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EIGENVALUES AND DOMINATION IN GRAPHS

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ABSTRACT. Let G be a finite connected simple graph with n vertices. We show a close relation between the domination number $\gamma(G)$ and the largest eigenvalue λ_n of the Laplacian matrix of G. If $\gamma(G) \geq 3$, then $\lambda_n < n - \left\lceil \frac{\gamma(G)-2}{2} \right\rceil$. If $\gamma(G) = 1$, then $\lambda_n = n$. If $\gamma(G) = 2$, no better bound than $\lambda_n \leq n$ exists. Furthermore we show that eigenvectors corresponding to large eigenvalues induce dominating sets in G.

1. Terminology

By $G(V, E)$ we denote a graph with vertex-set $V(G)$ and edge-set $E(G)$. Graphs considered in this paper are finite, undirected and contain neither loops nor multiple edges – they are so-called *simple graphs*. For $v \in V(G)$ we denote by N_v the *neighbourhood* of v in G. The *valency* d_v of v is the cardinality of N_v, and $\Delta = \max\limits_{v \in V(G)} d_v$. The *complement* $\overline{G}$ of a graph G is given by $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{(v,w) \mid v, w \in V(G),\ (v,w) \notin E(G)\}$. A subset $D \subset V(G)$ is called *dominating set* if every vertex $v \notin D$ is adjacent to at least one vertex of D. The *domination number* $\gamma(G)$ is the cardinality of a smallest dominating set. A dominating set D with $|D| = \gamma(G)$ is called a *minimum dominating set.*

By $L(G)$ we denote the *Laplacian matrix* of G, i.e., $L(G) = D(G) - A(G)$, where $A(G)$ is the adjacency matrix of G and $D(G)$ is the diagonal matrix with the corresponding valencies in the main diagonal. The eigenvalues of $L(G)$, $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq n$, $n = |V(G)|$, are simply called *eigenvalues of* G. If necessary, we write $\lambda_k(G)$ to emphasize that we consider the k-th smallest eigenvalue of a particular graph G.

The *usual eigenvector partition* with respect to an eigenvector is given as follows: Let $\vec{x}_k$ denote an eigenvector of $L(G)$ with respect to $\lambda_k \neq 0$. (If G

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is connected, $\lambda_k > 0$ for $k \geq 2$.) Let $V_1 \subset V(G)$ now denote the set that contains all vertices whose corresponding entries in $\vec{x}_k$ are negative, and let V_2 be the set that contains all vertices with entries ≥ 0. Clearly, $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = V(G)$.

2. Introduction

Since Fiedler's fundamental paper [2] on eigenvalues of the Laplacian matrix of graphs appeared, many papers relating eigenvalues to various properties of graphs were published (for surveys see, e.g., [3], [4]). But as far as we know, the relationship between domination numbers and eigenvalues has not yet been studied.

The main result of this paper shows that there is a close connection between the domination number $\gamma(G)$ of a graph G, $|V(G)| = n$, and its largest eigenvalue λ_n. We prove that $\lambda_n < n - \left\lceil \frac{\gamma(G)-2}{2} \right\rceil$ if $\gamma(G) \geq 3$, which is best possible. If $\gamma(G) = 1$, then $\lambda_n = n$, if $\gamma(G) = 2$, no better bound than $\lambda_n \leq n$ exists. Roughly spoken this means that large eigenvalues always imply a small domination number. If λ_n is large, say $\lambda_n \geq n - m$ for some fixed $m > 0$, a minimum dominating set can always be found in a polynomial time, simply by checking all subsets of $V(G)$ of cardinality less than $2(m+1)$. On the other hand, this result does not supply any information on the domination number if the largest eigenvalue of a graph is small, e.g., if $\lambda_n \leq \frac{n}{2}$.

Recently (see, e.g., [5], [6]) eigenvectors corresponding to the second smallest eigenvalue λ_2 were used to obtain "good" domain decompositions of finite element meshes. First results concerning this problem were already shown in Fiedler's paper [2] mentioned above.

Motivated by the connection between eigenvalues and domination numbers – in particular, by the fact that we involved the second smallest eigenvalue of the complement of a graph to prove it – we looked for possibilities to obtain dominating sets from eigenvectors. Of course, in general, we cannot expect to obtain minimum dominating sets from eigenvectors since the problem of finding such sets was shown to be NP-complete – even for subgraphs of grid graphs (a *grid graph* is the cartesian product of two paths). For specific eigenvector partitions (in general, however, not for the usual eigenvector partition) it is possible to obtain dominating sets.

3. The results

We first prove several lemmas. Then the proof of our main result is an immediate consequence of those lemmas.

Lemma 3.1. *Let $\gamma(G) = k \geq 3$. For every subset $W \subset V(G)$ with $|W| = m < k$ there exist at least $k - m$ vertices that are common neighbours in $\overline{G}$ to the vertices of W.*

Proof. By induction. If we start with $W_m = W$, then W_m is too small to be a dominating set in G. Thus there exists some $v \notin W_m$ not adjacent to any vertex of W_m. Let $W_{m+1} = W_m \cup \{v\}$ and iterate until W_k is obtained. The $k - m$ vertices in $W_k \setminus W_m$ are not adjacent to any vertex of W_m. □

Lemma 3.2. *Let $\gamma(G) = k \geq 3$, and let $V_1, V_2 \neq \emptyset$, $V_1 \cap V_2 = \emptyset$, $V_1 \cup V_2 = V(G)$ denote a partition of $V(G) = V(\overline{G})$. Then every $w \in V_1$ is not adjacent to at least $j = \left\lceil \frac{k-2}{2} \right\rceil$ vertices in V_2, or, symmetrically, every $w \in V_2$ is not adjacent to at least j vertices in V_1.*

Proof. Suppose there is a $w \in V_1$ that has only $m < j$ neighbours in $\overline{G}$ belonging to V_2. By Lemma 3.1, any pair of vertices (w, v), $v \in V_2$, has at least $k - 2$ common neighbours in $\overline{G}$. This implies that any $v \in V_2$ has at least $k - 2 - m \geq j$ neighbours in $\overline{G}$ that are contained in V_1. □

To fix terminology, we emphasize that for a vertex v we denote the corresponding component in $\vec{x}$ by x_v, while for vertices v_i we simply write x_i.

Lemma 3.3. *Let V_1, V_2 denote the usual partition of $V(G)$ with respect to an eigenvector $\vec{x} = (x_1, \ldots, x_n)^T$, $L(G)\vec{x} = \lambda\vec{x}$. Let $|N_w \cap V_1| = r \geq 1$, where $w \in V_2$ satisfies $x_w = \max_{v \in V_2} x_v$. Then $\lambda > r$.*

Proof. We assume that $N_w \cap V_1 = \{v_1, \ldots, v_r\}$ and $N_w \cap V_2 = \{v_{r+1}, \ldots, v_{r+s}\}$, $s \geq 0$. By x_j, $1 \leq j \leq r + s$, we denote the corresponding entries of $\vec{x}$. Then, in the equation $L(G)\vec{x} = \lambda\vec{x}$, the row corresponding to w is

$$\begin{aligned} & d_w x_w - x_1 - \cdots - x_r - x_{r+1} - \cdots - x_{r+s} \\ & \quad = (x_w - x_1) + \cdots + (x_w - x_r) + (x_w - x_{r+1}) + \cdots + (x_w - x_{r+s}) = \lambda x_w . \end{aligned}$$

Since $x_w - x_m > x_w$ for $1 \leq m \leq r$ and $x_w - x_{r+l} \geq 0$ for $1 \leq l \leq s$, our result follows immediately. □

We observe that an analogous result can be shown if $|N_w \cap V_2| = r \geq 1$, where $w \in V_1$ satisfies $x_w = \min_{v \in V_1} x_v$. But, in this case, equality can occur, i.e., $\lambda \geq r$, since w may be adjacent only to vertices $v \in V_2$ with $x_v = 0$. However, if λ is a maximal eigenvalue, then a vertex satisfying $x_w = \min_{v \in V_1} x_v$ must be adjacent to at least one vertex u with $x_u > 0$. Thus, for a maximum eigenvalue strict inequality, i.e., $\lambda > r$, always holds.

THEOREM 3.4. *Let G be a connected graph with $|V(G)| = n$. If $\gamma(G) = k \geq 3$, then $\lambda_n < n - \left\lceil \frac{k-2}{2} \right\rceil$. If $\gamma(G) = 1$, then $\lambda_n = n$, if $\gamma(G) = 2$, no better bound than $\lambda_n \leq n$ exists.*

P r o o f. If $\gamma(G) = 1$, then G contains a vertex of degree $n-1$. This means that G contains a subgraph isomorphic to the star on n vertices. Clearly, $\lambda_n = n$ holds for that star, and the interlacing theorem (see, e.g., [3]) completes the proof.

Complete bipartite graphs have domination number 2 and $\lambda_n = n$. Thus, in the case $\gamma(G) = 2$, no better bound than $\lambda_n \leq n$ exists.

Let $k \geq 3$. Perform the usual eigenvector partition V_1, V_2 of $V(G)$ with respect to an eigenvector $\vec{x}$ and the corresponding eigenvalue λ. Let $v \in V_1$ and $w \in V_2$ be the vertices corresponding to a minimum and a maximum component in $\vec{x}$, respectively.

Lemma 3.2 states that v is not adjacent to at least $\left\lceil \frac{k-2}{2} \right\rceil$ vertices in V_2, or that w is not adjacent to at least $\left\lceil \frac{k-2}{2} \right\rceil$ vertices in V_1. We may assume that the second clause holds.

Thus, there are at least $\left\lceil \frac{k-2}{2} \right\rceil$ edges in the complement $\overline{G}$ between w and V_1. Moreover, $\vec{x}$ is an eigenvector of $\overline{G}$ with eigenvalue $n - \lambda$. Lemma 3.3 applied to $\overline{G}$ ensures that $n - \lambda > \left\lceil \frac{k-2}{2} \right\rceil$.

If we consider $v \in V_1$, then these conclusions – according to the remarks following Lemma 3.3 – only hold if λ is a maximal eigenvalue, which is still sufficient for this proof. □

We observe that the bound given in Theorem 3.4 is best possible. Examples of graphs G with $\gamma(G) = 3$ and largest eigenvalue less than but arbitrarily close to $n - 1$ are given below (similar constructions are possible for $\gamma(G) = 4$):

Take a complete graph on the vertices $\{v_1, \dots, v_{2m+1}\}$. Add three new vertices v_{2m+2}, v_{2m+3} and v_{2m+4}. Let v_{2m+2} be adjacent to $v_1, \dots, v_m$, v_{2m+3} to $v_{m+1}, \dots, v_{2m}$, and v_{2m+3} to v_{2m+1}. The vector $\vec{x} = (a, \dots, a, -a, \dots \dots, -a, 1, -1, 0)$ is an eigenvector for $a = \frac{n \pm \sqrt{n^2 + 8n - 32}}{2n - 8}$. The corresponding eigenvalues are $\lambda_{1,2} = \left(3n - 8 \pm \sqrt{n^2 + 8n - 32}\right)/4$. For $n \to \infty$, $\lambda_1 = n - 1 - \frac{6}{n} + O\left(\frac{1}{n^2}\right)$. All other eigenvalues, which can be found by exploiting symmetry and equitable partitions, are smaller than λ_1.

The knowledge of this close relationship between the domination number and the largest eigenvalue of a graph suggests to exploit it algorithmically. The following considerations show that there is also a close relationship between eigenvector partitions and dominating sets. Unfortunately, the dominating sets we obtain by eigenvector partitions are in general far away from minimum dominating sets, although we obtain minimum sets in some special cases.

The eigenvector partition we have described in the introduction in general gives no partition into dominating sets. To prove that for certain eigenvalues we always obtain a corresponding eigenvector partition into dominating sets, we first describe one of the possible eigenvector partitions which lead to dominating sets:

Let $\vec{x}$ be an eigenvector of G, $\vec{x} \neq (1,\dots,1)^T$. Then $D_+^0 = \{v \in V(G) \mid x_v > 0\}$, $D_-^0 = \{v \in V(G) \mid x_v < 0\}$ and $D_0^0 = \{v \in V(G) \mid x_v = 0\}$. For $i \geq 0$ we now define:

$$\begin{aligned}
D_+^{i+1} &= D_+^i \cup \{v \in D_0^i \mid (v,w) \in E(G) \text{ for some } w \in D_-^i\},\\
D_0^{(i+1)'} &= D_0^i \setminus D_+^{i+1},\\
D_-^{i+1} &= D_-^i \cup \{v \in D_0^{(i+1)'} \mid (v,w) \in E(G) \text{ for some } w \in D_+^{i+1}\},\\
D_0^{i+1} &= D_0^{(i+1)'} \setminus D_-^i.
\end{aligned}$$

The recursive definition of these sets stops whenever $D_0^m = \emptyset$ for some $m \geq 0$. For convenience, we set $D_+ = D_+^m$ and $D_- = D_-^m$.

PROPOSITION 3.5. *Let $L(G)\vec{x} = \lambda\vec{x}$, where $\lambda > \Delta$. Then the sets D_+ and D_- corresponding to $\vec{x}$ are dominating sets.*

Proof. Assume first that $v \in D_+^0$. Then, since

$$d_v x_v - \sum_{(v,w)\in E(G)} x_w = \lambda x_v$$

and $\lambda > \Delta \geq d_v$, at least one x_w in the above sum must be negative. Hence, every vertex of D_+^0 is adjacent to a vertex of D_-^0.

Analogously, we show that every $v \in D_-^0$ is adjacent to at least one $w \in D_+^0$.

Let $v \in D_0^0$. If v is adjacent to at least one $w \in D_+^0$, then the above equation implies that it is also adjacent to a vertex of D_-^0.

Hence, if $D_0^0 = \emptyset$, the sets D_-^0 and D_+^0 are already dominating sets. If $D_0^0 \neq \emptyset$, then the fact that every vertex $v \in D_0^0$ which is adjacent to a vertex of D_+^0 is also adjacent to a vertex of D_-^0 immediately implies that the above defined sets D_- and D_+ are always dominating sets. □

We observe that graphs with $D_0^0 \neq \emptyset$ exist – even if $\vec{x}$ is an eigenvector corresponding to the largest eigenvalue. Hence, the usual eigenvector partition will not always lead to dominating sets.

Also, an alternative partition of $V(G)$ into two dominating sets can be found as follows:

We define D_-^0, D_+^0 and D_0^0 as above with respect to an eigenvector whose corresponding eigenvalue satisfies the assumptions of Proposition 3.5. Then, we take all connected subgraphs of G which are induced by D_0^0. For each of these graphs we determine an eigenvalue which satisfies the assumptions of Proposition 3.5 and a corresponding eigenvector $\vec{y}$. Then join the vertices with positive entries in $\vec{y}$ to D_+^0 and those with negative entries to D_-^0. If there are still some vertices neither in D_-^0 nor in D_+^0, we repeat this procedure until we obtain a partition. Because of the adjacencies we have proven above, this final partition of $V(G)$ is again a partition into two dominating sets.

Graphs G for which at least the partition with respect to an eigenvector corresponding to the largest eigenvalue gives a minimum dominating set are the following:

G is bipartite, and the bipartition is given by D and $V(G) \setminus D$, where D is a minimum dominating set.

To see that the algorithm determines a minimal dominating set for these graphs, we consider the Rayleigh quotient of the Laplacian $L(G)$. For the largest eigenvalue λ we know that $\lambda = \max\limits_{y \in \mathbb{R}^n} \dfrac{y^T L(G) y}{y^T y}$. For bipartite graphs this maximum is attained for a vector $\vec{x}$, where the entries of $\vec{x}$ corresponding to one of the sets of the bipartition all have the same sign. Hence, if one of these sets corresponds to a minimum dominating set, the algorithm returns this set.

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