y$ or $f(y) = y^{\alpha}$, $h(z) = 1$ (see the papers [1], [4—7]).

This paper is concerned with the oscillatory behaviour of the solutions of equation (1). We shall assume the validity of the following conditions:

1) a) $r(t) > 0$, $p(t) \leq 0$

b) $r(t) > 0$, $p(t) \geq 0$

where $r(t)$, $p(t)$ are continuous functions on $J = \langle t_0, \infty)$, $t_0 \in R = (-\infty, \infty)$;

2) $f(y)y > 0$ for $y \in R$, $y \neq 0$, continuous function on R;

3) $h(z) > 0$ and continuous on R;

4) $\varrho_i(t) \leq t$, $\varrho_i(t) \to \infty$ for $t \to \infty$, $i = 1, 2$ are continuous functions on J.

We restrict our consideration to those solutions $y(t)$ of (1) which exist on some interval J and satisfy

$$\sup \{|y(s)| : t \leq s < \infty\} > 0$$

for any $t \in J$. Such a solution is said to be oscillatory if the set of zeros of $y(t)$ is not bounded from the right. Otherwise, the solution $y(t)$ is said to be nonoscillatory. Let us denote $\gamma(t) = \sup\{s \geq t_0;\ \varrho_1(s) \leq t\}$ for $t \geq t_0$. We see that $t \leq \gamma(t)$ and $\varrho_1(\gamma(t)) = t$. Another property of the function $\gamma(t)$ is given in the following lemma:

Lemma 1. *For every t such that $t_0 \leq t < \infty$, the value $\gamma(t)$ is finite.*

Proof of Lemma may be found in [9].

I.

The first part of the present paper deals with the oscillatoriness of the solutions of equation (1) under the assumptions 1a), 2)—4).

The following theorem is a generalization of Theorem 1 in [8] and Lemma 2.1 of [3].

Theorem 1. *Suppose that for all* $t\in J$

$$r(t)\geqq r_0>0,\quad r_0\in R$$

and

$$\int^{\infty}\frac{\mathrm{d}t}{r(t)}=+\infty. \tag{2}$$

Let there exist a differentiable function $a(t)$, *non-negative on* J *such* $a'(t)r(t)\leqslant K$, $K\in R$ *and*

$$\int^{\infty}a(t)p(t)\,\mathrm{d}t=-\infty. \tag{3}$$

Then every non-oscillatory solution $y(t)$ *of* (1) *is either* $|y(t)|\to\infty$ *for* $t\to\infty$ *or*

$$\lim_{t\to\infty}y(t)=\lim_{t\to\infty}r(t)y'(t)=0.$$

Proof. Let $y(t)$ be a non-oscillatory solution of (1). Then there exists $t_1\geqslant t_0$ such that $y(t)\neq 0$ and $y(\varrho_1(t))\neq 0$ for every $t\geqslant t_1$. Let $y(t)>0$, $y(\varrho_1(t))>0$. Then

$$[r(t)y'(t)]'=-p(t)f(y(\varrho_1(t)))h(y'(\varrho_2(t)))\geqq 0.$$

We have to investigate the following cases:

i) $y(t)>0$, $y'(t)\leqslant 0$ for every $t\geqslant t_1$;

ii) there exists $t_2\geqslant t_1$ such that for $t\geqslant t_2$, $y'(t)>0$.

If case ii) takes place, then for $t\geqslant t_2$ we have

$$y'(t)\geqslant\frac{r(t_2)y'(t_2)}{r(t)}.$$

Using (2) we see that $y(t)\to\infty$ for $t\to\infty$.

If i) holds, then from (1) we get

$$\begin{aligned}\int_{t_1}^{t}a(s)[r(s)y'(s)]'\,\mathrm{d}s&=a(t)r(t)y'(t)-\int_{t_1}^{t}a'(s)r(s)y'(s)\,\mathrm{d}s=\\&=a(t_1)r(t_1)y'(t_1)-\int_{t_1}^{t}a(s)p(s)f(y(\varrho_1(s)))h(y'(\varrho_2(s)))\,\mathrm{d}s\end{aligned} \tag{4}$$

for $t\geqslant t_1$. Since $h(z)$ is continuous and for $t\geqslant t_1$

$$\frac{r(\varrho_2(t_1))y'(\varrho_2(t_1))}{r_0}\leqslant y'(t)\leqslant 0$$

holds, there exists $\beta \in \left\langle \frac{r(\varrho_2(t_1)),\ y'(\varrho_2(t_1))}{r_0};\ 0 \right\rangle$ such that for $t \geqslant t_1$

$$h(\beta) \leqq h(y'(\varrho_2(t))).$$

Let now $\lim_{t\to\infty} y(t) = c > 0$. Then there exists a number $\alpha \in \langle c, y(t_1)\rangle$ such that

$$f(\alpha) \leqslant f(y(\varrho_1(t))), \quad \text{for every} \quad t \geqslant t_2 = \gamma(t_1).$$

From (4) we have

$$a(t)r(t)y'(t) \geqq k_0 + K[y(t) - y(t_2)] - f(\alpha)h(\beta)\int_{t_2}^{t} a(s)p(s)\,\mathrm{d}s, \tag{5}$$

where $k_0 = a(t_2)r(t_2)y'(t_2)$. Using (5) we see that $a(t)r(t)y'(t) \to +\infty$ for $t \to \infty$, which contradicts the fact that $y'(t) \leqslant 0$. Therefore, $c = 0$.

From the equation (1) it follows that

$$[r(t)y'(t)]' \geqq 0$$

and therefore the limit $\lim_{t\to\infty} r(t)y'(t) = c_1 \leqslant 0$ exists. Let $c_1 < 0$, then for every $t \geqslant t_2$ there is $r(t)y'(t) \leqslant c_1$ and

$$y(t) \leqslant y(t_2) + c_1 \int_{t_2}^{t} \frac{\mathrm{d}s}{r(s)} \to -\infty \quad \text{for} \quad t \to \infty.$$

This is a contradiction.

Theorem 2. *Suppose that $\varrho_1(t)$ is non-decreasing in J and there exists a number $k_0 > 0$ such that*

$$\liminf_{y\to 0} \frac{f(y)}{y} > k_0. \tag{6}$$

Let there further exist a sequence $\{t_n\}_{n=1}^{\infty}$, $t_n \to \infty$ so that for sufficiently large n

$$\int_{\varrho_1(t_n)}^{t_n} [R(s) - R(\varrho_1(t_n))]p(s)\,\mathrm{d}s \leqslant -\frac{1}{k_0 h_0} \tag{7}$$

is true, where $r(t) = \int_{t_0}^{t} \frac{\mathrm{d}s}{r(s)}$ and $0 < h_0 = \inf_{z\in R} h(z)$.

If (2) holds, then any bounded solution $y(t)$ of (1) is oscillatory on J.

Proof. Suppose that $y(t)$ is a bounded solution of (1), e.g. such that $y(t) > 0$, $y(\varrho_1(t)) > 0$ for $t \geqslant t_1 \geqslant t_0$. The equation (1) yields

$$[r(t)y'(t)]' \geqslant 0.$$

Analogously with Theorem 1 we have two cases:

i) $y'(t) \leqslant 0$ for $t \geqslant t_1$

ii) there exists $t_2 \geqslant t_1$ such that $y'(t) > 0$ for $t \geqslant t_2$. Suppose that i) holds true.

Integrating the equation (1) from s to $t \geqslant s$, $s \geqslant t_1$, and then from $\varrho_1(t)$ to $t \geqslant \varrho_1(t)$, we get

$$y(\varrho_1(t)) \geqslant y(t) - h_0 \int_{\varrho_1(t)}^{t} \frac{1}{r(s)} \int_s^t p(u) f(y(\varrho_1(u)))\, du\, ds. \tag{8}$$

Let $\lim\limits_{t\to\infty} y(t) = L > 0$. Then there exists a number $\alpha \in \langle L, y(\varrho_1(t_1))\rangle$ such that for every $t \geqslant t_1$

$$0 < f(\alpha) \leqslant f(y(\varrho_1(t)))$$

is true. The inequality (8) implies

$$\frac{y(\varrho_1(t)) - y(t)}{h_0 f(\alpha)} \geqq - \int_{\varrho_1(t)}^{t} [R(s) - R(\varrho_1(t))] p(s)\, ds. \tag{9}$$

Since

$$\lim_{t\to\infty} \frac{y(\varrho_1(t)) - y(t)}{h_0 f(\alpha)} = 0$$

there exists a $T \geqslant t_1$ such that

$$\frac{y(\varrho_1(t)) - y(t)}{h_0 f(\alpha)} < \frac{1}{h_0 k_0} \quad \text{for every} \quad t \geqslant T.$$

From (9) for sufficiently large n, we may put $t = t_n \geqslant T$, we obtain a contradiction with (7).

Suppose now that $\lim\limits_{t\to\infty} y(t) = 0$. Then (8) yields

$$1 \geqslant -h_0 \int_{\varrho_1(t)}^{t} [R(s) - R(\varrho_1(t))] p(s) \frac{f(y(\varrho_1(s)))}{y(\varrho_1(s))}\, ds.$$

Using (6) we see that there exists $T_1 > T$ such that for every $t \geqslant T_1$

$$\frac{f(y(\varrho_1(t)))}{y(\varrho_1(t))} > k_0$$

is true, which means that from the last two inequalities we have

$$1 > -k_0 h_0 \int_{\varrho_1(t)}^{t} [R(s) - R(\varrho_1(t))] p(s)\, ds.$$

If we put $t = t_n$, this again leads to a contradiction with (7) for sufficiently large n.

If case ii) takes place, then

$$r(t) y'(t) \geqslant r(t_2) y'(t_2) > 0 \quad \text{for} \quad t \geqslant t_2.$$

Considering the assumption (2) we have a contradiction with the boundedness of the solution.

Remark 1. Theorem 2 is a generalization of Theorem 3.1 in [2].

Theorem 3. *The hypotheses of this theorem are the same as those for Theorem 2 except that instead of* (2) *and* (7) *we suppose*

$$0<\limsup_{t\to\infty} r(t)\int_{\varrho_1(t)}^{t}\frac{ds}{r(s)}=K_0<\infty \tag{2'}$$

and

$$\limsup_{t\to\infty}\int_{\varrho_1(t)}^{t}[R(s)-R(\varrho_1(t))]p(s)\,ds\leq -\frac{1}{h_0k_0}. \tag{7'}$$

Then all bounded solutions of (1) *are oscillatory.*

Proof. Analogously to Theorem 2 in case i) we have from (8)

$$\frac{1}{k_0h_0}>-\int_{\varrho_1(t)}^{t}[R(s)-R(\varrho_1(r))]p(s)\,ds \quad \text{for} \quad t\geq t_1$$

which contradicts (7′).

Suppose that ii) obtains. From equation (1) we get for $t\geq s\geq t_2$

$$y'(t)r(t)\int_{\varrho_1(t)}^{t}\frac{ds}{r(s)}\geqq y(t)-y(\varrho_1(t))-h_0\int_{\varrho_1(t)}^{t}\frac{1}{r(s)}\int_s^t p(u)f(y(\varrho_1(u)))\,du\,ds.$$

Since $y(t)$ is bounded, there exists a number

$$\alpha\in\langle y(\varrho_1(t_2)),K\rangle$$

such that for $t\geq t_3=\gamma(t_2)$

$$0<f(\alpha)\leq f(y(\varrho_1(t)))$$

is true. From the last two inequalities we get

$$y'(t)r(t)\int_{\varrho_1(t)}^{t}\frac{ds}{r(s)}\geqq y(t)-y(\varrho_1(t))-h_0f(\alpha)\int_{\varrho_1(t)}^{t}[R(s)-R(\varrho_1(t))]p(s)\,ds. \tag{10}$$

According to the hypotheses (2′) and (7′) there exists $t_4\geq t_3$ such that for $t\geq t_4$

$$r(t)\int_{\varrho_1(t)}^{t}\frac{ds}{r(s)}\leq 2K_0$$

and

$$\int_{\varrho_1(t)}^{t}[R(s)-R(\varrho_1(t))]p(s)\,ds\leq -\frac{1}{2k_0h_0}.$$

Hence, in view of (10) we have

$$y'(t)\geq\frac{f(\alpha)}{4k_0K_0}>0 \quad \text{for} \quad t\geq t_4,$$

which again contradicts the fact that $y(t)$ is a bounded solution of (1).

II.

The next part of the present paper contains some sufficient conditions for the oscillatory properties of the solutions of equation (1) under the conditions 1b), 2)—4).

Theorem 4. *Let for every* $t \in J$ $r(t) \geqslant r_0 > 0$, $r_0 \in R$ *hold and let* $a(t)$ *be a differentiable non-negative function such that for every* $t \in J$

$$a'(t)r(t) \leqq K < \infty$$

If

$$\int^{\infty} a(s)p(s)\,\mathrm{d}s = +\infty \tag{11}$$

and (2) *hold, then any non-oscillatory solution* $y(t)$ *of* (1) *is unbounded.*

Proof. Let $y(t)$ be a solution of (1), e.g. such that $y(t)>0$, $y(\varrho_1(t))>0$ for $t \geqslant t_1 \geqslant t_0$. We have to investigate the following cases:

i) $y(t)>0$, $y'(t) \geqslant 0$ for $t \geqslant t_1$;

ii) then there exists $t_2 \geqslant t_1$ such that $y(t)>0$, $y'(t)<0$ for $t \geqslant t_2$.

If ii) holds, then (1) yields

$$r(t)y'(t) \leqslant r(t_2)y'(t_2) \quad \text{for} \quad t \geqslant t_2.$$

Using (2) we see that $y(t) \to -\infty$ for $t \to \infty$, which contradicts the positivity of $y(t)$ for $t \geqslant t_2$.

Let i) hold and $y(t)$ is a bounded solution. Then there exist numbers $k_1>0$, $K_1>0$ and $\alpha \in \langle k_1, K_1 \rangle$, such that

$$0 < f(\alpha) \leqslant f(y(\varrho_1(t))) \quad \text{for} \quad t \geqq t_2 = \gamma(t_1)$$

Evidently for $t \geqslant t_1$ we have also

$$0 \leqslant y'(t) \leqslant \frac{r(t_1)y'(t_1)}{r_0}$$

and there exists β such that

$$h(\beta) \leqslant h(y'(\varrho_2(t))) \quad \text{for} \quad t \geqslant t_2.$$

Therefore we have from (1)

$$a(t)[r(t)y'(t)]' + f(\alpha)h(\beta)a(t)p(t) \leqslant 0$$

and integrating this inequality from t_2 to $t \geqslant t_2$ we get

$$a(t)r(t)y'(t) + f(\alpha)h(\beta)\int_{t_2}^{t} a(s)p(s)\,\mathrm{d}s \leqslant a(t_2)r(t_2)y'(t_2) + 2KK_1,$$

which contradicts the positivity of $y'(t)$ for $t \to \infty$.

Theorem 5. *Let the hypotheses of* Theorem 4 *be satisfied and instead of the assumption* $r(t) \geq r_0 > 0$ *we suppose that*

$$\inf_{z \in R} h(z) = h_0 > 0, \quad h_0 \in R.$$

Then all bounded solutions $y(t)$ of (1) are oscillatory.

Proof. The proof is analogous to proof of Theorem 4.

Theorem 6. *Let* $a(t)$ *be a differentiable, positive function on J such that* (11) *and*

$$\int_{t_0}^{\infty} \frac{\{a'(s)\}_+}{a(s)} \, ds = A < \infty$$

hold. Suppose further that $f(y)$ *is non-decreasing on R,* $\inf_{z \in R} h(z) = h_0 > 0$ *and* (2) *holds. Then every solution* $y(t)$ *of* (1) *is oscillatory.*

Proof. Suppose that (1) has a non-oscillatory solution $y(t)$, e.g. that $y(t) > 0$, $y(\varrho_1(t)) > 0$ for all $t \geq t_1 \geq t_0$. In view of (2) it is sufficient to consider the case i), it means $y(t) > 0$, $y'(t) \geq 0$ for $t \geq t_1$. From (1) we get

$$\begin{aligned} &a(t)r(t)y'(t) - \int_{t_2}^{t} a'(s)r(s)y'(s) \, ds + \\ &+ f(y(\varrho_1(t_2)))h_0 \int_{t_2}^{t} a(s)p(s) \, ds \leq a(t_2)r(t_2)y'(t_2) = c_1 \end{aligned} \tag{12}$$

for $t \geq t_2 = \gamma(t_1)$ and then (12) yields

$$a(t)r(t)y'(t) \leq c_1 + \int_{t_2}^{t} \frac{\{a'(s)\}_+}{a(s)} a(s)r(s)y'(s) \, ds.$$

Using the Gronwall inequality we get

$$a(t)r(t)y'(t) \leq c_1 \exp \int_{t_2}^{t} \frac{\{a'(s)\}_+}{a(s)} \, ds \leq c_1 \exp A.$$

We further have from (12) for $t \geq t_2$

$$a(t)r(t)y'(t) + f(y(\varrho_1(t_2)))h_0 \int_{t_2}^{t} a(s)p(s) \, ds \leq c_1 + Ac_1 \exp A$$

and so using (11) we get that

$$a(t)r(t)y'(t) \to -\infty \quad \text{for} \quad t \to \infty.$$

This is a contradiction with $y'(t) > 0$.

Remark 2. If we put $a(t) \equiv 1$, we have Theorem 3 in [8].

Theorem 7. *Let the assumptions of* Theorem 6 *be satisfied with the exception that instead of* $f(y)$ *to be non-decreasing we suppose that*

$$\int^{\infty} \frac{\mathrm{d}s}{a(s)r(s)} < \infty. \tag{13}$$

Then any solution $y(t)$ *of* (1) *is oscillatory.*

Proof. Analogously to Theorem 6 it is easy to verify that for $t \geq t_2 = \gamma(t_1)$

$$a(t)r(t)y'(t) + h_0 \int_{t_2}^{t} a(s)p(s)f(y(\varrho_1(s)))\,\mathrm{d}s \leq c_1 + Ac_1 \exp A = B \tag{14}$$

holds. From (13) and (14) it follows

$$0 < y(t) \leq y(t_2) + B\int_{t_2}^{t} \frac{\mathrm{d}s}{a(s)r(s)},$$

which means that $y(t)$ is a bounded solution. Thus from (14) we get

$$a(t)r(t)y'(t) + f(\alpha)h_0 \int_{t_2}^{t} a(s)p(s)\,\mathrm{d}s \leq B, \tag{15}$$

where α is such a number that for $t \geq t_2 = \gamma(t_1)$

$$f(\alpha) \leq f(y(\varrho_1(t))).$$

From (15) we have for $t \to \infty$ a contradiction with $y'(t) > 0$.

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КОЛЕБЛЕМОСТЬ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С ЗАПАЗДЫВАЮЩИМ АРГУМЕНТОМ

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Резюме

В статье приведены достаточные условия для того, чтобы решения дифференциального уравнения

$$(r(t)y'(t))' + p(t)f(y(\varrho_1(t)))h(y'(\varrho_2(t))) = 0$$

были колеблющиеся.